Three-Step and Four-Step Random Walk Integrals

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Abstract

We investigate the moments of 3-step and 4-step uniform random walk in the plane. In particular, we further analyse a formula conjectured in [BNSW09] expressing 4-step moments in terms of 3-step moments. Diverse related results including hypergeometric and elliptic closed forms for $W_4(\pm 1)$ are given and two new conjectures are recorded.

1 Introduction and Preliminaries

Continuing research commenced in [BNSW09], for complex s, we consider the n-dimensional integral

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x} \tag{1}$$

which occurs in the theory of uniform random walk integrals in the plane, where at each step a unit-step is taken in a random direction. As such, the integral (1) expresses the s-th moment of the distance to the origin after n steps. The study of such walks largely originated with Pearson more than a century ago [Pea1905, Pea1905b]. In his honor we call such integrals *ramble integrals*, as he posed such questions for a walker or rambler. As discussed in [BNSW09], and illustrated further herein, such ramble integrals are approachable by a mixture of analytic, combinatoric, algebraic and probabilistic methods. They provide interesting numeric and symbolic computation challenges. Indeed, nearly all of our results were discovered experimentally.

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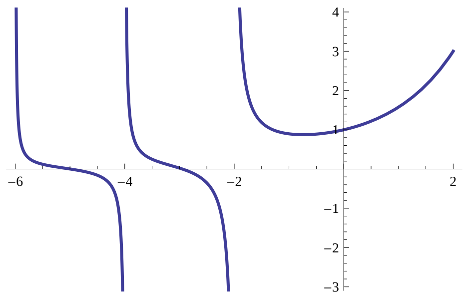
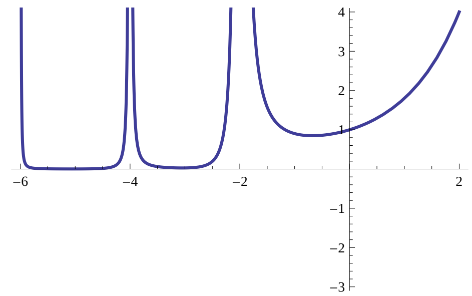
For $n \geqslant 3$, the integral (1) is well-defined and analytic for $\operatorname{Re} s > -2$, and admits an interesting analytic continuation to the complex plane with poles at certain negative integers, see [BNSW09]. We shall also write W_n for these continuations. In Figure 1 we show the continuations of W_3 and W_4 on the negative real axis. Observe the poles of W_3 at negative even integers (but note that neither function has zeroes at negative odd integers even though the graphs may suggest otherwise).



(a) W_3



(b) W_4

Figure 1: W_3, W_4 analytically continued to the real line.

It is easy to determine that $W_1(s) = 1$, and $W_2(s) = \binom{s}{s/2}$. Furthermore, it is proven in [BNSW09] that, for k a nonnegative integer, in terms of the generalized hypergeometric function, we have

$$W_3(k) = \operatorname{Re}\ {}_3F_2\left(\left. {\frac{1}{2}, -\frac{k}{2}, -\frac{k}{2} \atop 1, 1} \right| 4 \right). \tag{2}$$

From here, the following expressions for $W_3(1)$ can be established:

$$\begin{aligned} W_3(1) &= \frac{4\sqrt{3}}{3}\left({}_3F_2\left(\left. {-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \atop 1, 1} \right| \frac{1}{4} \right) - \frac{1}{\pi} \right) + \frac{\sqrt{3}}{24}\, {}_3F_2\left(\left. {\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 2, 2} \right| \frac{1}{4} \right) && (3) \\ &= 2\sqrt{3}\,\frac{K^2(k_3)}{\pi^2} + \sqrt{3}\,\frac{1}{K^2(k_3)} && (4) \\ &= \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right) && (5) \\ &= \frac{1}{\pi^2}\left(\frac{2^{1/3}}{4}\beta^2\left(\frac{1}{3}\right) + 2^{2/3}\beta^2\left(\frac{2}{3}\right) \right), && (6) \end{aligned}$$

where K is the complete elliptic integral of the first kind, $k_3 := \frac{\sqrt{3}-1}{2\sqrt{2}}$ is the *third singular value* as in [BB87], and $\beta(x) := B(x,x)$ is a central Beta-function value. More simply, but similarly,

$$W_3(-1) = 2\sqrt{3}\,\frac{K^2\,(k_3)}{\pi^2} = \frac{3}{16}\,\frac{2^{1/3}}{\pi^4}\,\Gamma^6\left(\frac{1}{3}\right) = \frac{2^{\frac{1}{3}}}{4\pi^2}\beta^2\left(\frac{1}{3}\right), \tag{7}$$

and, using the two-term recurrence for $W_3(n)$ given in [BNSW09], it follows that similar expressions can be given for W_3 evaluated at odd integers. It is one of the goals of this paper to give similar evaluations for a 4-step walk.

For s an even positive integer, the moments $W_n(s)$ take explicit integer values. In fact, for integers $k \geqslant 0$,

$$W_n(2k) = \sum_{a_1+\cdots+a_n=k} \binom{k}{a_1,\ldots,a_n}^2. \tag{8}$$

Based on the combinatorial properties of this evaluation, the following conjecture was made in [BNSW09]. Note that the case $n = 1$ is easily resolved.

Conjecture 1. *For positive integers n and complex s,*

$$W_{2n}(s) \overset{?[1]}{=} \sum_{j\geqslant 0} \binom{s/2}{j}^2 W_{2n-1}(s-2j). \tag{9}$$

We investigate this conjecture in some detail in Section 4. For $n = 2$, in conjunction with (3) this leads to a very efficient computation of W_4 at integers, yielding roughly a digit per term.

2 Bessel integral representations

We start with the result of Kluyver [Klu1906], amplified in [Watson32, §31.48] and exploited in [BNSW09], to the effect that the probability that an n-step walk ends up within a disc of radius α is

$$P_n(\alpha) = \alpha \int_0^\infty J_1(\alpha x) J_0^n(x)\,\mathrm{d}x. \tag{10}$$

From this, David Broadhurst [Bro09] obtains

$$W_n(s) = 2^{s+1-k}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(k-\frac{s}{2})}\int_0^\infty x^{2k-s-1}\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x)\,\mathrm{d}x \tag{11}$$

valid as long as $2k > s > -n/2$. Here and below $J_\nu(z)$ denotes the *Bessel function* of the first kind.

Example 1 ($W_n(\pm 1)$)**.** In particular, from (11), for $n \geqslant 2$, we can write:

$$W_n(-1) = \int_0^\infty J_0^n(x)\,\mathrm{d}x, \quad W_n(1) = n\int_0^\infty J_1(x)J_0(x)^{n-1}\frac{\mathrm{d}x}{x}. \tag{12}$$

$\diamondsuit$

Equation (11) enabled Broadhurst to verify Conjecture 1 for $n = 2, 3, 4, 5$ and odd positive $s < 50$ to a precision of 50 digits. A different proof of (11) is outlined in Remark 1 below. In particular, for $0 < s < n/2$, we have

$$W_n(-s) = 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\int_0^\infty x^{s-1}J_0^n(x)\,\mathrm{d}x, \tag{13}$$

so that $W_n(-s)$ essentially is the Mellin transform of (the analytic continuation of) the nth power of the Bessel function J_0.

Example 2. Using (13), the evaluations $W_1(s) = 1$ and $W_2(s) = \binom{s}{s/2}$ translate into

$$\begin{aligned}
\int_0^\infty x^{s-1}J_0(x)\,\mathrm{d}x &= 2^{s-1}\frac{\Gamma(s/2)}{\Gamma(1-s/2)},\\
\int_0^\infty x^{s-1}J_0^2(x)\,\mathrm{d}x &= \frac{1}{2\Gamma(1/2)}\frac{\Gamma(s/2)\Gamma(1/2-s/2)}{\Gamma(1-s/2)^2}
\end{aligned}$$

in the region where the left-hand side converges.

The Mellin transforms of J_0^3 and J_0^4 in terms of Meijer G-functions appear in the proofs of Theorems 2 and 3. $\diamondsuit$

Remark 1. Here, we demonstrate how Ramanujan's "master theorem" may be applied to find the Bessel integral representation (11) in a natural way; this and more applications of Ramanujan's master theorem will appear in [RMT10]. For an alternative proof see [Bro09].

Ramanujan's master theorem [Har78] states that, under certain conditions on the analytic function φ,

$$\int_0^\infty x^{\nu-1}\left(\sum_{k=0}^\infty \frac{(-1)^k}{k!}\varphi(k)x^k\right) = \Gamma(\nu)\varphi(-\nu). \tag{14}$$

Based on the evaluation (8), we have, as noted in [BNSW09], the generating function

$$\sum_{k\geqslant 0} W_n(2k)\frac{(-x)^k}{(k!)^2} = \left(\sum_{k\geqslant 0}\frac{(-x)^k}{(k!)^2}\right)^n = J_0(2\sqrt{x})^n \tag{15}$$

for the even moments. Applying Ramanujan's master theorem (14) to $\varphi(k) := W_n(2k)/k!$, we find

$$\Gamma(\nu)\varphi(-\nu) = \int_0^\infty x^{\nu-1} J_0^n(2\sqrt{x})\,\mathrm{d}x. \tag{16}$$

Upon a change of variables and setting $s = 2\nu$,

$$W_n(-s) = 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\int_0^\infty x^{s-1} J_0^n(x)\,\mathrm{d}x. \tag{17}$$

This is the case $k = 0$ of (11). The general case follows from the fact that if $F(s)$ is the Mellin transform of $f(x)$, then $(s-2)(s-4)\cdots(s-2k)F(s-2k)$ is the Mellin transform of $\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x)$. $\Diamond$

2.1 Pole structure

A very useful consequence of equation (13) is the following proposition.

Proposition 1 (Poles)**.** *The structure of the poles of W_n is as follows:*

(a) (Reflection) *For $n = 3$, we have explicitly for $k = 0, 1, 2, \ldots$ that*

$$\mathrm{Res}_{(-2k-2)}(W_3) = \frac{2}{\pi\sqrt{3}}\frac{W_3(2k)}{3^{2k}} > 0,$$

and the corresponding poles are simple.

(b) For each integer $n \geqslant 5$, the function $W_n(s)$ has a simple pole at $-2k-2$ for integers $0 \leqslant k < (n-1)/4$ with residue given by

$$\mathrm{Res}_{(-2k-2)}(W_n) = \frac{(-1)^k}{2^{2k}(k!)^2}\int_0^\infty x^{2k+1} J_0^n(x)\,\mathrm{d}x. \tag{18}$$

(c) Moreover, for odd $n \geqslant 5$, all poles of $W_n(s)$ are simple as soon as the first $(n-1)/2$ are.

In fact, we believe that for odd n, all poles of $W_n(s)$ are simple as stated in Conjecture 2. For individual n this may be verified as in Example 3.

Proof. (a) For $n = 3$ it was shown in [BNSW09] that $\mathrm{Res}_{-2}(W_3) = 2/(\sqrt{3}\pi)$. This also follows from (26) of Corollary 1. We remark that from [Watson32, (4) p. 412] this is also the value of the conditional integral $\int_0^\infty x J_0^3(x)\,\mathrm{d}x$ in accordance with (18). Letting $r_3(k) := \mathrm{Res}_{(-2k)}(W_n)$, the explicit residue equation is

$$r_3\,(k) = \frac{(10\,k^2 - 30\,k + 23)\,r_3(k-1) - (k-2)^2 r_3(k-2)}{9\,(k-1)^2},$$

which has the asserted solution, when compared to the recursion for $W_3(s)$:

$$(s+4)^2 W_3(s+4) - 2(5s^2 + 30s + 46)W_3(s+2) + 9(s+2)^2 W_3(s) = 0. \quad (19)$$

We give another derivation in Example 7 in Section 3.

(b) For $n \geqslant 5$ we note that the integral in (18) is absolutely convergent since $|J_0(x)| \leqslant 1$ on the real axis and $J_0(x) \approx \sqrt{2/(\pi x)}\,\cos(x - \pi/4)$ (see [AS72, (9.2.1)]). Since

$$\lim_{s\to 2k}(s-2k)\Gamma(1-s/2) = 2\frac{(-1)^k}{(k-1)!},$$

the residue is as claimed by (17).

(c) As shown in [BNSW09] W_n, for odd n, satisfies a recursion of the form

$$(-1)^\lambda\,(n!!)^2 \prod_{j=1}^{\lambda-1}(s+2j)^2\; W_n(s) + c_1(s)W_n(s+2) + \cdots + (s+2\lambda)^{n-1}\,W_n(s+2\lambda) = 0,$$

with polynomial coefficients of degree $n-1$ where $\lambda := (n+1)/2$. From this, on multiplying by $(s+2k)(s+2k-2)\cdots(s-2k+2\lambda)$ one may derive a corresponding recursion for $\mathrm{Res}_{(-2k)}(W_n)$ for $k = 1, 2, \ldots$ Inductively, this lets us establish that the poles are simple. The argument breaks down if one of the initial values is infinite as it is when $4|n$.

□

Example 3 (Poles of W_5)**.** We illustrate Proposition 1 in the case $n = 5$. In particular, we demonstrate how to show that all poles are indeed simple. To this end, we start with the recursion:

$$\begin{gathered}(s+6)^4 W_5(s+6) - (35(s+5)^4 + 42(s+5)^2 + 3)W_5(s+4) \\ +\,(s+4)^2(259(s+4)^2 + 104)W_5(s+2) = 225(s+4)^2(s+2)^2 W_5(s).\end{gathered}$$

From here,

$$\lim_{s\to-2}(s+2)^2W_5(s) = \frac{4}{225}\left(285W_5(0) - 201W_5(2) + 16W_5(4)\right) = 0$$

which shows that the first pole is indeed simple as is also guaranteed by Proposition 1b. Similarly,

$$\lim_{s\to-4}(s+4)^2W_5(s) = -\frac{4}{225}\left(5W_5(0) - W_5(2)\right) = 0$$

showing that the second pole is simple as well. It follows from Proposition 1c that all poles of W_5 are simple. More specifically, let $r_5(k) := \mathrm{Res}_{(-2k)}(W_5)$. With initial values $r_5(0) = 0, r_5(1)$ and $r_5(2)$, we derive that

$$\begin{aligned} r_5(k+3) = {} & \frac{k^4 r_5(k) - (5 + 28\,k + 63\,k^2 + 70\,k^3 + 35\,k^4)\, r_5(k+1)}{225(k+1)^2(k+2)^2} \\ & + \frac{(285 + 518\,k + 259\,k^2)\, r_5(k+2)}{225(k+2)^2}. \end{aligned}$$

$\diamondsuit$

Example 4 (Poles of W_4)**.** Let $r_4(k) := \lim_{s\to-2k}(s+2k)^2W_4(s)$, then the recursion for $W_4(s)$

$$(s+4)^3W_4(s+4) - 4(s+3)(5s^2+30s+48)W_4(s+2) + 64(s+2)^3W_4(s) = 0$$

gives us

$$r_4(k+2) = \frac{1}{32}\,\frac{(2k+1)(5k^2+5k+2)}{(k+1)^3}\, r_4(k+1) - \frac{1}{64}\,\frac{k^3}{(k+1)^3}\, r_4(k).$$

We also compute that

$$\frac{3}{2\pi^2} = r_4(1) = \lim_{s\to-2}(s+2)^2W_4(s) = \frac{3 + 4W_4'(0) - W_4'(2)}{8}.$$

The first equality is obtainable from (27). Further, L'Hôpital's rule shows that the residue at $s = -2$ is

$$\lim_{s\to-2}\frac{\mathrm{d}}{\mathrm{d}s}\left((s+2)^2W_4(s)\right) = \frac{9 + 18W_4'(0) - 3W_4'(2) + 4W_4''(0) - W_4''(2)}{16}$$

with a numerical value of $0.316037\ldots$ which we were able to identify as $\frac{9}{2\pi^2}\log(2)$. Similarly, the second residue is found to be $\frac{9}{128\pi^2}\left(4\log(2)-1\right)$ with similar formulae for the other residues.

We finally record a remarkable identity related to the pole of W_4 at -2 that was established in [Watson32, (10) p. 415]. It is

$$\int_0^\infty J_\nu^4(x)x^{1-2\nu}\,\mathrm{d}x = \frac{1}{2\pi}\frac{\Gamma(2\nu)\Gamma(\nu)}{\Gamma(3\nu)\Gamma(\nu+1/2)} \tag{20}$$

for $\mathrm{Re}\,\nu > 0$. Hence $\int_0^\infty J_\nu^4(x)x^{1-2\nu}\,\mathrm{d}x \approx \frac{3/\nu}{4\pi^{3/2}}$ as $\nu \to 0$. $\diamondsuit$

2.2 Meijer G-function representations

We recall that the *Meijer G-function* – introduced in 1936 by the Dutch mathematician Cornelis Simon Meijer (1904-1974) – is defined, for parameter vectors $\mathbf{a}$ and $\mathbf{b}$ [AAR99], by

$$G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = G^{m,n}_{p,q}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle| x\right) \tag{21}$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{k=1}^m \Gamma(b_k - t)\prod_{k=1}^n \Gamma(1-a_k+t)}{\prod_{k=m+1}^q \Gamma(1-b_k+t)\prod_{k=n+1}^p \Gamma(a_k-t)} x^t\,\mathrm{d}t. \tag{22}$$

In the case $|x|<1$ and $p=q$ the contour L is a loop that starts at infinity on a line parallel to the positive real axis, encircles the poles of the $\Gamma(b_k-t)$ once in the negative sense and returns to infinity on another line parallel to the positive real axis; with a similar contour when $|x|>1$. Moreover $G^{p,q}_{m,n}$ is analytic in each parameter; in consequence so are the compositions arising below.

Our main tool below is the following special case of Parseval's formula giving the Mellin transform of a product.

Theorem 1 (Mellin transform)**.** *Let $G(s)$ and $H(s)$ be the Mellin transforms of $g(x)$ and $h(x)$ respectively. Then*

$$\int_0^\infty x^{s-1}g(x)h(x)\,\mathrm{d}x = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} G(z)H(s-z)\,\mathrm{d}z \tag{23}$$

for any real number δ in the common region of analyticity.

This leads to:

Theorem 2 (Meijer form for W_3). *For all complex s*

$$W_3(s) = \frac{\Gamma(1+s/2)}{\Gamma(1/2)\Gamma(-s/2)} G^{2,1}_{3,3}\left(\begin{matrix}1,1,1\\1/2,-s/2,-s/2\end{matrix}\middle|\frac{1}{4}\right). \tag{24}$$

Proof. We apply Theorem 1 to $J_0^3 = J_0^2 \cdot J_0$ for s in a vertical strip. Using Example 2 we then obtain

$$\begin{aligned}\int_0^\infty x^{s-1} J_0^3(x)\,\mathrm{d}x &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{2^{s-z-2}}{\Gamma(1/2)} \frac{\Gamma(z/2)\Gamma(1/2-z/2)}{\Gamma(1-z/2)^2} \frac{\Gamma(s/2-z/2)}{\Gamma(1-s/2+z/2)}\,\mathrm{d}z\\ &= \frac{2^s}{2\Gamma(1/2)}\frac{1}{2\pi i}\int_{\delta/2-i\infty}^{\delta/2+i\infty} 4^{-t}\frac{\Gamma(t)\Gamma(1/2-t)\Gamma(s/2-t)}{\Gamma(1-t)^2\Gamma(1-s/2+t)}\,\mathrm{d}t\\ &= \frac{2^s}{2\Gamma(1/2)} G^{2,1}_{3,3}\left(\begin{matrix}1,1,1\\1/2,s/2,s/2\end{matrix}\middle|\frac{1}{4}\right)\end{aligned}$$

where $0<\delta<1$. The claim follows from (17) by analytic continuation. □

Similarly we obtain:

Theorem 3 (Meijer form for W_4). *For all complex s with* $\operatorname{Re} s > -2$

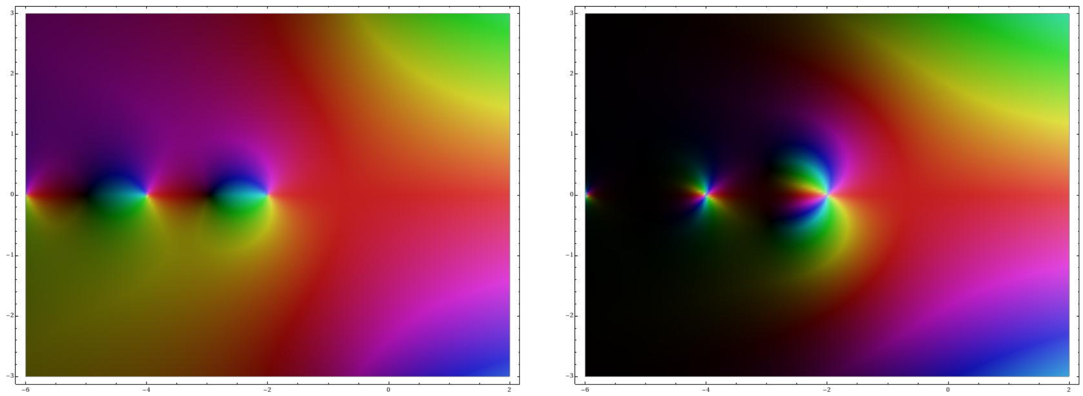
$$W_4(s) = \frac{2^s}{\pi}\frac{\Gamma(1+s/2)}{\Gamma(-s/2)} G^{2,2}_{4,4}\left(\begin{matrix}1,(1-s)/2,1,1\\1/2,-s/2,-s/2,-s/2\end{matrix}\middle|1\right). \tag{25}$$

Proof. We now apply Theorem 1 to $J_0^4 = J_0^2 \cdot J_0^2$, again for s in a vertical strip. Using once more Example 2, we obtain

$$\begin{aligned}\int_0^\infty x^{s-1} J_0^4(x)\,\mathrm{d}x &= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{1}{4\pi}\frac{\Gamma(z/2)\Gamma(1/2-z/2)}{\Gamma(1-z/2)^2}\frac{\Gamma(s/2-z/2)\Gamma(1/2-s/2+z/2)}{\Gamma(1-s/2+z/2)^2}\,\mathrm{d}z\\ &= \frac{1}{2\pi} G^{2,2}_{4,4}\left(\begin{matrix}1,(1+s)/2,1,1\\1/2,s/2,s/2,s/2\end{matrix}\middle|1\right)\end{aligned}$$

where $0<\delta<1$. The claim again follows from (17). □

We illustrate with graphs of W_3, W_4 in the complex plane in Figure 2. Note the poles and removable singularities. These graphs were produced employing the Meijer forms in their hypergeometric form as presented in the next section. In the case $n=4$, the functional equation is employed for s with $\operatorname{Re} s \leqslant -2$.



(a) W_3 (b) W_4

Figure 2: W_3 via (24) and W_4 via (25) in the complex plane.

2.3 Hypergeometric representations

By Slater's theorem [Mar83, p. 57], the Meijer G-function representations for $W_3(s)$ and $W_4(s)$ given in Theorems 2 and 3 can be expanded in terms of generalized hypergeometric functions.

For $n = 3, 4$ we obtain the following:

Corollary 1 (Hypergeometric forms)**.** *For s not an odd integer, we have*

$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{s+3}{2}, \frac{s+3}{2}}\right| \frac{1}{4}\right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\left.{-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, -\frac{s-1}{2}}\right| \frac{1}{4}\right), \tag{26}$$

and, if also $\operatorname{Re} s > -2$, *we have*

$$W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^3 {}_4F_3\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{s}{2}+1 \atop \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2}}\right| 1\right) + \binom{s}{\frac{s}{2}} {}_4F_3\left(\left.{\frac{1}{2}, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, 1, -\frac{s-1}{2}}\right| 1\right). \tag{27}$$

These lovely analytic continuations of W_3 and W_4, first found in [Cra09] using *Mathematica*, can also be obtained by symbolic integration of (11) in *Mathematica*.

Example 5. From (26) and taking the limit using L'Hôpital's rule, we have

$$W_3(-1) = \frac{16}{\pi^3} \mathrm{K}^2\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \log 2 + \frac{3}{\pi} \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{\sum_{k=1}^{2n} \frac{(-1)^k}{k}}{4^{4n}}. \tag{28}$$

In conjunction with (7) we obtain

$$\sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{\sum_{k=1}^{2n} \frac{(-1)^k}{k}}{4^{4n}} = \left(\frac{2}{\sqrt{3}\pi} - \frac{16}{3\pi^2} \log 2 \right) \mathrm{K}^2 \left(\frac{\sqrt{3}-1}{2\sqrt{2}} \right). \tag{29}$$

For comparison, (27) produces

$$W_4(-1) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4}{4^{4n}} \sum_{k=2n+1}^{\infty} \frac{(-1)^{k+1}}{k}.$$

◊

We see that while Corollary 1 makes it easy to analyse the poles, the provably removable singularities at odd integers are much harder to resolve explicitly [Cra09]. For $W_4(-1)$ we proceed as follows:

Theorem 4 (Hypergeometric form for $W_4(-1)$)**.**

$$W_4(-1) = \frac{\pi}{4} \, {}_7F_6 \left(\left. \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \right| 1 \right). \tag{30}$$

Proof. Using Theorem 3 we write

$$W_4(-1) = \frac{1}{2\pi} G_{4,4}^{2,2} \left(\left. \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \right| 1 \right).$$

Using the definition (21) of the Meijer G-function as a contour-integral, we see that the corresponding integrand is

$$\frac{\Gamma(\frac{1}{2}-t)^2 \Gamma(t)^2}{\Gamma(\frac{1}{2}+t)^2 \Gamma(1-t)^2} x^t = \frac{\Gamma(\frac{1}{2}-t)^2 \Gamma(t)^4}{\Gamma(\frac{1}{2}+t)^2} \cdot \frac{\sin^2(\pi t)}{\pi^2} x^t, \tag{31}$$

where we have used $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}$. We choose the contour of integration to enclose the poles of $\Gamma(\frac{1}{2}-t)$. Note then that the presence of $\sin^2(\pi t)$ does not interfere with the contour or the residues (for $\sin^2(\pi t) = 1$ at half integers). Hence we may ignore $\sin^2(\pi t)$ in the integrand altogether. Then the right-hand side of (31) is the integrand of another Meijer G-function; thus we have shown that

$$G_{4,4}^{2,2} \left(\left. \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \right| 1 \right) = \frac{1}{\pi^2} G_{4,4}^{2,4} \left(\left. \begin{matrix} 1, 1, 1, 1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{matrix} \right| 1 \right). \tag{32}$$

The same argument shows that the factor of $\frac{1}{\pi^2}$ applies to all $W_4(s)$ when we change from $G^{2,2}_{4,4}$ to $G^{2,4}_{4,4}$.

Now, using the transformation

$$x^\alpha G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) = G^{m,n}_{p,q}\left(\begin{matrix}\mathbf{a}+\alpha\\ \mathbf{b}+\alpha\end{matrix}\middle| x\right) \tag{33}$$

we deduce that

$$W_4(-1) = \frac{1}{2\pi^3} G^{2,4}_{4,4}\left(\begin{matrix}\frac12,\frac12,\frac12,\frac12\\ 0,0,0,0\end{matrix}\middle| 1\right).$$

Finally, we appeal to Bailey's identity [Bai32, Formula (3.4)]:

$$\begin{aligned}&{}_7F_6\left(\begin{matrix}a,1+\frac{a}{2},b,c,d,e,f\\ \frac{a}{2},1+a-b,1+a-c,1+a-d,1+a-e,1+a-f\end{matrix}\middle| 1\right)\\ &= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)\Gamma(1+a-e-f)}\\ &\quad\times G^{2,4}_{4,4}\left(\begin{matrix}e+f-a,1-b,1-c,1-d\\ 0,1+a-b-c-d,e-a,f-a\end{matrix}\middle| 1\right).\end{aligned} \tag{34}$$

The claim follows upon setting all parameters to 1/2. $\square$

An attempt to analogously apply Bailey's identity for $W_4(1)$ fails, since its Meijer G representation as obtained from Theorem 3 does not meet the precise form required in the formula. Nevertheless, a combination of Nesterenko's theorem ([Nest]) and Zudilin's theorem ([Zudilin02]) gives the following result:

Theorem 5 (Hypergeometric form for $W_4(1)$)**.**

$$W_4(1) = \frac{3\pi}{4}\,{}_7F_6\left(\begin{matrix}\frac74,\frac32,\frac32,\frac32,\frac12,\frac12,\frac12\\ \frac34,2,2,2,1,1\end{matrix}\middle| 1\right) - \frac{3\pi}{8}\,{}_7F_6\left(\begin{matrix}\frac74,\frac32,\frac32,\frac12,\frac12,\frac12,\frac12\\ \frac34,2,2,2,2,1\end{matrix}\middle| 1\right). \tag{35}$$

Proof. We first prove a result that will allow us to use Nesterenko's theorem, which converts the Meijer G form of $W_4(1)$ to a triple integral. We need the following identities which can be readily verified:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{-b_1} G^{2,2}_{4,4}\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3,b_4\end{matrix}\middle| z\right)\right) = -z^{-1-b_1} G^{2,2}_{4,4}\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1+1,b_2,b_3,b_4\end{matrix}\middle| z\right) \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{1-a_1} G^{2,2}_{4,4}\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3,b_4\end{matrix}\middle| z\right)\right) = z^{-a_1} G^{2,2}_{4,4}\left(\begin{matrix}a_1-1,a_2,a_3,a_4\\ b_1,b_2,b_3,b_4\end{matrix}\middle| z\right) \tag{37}$$

Let $a(z) := G^{2,2}_{4,4}\left({0,1,1,1 \atop -\frac12,\frac12,-\frac12,-\frac12}\middle|z\right)$. Note that $a(1) = -2\pi W_4(1)$ by Theorem 3. Applying (36) to $a(z)$ and using the product rule, we get $\frac12 a(1) + a'(1) = c_1$, where $c_1 := -G^{2,2}_{4,4}\left({0,1,1,1 \atop \frac12,\frac12,-\frac12,-\frac12}\middle|1\right)$. Applying (37) and (33) to $a(z)$, we obtain $a'(1) = b_1$ where $b_1 := G^{2,2}_{4,4}\left({-\frac12,-\frac12,\frac12,\frac12 \atop 0,-1,-1,-1}\middle|1\right)$. Appealing to equation (72), we see that $b_1 = -c_1$. Hence $a(1) = 4c_1$. Converting c_1 to a $G^{2,4}_{4,4}$ as in (32), which finally satisfies the conditions of Nesterenko's theorem, we obtain:

$$W_4(1) = \frac{4}{\pi^3}\int_0^1\int_0^1\int_0^1 \sqrt{\frac{x(1-y)(1-z)}{(1-x)yz(1-x(1-yz))}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We now make a change of variable $z' = 1-z$. Writing

$$(z')^{\frac12} = (z')^{-\frac12}(1-(1-z')) = (z')^{-\frac12} - (z')^{-\frac12}(1-z')$$

splits the previous triple integral into two terms. Each term satisfies Zudilin's theorem and so can be written as a ${}_7F_6$. We thence obtain the result as claimed. □

The following alternative relation was first predicted by the *integer relation algorithm* PSLQ in a computational hunt for results similar to that in Theorem 4:

Theorem 6 (Alternative hypergeometric form for $W_4(1)$)**.**

$$W_4(1) = \frac{9\pi}{4}\,{}_7F_6\left({\frac74,\frac32,\frac32,\frac32,\frac12,\frac12,\frac12 \atop \frac34,2,2,2,1,1}\middle|1\right) - 2\pi\,{}_7F_6\left({\frac54,\frac12,\frac12,\frac12,\frac12,\frac12,\frac12 \atop \frac14,1,1,1,1,1}\middle|1\right). \tag{38}$$

Proof. For notational convenience, let

$$A := \frac{3\pi^4}{128}\,{}_7F_6\left({\frac74,\frac32,\frac32,\frac32,\frac12,\frac12,\frac12 \atop \frac34,2,2,2,1,1}\middle|1\right),$$

$$B := \frac{3\pi^4}{256}\,{}_7F_6\left({\frac74,\frac32,\frac32,\frac12,\frac12,\frac12,\frac12 \atop \frac34,2,2,2,2,1}\middle|1\right),$$

$$C := \frac{\pi^4}{16}\,{}_7F_6\left({\frac54,\frac12,\frac12,\frac12,\frac12,\frac12,\frac12 \atop \frac14,1,1,1,1,1}\middle|1\right).$$

By (35), $W_4(1) = (32/\pi^3)(A-B)$, and the truth of (38) is equivalent to the evaluation $W_4(1) = (32/\pi^3)(3A-C)$. Thus, we only need to show $2A+B-C=0$.

The triple integral for A encountered in the application of Zudilin's theorem is

$$A = \frac18\int_0^1\int_0^1\int_0^1 \sqrt{\frac{x(1-y)}{(1-x)yz(1-z)(1-x(1-yz))}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

and can be reduced to a one dimensional integral:

$$A = A_1 := \int_0^1 \frac{(K'(k) - E'(k))^2}{1-k^2}\,\mathrm{d}k,$$

Here, as usual, $K'(k) := K\left(\sqrt{1-k^2}\right)$ and $E'(k) := E(\sqrt{1-k^2})$.

Happily, we may apply a non-trivial action on the exponents of x, y, z and leave the value of the integral unchanged (see [Zudilin04], remark after lemma 8). We obtain:

$$\begin{aligned} A &= \frac{1}{8}\int_0^1\int_0^1\int_0^1 \sqrt{\frac{1-x(1-yz)}{xyz(1-x)(1-y)(1-z)}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= A_2 := \int_0^1 K'(k)E'(k)\,\mathrm{d}k. \end{aligned}$$

The like integral for B can also be reduced to a one dimensional integral,

$$B = B_2 := \int_0^1 k^2 K'(k)^2\,\mathrm{d}k.$$

But B also satisfies the conditions of Bailey's identity and Nesterenko's theorem, from which we are able to produce an alternative triple integral, and reduce it to:

$$B = B_1 := \int_0^1 (K'(k) - E'(k))\left(E'(k) - k^2K'(k)\right)\frac{\mathrm{d}k}{1-k^2}.$$

As for C, equation (64) details its evaluation, which we also record here:

$$C = \int_0^1 K'(k)^2\,\mathrm{d}k.$$

Now $2A + B - C = A_1 + A_2 + B_1 - C = 0$, because the integrand of the later expression is zero. □

Note that the theorem gives the identity

$$2\int_0^1 K'(k)E'(k)\,\mathrm{d}k = \int_0^1 (1-k^2)K'(k)^2\,\mathrm{d}k, \tag{39}$$

among others.

Remark 2. Note that each of the ${}_7F_6$'s involved in Theorems 4, 5 and 6 can also be easily written as a sum of two ${}_6F_5$'s.

Also note that the first ${}_7F_6$ term in Theorem 6 satisfies the conditions of Bailey's identity (34) (with $a = e = f = \frac{3}{2}, b = c = d = \frac{1}{2}$):

$$ {}_7F_6\left(\left.{\frac{7}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{3}{4}, 2, 2, 2, 1, 1}\right| 1\right) = -\frac{16}{3\pi^4} G^{2,4}_{4,4}\left(\left.{\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 1, 0, 0, 0}\right| 1\right). \tag{40}$$

We can thence convert the right-hand side to a Meijer G form. On the other hand,

$$W_4(1) = -\frac{1}{2\pi^3} G^{2,4}_{4,4}\left(\left.{0, 1, 1, 1 \atop \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}\right| 1\right).$$

We thus obtain the non-trivial identity:

$$G^{2,4}_{4,4}\left(\left.{\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \atop 1, 0, 0, 0}\right| 1\right) = 24\, G^{2,4}_{4,4}\left(\left.{\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 1, 0, 0, 0}\right| 1\right) + 8\, G^{2,4}_{4,4}\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 0, 0, 0, 0}\right| 1\right). \tag{41}$$

$\Diamond$

Corollary 2 (Elliptic integral representation for $W_4(1)$)**.** *We have*

$$W_4(1) = \frac{16}{\pi^3} \int_0^1 (1 - 3k^2) K'(k)^2 \, \mathrm{d}k. \tag{42}$$

Proof. The conclusion of Theorem 6 implies $(\pi^3/16)\, W_4(1) = C - 3B = C - 3B_2$. $\square$

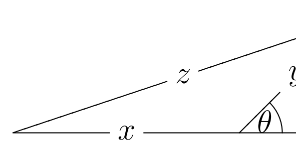
3 Probabilistically inspired representations

In this section, we build on the probabilistic approach taken in Section 6 of [BNSW09]. We may profitably view a $(m+n)$-step walk as a composition of an m-step and n-step walk for $m, n \geqslant 1$. Different decompositions make different structures apparent.

We express the distance z of an $(n+m)$-step walk conditioned on a given distance x of the first n steps as well as the distance y of the remaining m steps. Then, by the cosine rule,

$$z^2 = x^2 + y^2 + 2xy\cos(\theta),$$

where θ is the outside angle of the triangle with sides of lengths x, y, and z:



It follows that for $s > 0$, the s-th moment of an $(n+m)$-step walk conditioned on the distance x of the first n steps and the distance y of the remaining m steps is

$$g_s(x,y) := \frac{1}{\pi}\int_0^\pi z^s \,\mathrm{d}\theta = |x-y|^s \, {}_2F_1\left(\begin{matrix}\frac{1}{2}, -\frac{s}{2}\\ 1\end{matrix}\middle| -\frac{4xy}{(x-y)^2}\right). \tag{43}$$

Here we appealed to symmetry to restrict the angle to $\theta \in [0,\pi)$. We then evaluated the integral in hypergeometric form which, for instance, can be done with the help of *Mathematica* or *Maple*.

Remark 3 (Alternate forms for g_s)**.** Using Kummer's quadratic transformation [AAR99], we obtain

$$g_s(x,y) = \operatorname{Re} y^s \, {}_2F_1\left(\begin{matrix}-\frac{s}{2}, -\frac{s}{2}\\ 1\end{matrix}\middle| \frac{x^2}{y^2}\right) \tag{44}$$

for general positive x, y. This provides an analytic continuation of $s \mapsto g_s(x,y)$. In particular, we have

$$g_{-1}(x,y) = \frac{2}{\pi}\operatorname{Re}\frac{1}{y}\,K\left(\frac{x}{y}\right) \tag{45}$$

and, with E the complete elliptic integral of the second kind, we have

$$g_1(x,y) = \frac{2}{\pi}\operatorname{Re} y\left\{2E\left(\frac{x}{y}\right) - \left(1-\frac{x^2}{y^2}\right)K\left(\frac{x}{y}\right)\right\}. \tag{46}$$

This later form has various re-expressions. $\Diamond$

Denote by $p_n(x)$ the density of the distance x for an n-step walk. Since $W_{n+m}(s)$ is the s-th moment of the distance of an $(n+m)$-step walk, we obtain

$$W_{n+m}(s) = \int_0^n \int_0^m g_s(x,y)\, p_n(x) p_m(y)\,\mathrm{d}y\,\mathrm{d}x, \tag{47}$$

for $s \geqslant 0$. Since for the 1-step walk we have $p_1(x) = \delta_1(x)$, this generalizes the corresponding formula given for $W_{n+1}(s)$ in [BNSW09].

In (47), if $n = 0$, then we may take $p_0(x) = \delta_0(x)$, and regard the limits of integration as from $-\epsilon$ and $+\epsilon$, $\epsilon \to 0$. Then $g_s = y^s$ as the hypergeometric collapses to 1, and we recover the basic form

$$W_m(s) = \int_0^m y^s p_m(y)\,\mathrm{d}y. \tag{48}$$

It is also easily shown that the probability density for a 2-step walk is given by

$$p_2(x) = \frac{2}{\pi\sqrt{4-x^2}}$$

for $0 \leqslant x \leqslant 2$ and 0 otherwise.

The density $p_3(x)$ for $0 \leqslant x \leqslant 3$ can be expressed by

$$p_3(x) = \operatorname{Re}\left(\frac{\sqrt{x}}{\pi^2}\, K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right), \tag{49}$$

see, e.g., [Pea1906]. To make (49) more accessible we need the following cubic identity.

Proposition 2. *For all $0 \leqslant x \leqslant 1$ we have*

$$K\left(\sqrt{\frac{16x^3}{(3-x)^3(1+x)}}\right) = \frac{3-x}{3+3x}\, K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right).$$

Proof. Both sides satisfy the differential equation

$$4x^2(x+3)^2 f(x) + (x-3)(x+1)^2((x^3-9x^2-9x+9)f'(x) + x(x^3-x^2-9x+9)f''(x)) = 0,$$

and both of their function values and derivative values agree at the origin. □

We can use Proposition 2 to deduce the more symmetrical formula

$$p_3(x) = \frac{2}{\pi} \operatorname{Re}\left(\frac{x}{\mathrm{AGM}\left(\sqrt{(3+x)(1-x)^3}, \sqrt{(3-x)(1+x)^3}\right)}\right) \tag{50}$$

since $\mathrm{AGM}(1, \sqrt{1-k^2}) = \frac{\pi}{2K(k)}$ denotes the Gaussian arithmetic-geometric mean.

We also apply Jacobi's imaginary transform ([BB87], p73), $\operatorname{Re} K(x) = \frac{1}{x} K\left(\frac{1}{x}\right)$ for $x > 1$ to express $p_3(x)$ as a real function over $[0,1]$ and $[1,3]$. This leads to

$$W_3(-1) = \int_0^3 \frac{p_3(x)}{x}\,\mathrm{d}x = \frac{4}{\pi^2}\int_0^1 \frac{K\left(\sqrt{\frac{16x}{(3-x)(1+x)^3}}\right)}{\sqrt{(3-x)(1+x)^3}}\,\mathrm{d}x + \frac{1}{\pi^2}\int_1^3 \frac{K\left(\sqrt{\frac{(3-x)(1+x)^3}{16x}}\right)}{\sqrt{x}}\,\mathrm{d}x.$$

Now in the last integral, we make the change of variable $x \to \frac{3-t}{1+t}$, and it transforms exactly into the second last integral. Therefore,

$$W_3(-1) = 2\int_0^1 \frac{p_3(x)}{x}\,\mathrm{d}x. \tag{51}$$

To make sense of what has happened more abstractly, let

$$\sigma(x) := \frac{3-x}{1+x}, \qquad \lambda(x) := \frac{(1+x)^3(3-x)}{16x}. \tag{52}$$

Then for $0 < x < 3$ we have $\sigma^2(x) = x$ and $\lambda(x)\lambda(\sigma(x)) = 1$. In consequence σ is an involution that sends $[0,1]$ to $[1,3]$ and

$$p_3(x) = \frac{4x}{(3-x)(x+1)}\, p_3(\sigma(x)). \tag{53}$$

Example 6 (Series for p_3 and $W_3(-1)$)**.** We know that

$$W_3(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

is the sum of squares of trinomials (see (8) and [BNSW09]). Using Proposition 2, we may now apply equation (184) in [BBBG08, Section 5.10] to obtain

$$p_3(x) = \frac{2x}{\pi\sqrt{3}} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{x}{3}\right)^{2k}, \tag{54}$$

with radius of convergence 1. For $1 < x < 3$, on using (53) we obtain

$$p_3(x) = \frac{8x}{\pi\sqrt{3}(x+1)^2} \sum_{k=0}^{\infty} W_3(2k) \left(\frac{3-x}{3+3x}\right)^{2k}. \tag{55}$$

From (54) and (51) we deduce that

$$W_3(-1) = \frac{4}{\pi\sqrt{3}} \sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(2k+1)}$$

as a type of reflection formula. ♢

We can use (10) to deduce for all $n \geqslant 4$ that

$$p_n(\alpha) = \alpha \int_0^\infty J_0(t)^n J_0(\alpha t) t \,\mathrm{d}t. \tag{56}$$

Alternatively, setting $\phi_n(r) := p_n(r)/(2\pi r)$, we have that for $n \geqslant 2$ ([Hug95])

$$\phi_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \phi_{n-1}\left(\sqrt{r^2+1-2r\cos t}\right) \mathrm{d}t. \tag{57}$$

The densities p_3 and p_4 are shown in Figure 3. Note that p_3 has a singularity at 1 as follows from (49). We remark that the derivative of p_4 has singularities at 0 and 4. We also record that $p_4^-(2) \approx .144687$ while $p_4^+(2) = -\infty$. This can be proven by using the large-order asymptotic expansion for J_ν to estimate $p_4'(s)$ for s near 2 as a combination of Fresnel integrals.

Further, computed from (56), the densities p_5, p_6, p_7, and p_8 are shown in Figure 4 to illustrate that the case of small n is very different from the case of large n. In particular, it should be noted that, for $n \geqslant 7$, the density p_n is already quite well approximated by the limiting $\frac{2x}{n}e^{-x^2/n}$. Also notice that, as the pictures suggest, p_n is continuously differentiable for $n \geqslant 6$ while p_5 is not. This, as well as similar statements about higher-order differentiability, may be proven using dominated convergence and (56).

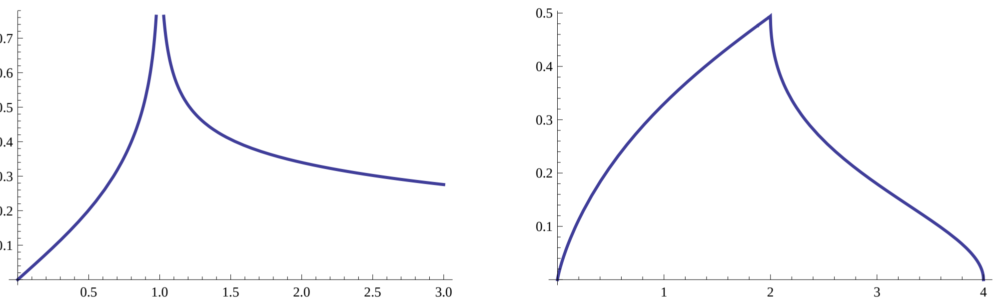
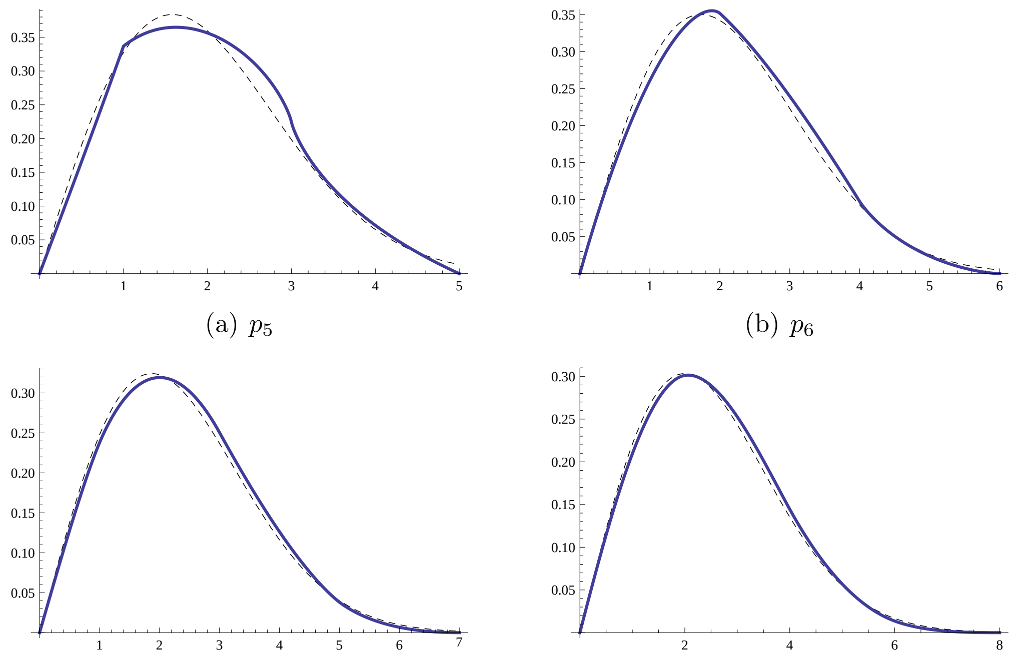


Figure 3: The densities p_3 (L) and p_4 (R).

Example 7 (Poles of W_3)**.** From here we may efficiently recover the explicit form for the residues of W_3 given in Proposition 1a. Fix integers $N > 2k > 0$ and $0 < \alpha < 1$. Use the series $p_3(x) = \sum_{j \geqslant 0} a_j x^{2j+1}$ in (54) to write

$$\begin{aligned} W_3(s) - \int_\alpha^3 p_3(x) x^s \,\mathrm{d}x - \int_0^\alpha \sum_{j=N}^{\infty} a_j x^{2j+1+s} \,\mathrm{d}x &= \int_0^\alpha \sum_{j=0}^{N-1} a_j x^{2j+1+s} \,\mathrm{d}x \\ &= \sum_{j=1}^{N} a_{j-1} \frac{\alpha^{2j+s}}{2j+s}, \end{aligned} \tag{58}$$

and observe that both sides are holomorphic and so (58) holds in a neighborhood of $s = -2k$. Since only the first term on the left has a pole at $-2k$ we may deduce that



(a) p_5 (b) p_6

(c) p_7 (d) p_8

Figure 4: Densities p_n with the limiting behaviour superimposed.

$\mathrm{Res}_{(-2k)}(W_3) = a_{k-1}$. Equivalently,

$$\mathrm{Res}_{(-2k-2)}(W_3) = \frac{2}{\pi\sqrt{3}} \frac{W_3(2k)}{3^{2k}},$$

which exposes an elegant reflection property. ◊

Remark 4 (W_5)**.** Using (47) we may express $W_5(s)$ and $W_6(s)$ as double integrals, for example,

$$W_5(-1) = \frac{4}{\pi^4} \int_0^3 \int_0^2 \frac{\sqrt{x}}{y\sqrt{4-y^2}} \,\mathrm{Re}\left(K\left(\frac{x}{y}\right)\right) \mathrm{Re}\left(K\left(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\right)\right) \mathrm{d}y\,\mathrm{d}x.$$

We also have an expression based on taking two 2-step walks and a 1-step walk:

$$\begin{aligned} W_5(-1) &= \frac{8}{\pi^4} \int_0^2 \int_0^2 \int_0^\pi \frac{\mathrm{Re}\, K(\sqrt{x^2+y^2+2xy\cos z})}{\sqrt{(4-x^2)(4-y^2)}} \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{8}{\pi^4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\pi \mathrm{Re}\, K\left(2\sqrt{\sin^2 a + \sin^2 b + 2\sin a \sin b \cos c}\right) \mathrm{d}c\,\mathrm{d}a\,\mathrm{d}b, \end{aligned}$$

but we have been unable to make further progress with these forms. ◊

3.1 Elliptic integral representations

From (47), we derive

$$\begin{aligned} W_4(s) &= \frac{2^{s+2}}{\pi^2} \int_0^1 \int_0^1 \frac{g_s(x,y)}{\sqrt{(1-x^2)(1-y^2)}} \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{2^{s+2}}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} g_s(\sin u, \sin v) \,\mathrm{d}u\,\mathrm{d}v. \end{aligned}$$

where $s > -2$. In particular, for $s = -1$, again using Jacobi's imaginary transformation, we have:

$$\begin{aligned} W_4(-1) &= \frac{4}{\pi^3} \mathrm{Re} \int_0^1 \int_0^1 \frac{K(x/y)}{y\sqrt{(1-x^2)(1-y^2)}} \,\mathrm{d}x\,\mathrm{d}y \qquad (59) \\ &= \frac{8}{\pi^3} \int_0^1 \int_0^1 \frac{K(t)}{\sqrt{(1-t^2y^2)(1-y^2)}} \,\mathrm{d}y\,\mathrm{d}t \\ &= \frac{8}{\pi^3} \int_0^1 K^2(k) \,\mathrm{d}k. \qquad (60) \end{aligned}$$

The corresponding integral at $s = 1$ is

$$W_4(1) = \frac{32}{\pi^3} \int_0^1 \frac{(k+1)(K(k) - E(k))}{k^2} \; E\left(\frac{2\sqrt{k}}{k+1}\right) \mathrm{d}k.$$

Also, the following Fourier series [BBBG08, Eqn (70)] allows one to apply the Parseval-Bessel formula

$$K(\sin\theta) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+n)^2}{\Gamma(n+1)^2} \sin\left((4n+1)\theta\right), \tag{61}$$

from which we may obtain

$$W_4(-1) = \frac{4}{\pi} \sum_{n,m\geqslant 0} \left(\frac{(\frac{1}{2})_n(\frac{1}{2})_m}{n!m!}\right)^2 \left(\frac{1}{1-(4(m-n))^2} + \frac{1}{1-(4(m+n)+2)^2}\right), \tag{62}$$

in terms of the rising factorial $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$.

Starting instead with Nesterenko's theorem [Nest] we have the following:

$$W_4(-1) = \frac{1}{2\pi^3} \int_{[0,1]^3} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\sqrt{xyz(1-x)(1-y)(1-z)(1-x(1-yz))}}. \tag{63}$$

(Such integrals are related to Beukers integrals, which were used in the elementary derivation of the irrationality of $\zeta(3)$.) Upon computing the $\mathrm{d}x$ integral, followed by the change of variable $k^2 = yz$, we have:

$$\begin{aligned} W_4(-1) &= \frac{1}{\pi^3} \int_0^1 \int_0^1 \frac{K(\sqrt{1-yz})}{\sqrt{yz(1-y)(1-z)}} \,\mathrm{d}y\,\mathrm{d}z && (64) \\ &= \frac{2}{\pi^3} \int_0^1 \int_{k^2}^1 \frac{K(\sqrt{1-k^2})}{\sqrt{y(1-y)(y-k^2)}} \,\mathrm{d}y\,\mathrm{d}k \\ &= \frac{4}{\pi^3} \int_0^1 K'(k)^2 \,\mathrm{d}k. && (65) \end{aligned}$$

Compare this with the corresponding (59). In particular, appealing to Theorem 4 we derive the closed forms:

$$2\int_0^1 K(k)^2 \,\mathrm{d}k = \int_0^1 K'(k)^2 \,\mathrm{d}k = \left(\frac{\pi}{2}\right)^4 {}_7F_6\left(\left.\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix}\right| 1\right). \tag{66}$$

Recalling Corollary 2 and equation (39) we also deduce that

$$W_4(1) = \frac{96}{\pi^3} \int_0^1 E'(k)K'(k)\,\mathrm{d}k - 8\,W_4(-1). \tag{67}$$

If we make a trigonometric change of variables in (64), we obtain

$$W_4(-1) = \frac{4}{\pi^3} \int_0^{\pi/2} \int_0^{\pi/2} K\left(\sqrt{1-\sin^2 x \sin^2 y}\right) \mathrm{d}x\,\mathrm{d}y. \tag{68}$$

We may rewrite the integrand as a sum, and then interchange integration and summation to arrive at a slowly convergent representation of the same general form as in Conjecture 1:

$$W_4(-1) = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/2}{n}^2 {}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, -n \atop 1, 1}\right| 1\right). \tag{69}$$

Remark 5 (Relation to Watson integrals)**.** From the evaluation (7) we note that $W_3(-1)$ equals twice the second of three triple integrals considered by Watson in [Watson39]:

$$W_3(-1) = \frac{1}{\pi^3} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{\mathrm{d}u\mathrm{d}v\mathrm{d}w}{3 - \cos v \cos w - \cos w \cos u - \cos u \cos v}. \tag{70}$$

This is derived in [BBG05] and various related extensions are to be found in [BBBG08]. It is not clear how to generalize this to $W_4(-1)$.

Watson's second integral (70) also gives the alternative representation:

$$W_3(-1) = \pi^{-5/2} G_{3,3}^{3,2}\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop 0, 0, 0}\right| 4\right). \tag{71}$$

The equivalence of this and the Meijer G representation coming from Theorem 2 can be established similarly to the proof of Theorem 4 upon using the Meijer G transformation

$$G_{p,q}^{m,n}\left(\left.{\mathbf{a} \atop \mathbf{b}}\right| x\right) = G_{q,p}^{n,m}\left(\left.{1-\mathbf{b} \atop 1-\mathbf{a}}\right| \frac{1}{x}\right). \tag{72}$$

$\Diamond$

Remark 6 (Probability of return to the unit disk)**.** By a simple geometric argument, there is a $\frac{1}{3}$ chance of returning to the unit disk in a 2-step walk. Similarly, for a

3-step walk, if the second step makes an angle θ with the first step, then the third step can only vary over a range of θ to return to the unit disk (it can be parallel to the first step, to the second step, or anywhere in between). Thus the probability of returning to the unit disk in three steps is

$$\frac{1}{4\pi^2}\int_{-\pi}^{\pi} |\theta| \,\mathrm{d}\theta = \frac{1}{4} = \int_0^1 p_3(x)\,\mathrm{d}x.$$

Appealing to (54) we deduce that

$$\sum_{k=0}^{\infty} \frac{W_3(2k)}{9^k(k+1)} = \frac{\sqrt{3}\pi}{4}.$$

In fact, as Kluyver showed [Klu1906], the probability of an n-step walk ending in the unit disk is $1/(n+1)$. This is easily obtained by setting $\alpha = 1$ in (10). ♢

4 Partial resolution of Conjecture 1

We may now investigate Conjecture 1 which is restated below for convenience.

Conjecture. For positive integers n and complex s,

$$W_{2n}(s) \overset{?[1]}{=} \sum_{j\geqslant 0} \binom{s/2}{j}^2 W_{2n-1}(s-2j). \tag{73}$$

We can resolve this conjecture modulo a conjectured technical estimate given in Conjecture 3. The proof outline below certainly explains Conjecture 1 by identifying the terms of the infinite sum as natural residues.

Proof. Using (17) we write W_{2n} as a Bessel integral

$$W_{2n}(-s) = 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\int_0^\infty x^{s-1}J_0^{2n}(x)\,\mathrm{d}x.$$

Then we apply Theorem 1 to $J_0^{2n} = J_0^{2n-1}\cdot J_0$ for s in a vertical strip. Since, again by (17), we have

$$\int_0^\infty x^{s-1}J_0^{2n}(x)\,\mathrm{d}x = 2^{s-1}\frac{\Gamma(s/2)}{\Gamma(1-s/2)}W_n(-s)$$

we obtain

$$\begin{aligned} W_{2n}(-s) &= 2^{1-s}\frac{\Gamma(1-s/2)}{\Gamma(s/2)}\int_0^\infty x^{s-1}J_0^{2n-1}(x)\cdot J_0(x)\,\mathrm{d}x \\ &= \frac{\Gamma(1-s/2)}{\Gamma(s/2)}\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{2}\frac{\Gamma(z/2)\Gamma(s/2-z/2)}{\Gamma(1-z/2)\Gamma(1-s/2+z/2)}W_{2n-1}(-z)\,\mathrm{d}z \end{aligned}$$

where $0<\delta<1$.

Observe that the integrand has poles at $z=s,s+2,s+4,\ldots$ coming from $\Gamma(s/2-z/2)$ as well as (irrelevant for current purposes) poles at $z=0,-2,-4,\ldots$ coming from $\Gamma(z/2)$. On the other hand, the term $W_{2n-1}(-z)$ has at most simple poles at $z=2,4,6,\ldots$ which are cancelled by the corresponding zeros of $\Gamma(1-z/2)$. This asserted pole structure of W_{2n-1} was shown in Proposition 1 for $n=3,4$ and that argument can be extended to $7,9,\ldots$, but this will not resolve the general case. (See Conjecture 2 below.)

Next, we determine the residue of the integrand at $z=s+2j$. Since $\Gamma(s/2-z/2)$ has a residue of $-2(-1)^j/j!$ at $z=s+2j$, the residue of the integrand is

$$-\frac{(-1)^j\Gamma(s/2+j)}{(j!)^2\Gamma(1-s/2-j)}W_{2n-1}(-(2j+s))=-\frac{\Gamma(s/2)}{\Gamma(1-s/2)}\binom{-s/2}{j}^2W_{2n-1}(-s-2j).$$

Thus it follows that

$$W_{2n}(-s)=\sum_{j\geqslant 0}\binom{-s/2}{j}^2W_{2n-1}(-s-2j), \tag{74}$$

which is what we want to prove, provided that we can close the contour in the right half-plane so as to show that

$$\liminf_{\alpha\to\infty}\int_{\gamma_\alpha}\frac{\Gamma(z/2)\Gamma(s/2-z/2)}{\Gamma(1-z/2)\Gamma(1-s/2+z/2)}W_{2n-1}(-z)\,\mathrm{d}z=0. \tag{75}$$

Here, γ_α is a right half-circle of radius r_α around δ. This follows from Conjecture 3 below. □

To make this proof rigorous we therefore need to show that the next two conjectures hold.

Conjecture 2 (Poles of W_{2n-1}). *For each $n\geqslant 1$ all poles of W_{2n-1} are simple.*

Conjecture 3 (Growth of W_{2n-1})**.** *For given s, the maximum modulus of*

$$\frac{\Gamma(z/2)\Gamma(s/2-z/2)}{\Gamma(1-z/2)\Gamma(1-s/2+z/2)}W_{2n-1}(-z)$$

over the half-circle γ_{α_m}, is achieved on the real axis at a point a_m; and the value $W_{2n-1}(a_m)$ tends to zero for properly chosen $r_{\alpha_m} \to \infty$.

Remark 7 (Other approaches to Conjecture 1)**.** We restrict ourself to the core case with $n = 2$. One can prove that both sides of the needed identity satisfy the recursion for W_4. Hence, it suffices to show that the conjecture is correct for $s = \pm 1$. Working entirely formally with (11) and ignoring the restriction on s we have:

$$\begin{aligned}\sum_{j\geqslant 0}\binom{-1/2}{j}^2 W_3(-1-2j) &= \sum_{j=0}^{\infty}\binom{-1/2}{j}^2 2^{-2j}\frac{\Gamma(\frac{1}{2}-j)}{\Gamma(\frac{1}{2}+j)}\int_0^\infty x^{2j}J_0^3(x)\,\mathrm{d}x \\ &= \int_0^\infty J_0^3(x)\sum_{j=0}^{\infty}\binom{-1/2}{j}^2\frac{\Gamma(\frac{1}{2}-j)}{\Gamma(\frac{1}{2}+j)}\left(\frac{x}{2}\right)^{2j}\mathrm{d}x \\ &= \int_0^\infty J_0^4(x)\,\mathrm{d}x \\ &= W_4(-1),\end{aligned}$$

on appealing to Example 1, since

$$\sum_{j=0}^{\infty}\binom{-1/2}{j}^2\frac{\Gamma(\frac{1}{2}-j)}{\Gamma(\frac{1}{2}+j)}x^{2j} = J_0(2x)$$

for $x > 0$. There is a corresponding manipulation for $s = 1$, but we cannot make them rigorous. However, in [BSW10], we prove the conjecture for $n = 2$ and s an integer. $\Diamond$

Conclusion

In addition to the two new conjectures made explicitly above, it would be fascinating to obtain closed forms for any of the residues in Proposition 1 with $n \geqslant 5$. It would likewise be very informative to obtain a closed form for $W_5(\pm 1)$.

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References

[AS72] M. Abramowitz and I. A. Stegun, ed. *Handbook of Mathematical Functions*. Dover, New York, 1972.

[AAR99] G. E. Andrews, R. Askey and R. Roy. *Special Functions*. Cambridge University Press, 1999.

[BBBG08] David H. Bailey, Jonathan M. Borwein, David J. Broadhurst, and M. L. Glasser. "Elliptic integral evaluations of Bessel moments and applications." *J. Phys. A: Math. Theor.*, **41** (2008), 5203–5231.

[Bai32] W. N. Bailey. "Some transformations of generalized hypergeometric series, and contour integrals of Barnes's type." *Quart. J. Math. Oxford* **3** (1932), 168–182.

[Bai35] W. N. Bailey. *Generalized Hypergeometric Series*. Cambridge University Press, 1935.

[BBG05] J. M. Borwein, D. H. Bailey and Roland Girgensohn. "Experimentation in Mathematics: Computational Paths to Discovery." A K Peters, 2005.

[BNSW09] J. Borwein, D. Nuyens, A. Straub, and J. Wan. "Random walk integrals." Preprint, October 2009.

[BSW10] J. Borwein, A. Straub and J. Wan. "Densities of few-step uniform random walks." Preprint, September 2010.

[BB87] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*. Wiley, 1987.

[BZB08] J.M. Borwein, I. J. Zucker and J. Boersma. "The evaluation of character Euler double sums," *Ramanujan Journal*, **15** (2008), 377–405.

[Bro09] David Broadhurst. "Bessel moments, random walks and Calabi-Yau equations." Preprint, November 2009.

[Cra09] R. E. Crandall. "Analytic representations for circle-jump moments." *PSIpress*, 21 Nov 09. Available at `www.perfscipress.com/papers/analyticWn_psipress.pdf`.

[Har78] G. H. Hardy. "Ramanujan. Twelve Lectures on subjects suggested by his life and work." Chelsea Publishing Company, New York, 1978.

[Hug95] B. D. Hughes. *Random Walks and Random Environments*, Volume 1. Oxford, 1995.

[Klu1906] J. C. Kluyver. "A local probability problem." *Nederl. Acad. Wetensch. Proc.* **8** (1906), 341–350.

[Luk69] Y. L. Luke. *The Special Functions and Their Approximations.* Volume I. Academic Press, 1969.

[Mar83] O.I. Marichev. *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables*, Ellis Horwood Limited, Chichester, England, 1983.

[Nest] Yu. V. Nesterenko. "Arithmetic properties of values of the Riemann Zeta function and generalized hypergeometric functions." (2001).

[Pea1905] Karl Pearson. "The random walk." *Nature*, **72** (1905), 294.

[Pea1905b] Karl Pearson. "The problem of the random walk." *Nature*, **72** (1905), 342.

[Pea1906] K. Pearson. *A Mathematical Theory of Random Migration*, Mathematical Contributions to the Theory of Evolution XV. Drapers, 1906.

[PWZ06] M. Petkovsek, H. Wilf and D. Zeilberger. *A=B*, AK Peters, (2006).

[Ray1905] Lord Rayleigh. "The problem of the random walk." *Nature*, **72** (1905), 318.

[RMT10] T. Amdeberhan, O. Espinosa, I. Gonzalez, M. Harrison, V. Moll, and A. Straub. "Ramanujan's Master Theorem." Preprint, 2010.

[RS09] L. B. Richmond and J. Shallit. "Counting abelian squares." *The Electronic Journal of Combinatorics*, **16**, 2009.

[Watson32] G. N. Watson. *A Treatise on the Theory of Bessel Functions*, Second edition, 1941.

[Watson39] G. N. Watson. "Three triple integrals." *Physica*, **5** (1938), 465.

[Zudilin02] W. Zudilin. "Very well-poised hypergeometric series and multiple integrals." *Russian Math. Surveys*, **57**:4 (2002), 824–826.

[Zudilin04] W. Zudilin. "Arithmetic of linear forms involving odd zeta values." *J. Théorie Nombres Bordeaux*, **16**:1 (2004), 251–291.