# UNIFORMLY CONVEX FUNCTIONS ON BANACH SPACES

J. BORWEIN[1], A. J. GUIRAO[2], P. HÁJEK[3], AND J. VANDERWERFF

Abstract. We study the connection between uniformly convex functions $f : X \to \mathbb{R}$ bounded above by $\|x\|^p$, and the existence of norms on $X$ with moduli of convexity of power type. In particular, we show that there exists a uniformly convex function $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^2$ if and only if $X$ admits a norm with modulus of convexity of power type 2.

## 1. Introduction

Uniformly convex functions on Banach spaces were introduced by Levitin and Poljak in [13]. Their properties were studied in depth by Zălinescu [17], and then later Azé and Penot [2] studied their duality with uniformly smooth convex functions; see also [18] for more details. Yet, surprisingly, little precise information seems to be known about when they can exist on Banach spaces. For example, [18, Theorem 3.5.13], shows that a Banach space admitting a uniformly convex function whose domain has nonempty interior is reflexive, and in fact, it can be shown that such a Banach space is superreflexive—see Theorem 2.5 (recall that a Banach space is *superreflexive* if and only if it admits an equivalent uniformly convex norm [9]). On the other hand, if one does not require the function to be globally uniformly convex, then [4] and [5] show $f(x) = \|x\|^r$ is totally convex and uniformly convex on bounded sets whenever $\|\cdot\|$ is uniformly convex and $r > 1$. See also [3, 6] for further applications of totally convex and other related convex functions.

In this note, we will focus on the class of *uniformly convex functions*, i.e. functions that are globally uniformly convex as studied in [17] and defined below. This is known to be a more restricted class of functions than those that are uniformly convex on bounded sets; see e.g. [18, Proposition 3.5.8] which shows that $f(x) = \|x\|^r$ cannot be uniformly convex for $r < 2$. We will sharpen this by establishing the precise connection between the uniform convexity of $f(x) = \|x\|^r$ and the modulus of convexity of $\|\cdot\|$. We also examine the more general converse problem: if $f : X \to \mathbb{R}$ is uniformly convex and bounded above by $\|\cdot\|^r$, does $X$ admit a norm with a modulus of convexity of power type related to $r$?

We work with a real Banach space $X$ with dual $X^*$, and let $B_X$ and $S_X$ denote the closed unit ball and sphere respectively. The *Fenchel conjugate* of $f$ is the function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in X\}.$$

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For a given convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ we define its *modulus of convexity* as the function $\delta_f : \mathbb{R}^+ \to [0, +\infty]$ defined by

$$\delta_f(t) := \inf \left\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right) : \|x - y\| = t,\ x, y \in \operatorname{dom} f \right\},$$

where the infimum over the empty set is $+\infty$. Similarly we consider the *modulus of smoothness* of $f : X \to \mathbb{R}$ as the function $\rho_f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\rho_f(t) := \sup \left\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\left(\frac{x+y}{2}\right) : \|x - y\| = t \right\}.$$

We say that $f$ is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$, and $f$ has a *modulus of convexity of power type* $p$ if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$. We will say $f$ is *uniformly smooth* if $\lim_{t \to 0^+} \rho_f(t)/t = 0$, and $f$ has a *modulus of smoothness of power type* $q$ if there is a constant $C > 0$, so that $\rho_f(t) \leq Ct^q$ for all $t > 0$. Let us note that these concepts are sometimes defined using the *gage of uniform convexity* and *gage of uniform smoothness* respectively as found in [18]; it is important to note that these alternate definitions using the respective gages are equivalent to those just given; cf. [17, Remark 2.1] and [18, p. 205].

## 2. Uniform Convexity of Functions and Norms

This section will demonstrate for $1 < p < \infty$ that $f(\cdot) = \|\cdot\|^p$ is uniformly convex if and only if $\|\cdot\|$ has modulus of convexity of power type $p$.

**Lemma 2.1.** *Let* $0 < r \leq 1$, *then* $|t^r - s^r| \leq |t - s|^r$ *for all* $s$, $t \in [0, \infty)$.

*Proof.* First, for $x \geq 0$, $(1 + x)^r \leq 1 + x^r$ (see [16, Example 4.20]). Setting $x = (t - s)/s$ with $t \geq s > 0$, and then multiplying by $s^r$, we get $t^r \leq s^r + (t - s)^r$. The conclusion follows from this. □

**Theorem 2.2.** *For* $1 < q \leq 2$, *the following are equivalent in a Banach space* $X$.
*(a) The norm* $\|\cdot\|$ *has modulus of smoothness of power type* $q$.
*(b) The function* $f(\cdot) = \|\cdot\|^q$ *has modulus of smoothness of power type* $q$.
*(c) The function* $f(\cdot) = \|\cdot\|^q$ *is uniformly smooth.*

*Proof.* (a) $\Rightarrow$ (b): We assume that $\|\cdot\|$ has modulus of smoothness of power type $q$. Then it has a (Fréchet) derivative satisfying a $(q-1)$-Hölder-condition on its sphere, see [7, Lemma IV.5.1]. Moreover, $f(x) = \|x\|^q$ satisfies $f'(0) = 0$ and $f'(x) = q\|x\|^{q-1}\phi_x$ where $\phi_x \in J(x)$, the duality map, for $x \neq 0$ (i.e. $\phi_x \in S_X$, and $\phi_x(x) = \|x\|$). Observe that if $x = 0$ or $y = 0$ then $\|f'(x) - f'(y)\| \leq q\|x - y\|^{q-1}$. Assuming that $x, y \in X \setminus \{0\}$ we compute,

$$\begin{aligned} f'(x) - f'(y) &= q\|x\|^{q-1}\phi_x - q\|y\|^{q-1}\phi_y \\ &= q\|x\|^{q-1}\phi_x - q\|x\|^{q-1}\phi_y + q\|x\|^{q-1}\phi_y - q\|y\|^{q-1}\phi_y \\ &= q\|x\|^{q-1}(\phi_x - \phi_y) + \big(q\|x\|^{q-1} - q\|y\|^{q-1}\big)\phi_y. \end{aligned} \tag{2.1}$$

Using Lemma 2.1 we also compute

$$\left| q\|x\|^{q-1} - q\|y\|^{q-1} \right| \leq q\big|\,\|x\| - \|y\|\,\big|^{q-1} \leq q\|x - y\|^{q-1}. \tag{2.2}$$

We now work on an estimate for $q\|x\|^{q-1}(\phi_x - \phi_y)$. First, consider the case where $0 < \|y\| \leq \|x\| \leq 1$. If $\|y\| \leq \|x\|/2$, then

$$q\|x\|^{q-1}\|\phi_x - \phi_y\| \leq 2q\|x\|^{q-1} \leq q2^q\|x - y\|^{q-1}.$$

If $\|x\| \leq 1$ and $\|y\| \geq \|x\|/2$, consider $x' = \lambda x$ where $\lambda = \|y\|/\|x\|$, so that $\|x'\| = \|y\|$. Then

$$\begin{aligned}\|x'-y\| &\leq \|x'-x\| + \|x-y\| \\ &= \|x\| - \|y\| + \|x-y\| \leq 2\|x-y\|.\end{aligned}$$

Thus, given the Hölder-condition for the derivative on spheres, there is $C > 0$ such that $\|\phi_u - \phi_v\| \leq C\|u-v\|^{q-1}/\|u\|^{q-1}$ when $\|u\| = \|v\|$, and so we have

$$\|\phi_x - \phi_y\| = \|\phi_{x'} - \phi_y\| \leq C\left(\frac{2\|x-y\|}{\|y\|}\right)^{q-1}.$$

Consequently,

$$q\|x\|^{q-1}\|\phi_x - \phi_y\| \leq Cq\frac{\|x\|^{q-1}}{\|y\|^{q-1}}2^{q-1}\|x-y\|^{q-1} \leq C2^{2q-2}q\|x-y\|^{q-1}.$$

Hence in either case we have $K > 0$ so that for $x$, $y \in B_X$ we have

$$q\|x\|^{q-1}\|\phi_x - \phi_y\| \leq K\|x-y\|^{q-1}.$$

Now consider the case $\|x\| > 1$, and let $\lambda = \|x\|$. Denote $u = x/\lambda$ and $v = y/\lambda$. Then $u$, $v \in B_X$ and $\|u - v\| = \|x-y\|/\lambda$. Thus one can write

$$\begin{aligned}q\|x\|^{q-1}\|\phi_x - \phi_y\| &= q\|x\|^{q-1}\|\phi_u - \phi_v\| \\ &\leq q\|x\|^{q-1}K\|u-v\|^{q-1} \\ &= q\|x\|^{q-1}\frac{1}{\|x\|^{q-1}}K\|x-y\|^{q-1} \\ &= Kq\|x-y\|^{q-1}.\end{aligned}$$

Consequently, in any case

$$q\|x\|^{q-1}\|\phi_x - \phi_y\| \leq Kq\|x-y\|^{q-1}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) shows that for $f(x) = \|x\|^q$, $f'(x)$ satisfies a $(q-1)$-Hölder-condition and hence that $\|x\|^q$ has modulus of smoothness of power type $q$, on appealing to [18, Corollary 3.5.7] (see also [7, Lemma V.3.5]).

Now (b) $\Rightarrow$ (c) is trivial, so we prove (c) $\Rightarrow$ (a). Suppose $\|\cdot\|$ does not have modulus of smoothness of power type q. Then using [7, Lemma IV.5.1] there are $x_n$, $y_n \in S_X$ such that $\|x_n - y_n\| \to 0$ while

$$\|\phi_{x_n} - \phi_{y_n}\| \geq n\|x_n - y_n\|^{q-1}.$$

Let $\delta_n = \|x_n - y_n\|$ and define $u_n = \frac{1}{\delta_n\sqrt{n}}x_n$ and $v_n = \frac{1}{\delta_n\sqrt{n}}y_n$. Then $\|u_n - v_n\| = \frac{1}{\sqrt{n}} \to 0$. However

$$\begin{aligned}\|f'(u_n) - f'(v_n)\| &= \left\|q\|u_n\|^{q-1}\phi_{u_n} - q\|v_n\|^{q-1}\phi_{v_n}\right\| \\ &= \left\|q\|u_n\|^{q-1}\phi_{x_n} - q\|v_n\|^{q-1}\phi_{y_n}\right\| \\ &= \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\|\phi_{x_n} - \phi_{y_n}\| \\ &\geq \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\left(n\delta_n^{q-1}\right) = qn^{\frac{3-q}{2}} \to \infty.\end{aligned}$$

Consequently, $f'$ is not uniformly continuous, and so [18, Theorem 3.5.6] (see also [7, Lemma IV.3.5]) shows that that $f(\cdot) = \|\cdot\|^q$ is not a uniformly smooth convex function. $\square$

The results in [2] enable us to derive the dual version of Theorem 2.2 for uniformly convex functions.

**Theorem 2.3.** *Let $X$ be a Banach space, and let $2 \le p < \infty$. Then the following are equivalent.*
*(a) The norm $\|\cdot\|$ on $X$ has modulus of convexity of power type $p$.*
*(b) The function $f(\cdot) = \|\cdot\|^p$ has modulus of convexity of power type $p$.*
*(c) The function $f(\cdot) = \|\cdot\|^p$ is uniformly convex.*

*Proof.* (a) $\Rightarrow$ (b): Let us assume that $\|\cdot\|$ has modulus of convexity of power type $p$, then the modulus of smoothness of the dual norm on $X^*$, namely $\|\cdot\|_*$, is of power type $q$ where $\frac{1}{p} + \frac{1}{q} = 1$; see [7]. By Theorem 2.2 the function $g(\cdot) = \frac{1}{q}\|\cdot\|_*^q$ has modulus of smoothness of power type $q$. The Fenchel conjugate of $g$ is $g^*(\cdot) = \frac{1}{p}\|\cdot\|^p$, see [2, 18]. Now $g^*$—and hence $\|\cdot\|^p$—has a modulus of convexity of power type $p$ according to [2] (see also [18, Corollary 3.5.11]).

Observe that (b) $\Rightarrow$ (c) is trivial, so we prove (c) $\Rightarrow$ (a). Indeed, assuming that $f(\cdot) = \|\cdot\|^p$ is a uniformly convex function, then [2] shows that $f^*$ (and hence $\|\cdot\|_*^q$) is a uniformly smooth function. According to Theorem 2.2, $\|\cdot\|_*$ has modulus of smoothness of power type $q$; therefore $\|\cdot\|$ has modulus of convexity of power type $p$, see [7]. $\square$

We close this section by confirming that the spaces with nontrivial uniformly convex functions are the superreflexive spaces. First, we record a simple symmetrization fact.

**Lemma 2.4.** *Suppose $f : X \to (-\infty, +\infty]$ is a l.s.c. uniformly convex function. Then there exist a uniformly convex l.s.c. function $h : X \to (-\infty, +\infty]$ that is centrally symmetric with*

$$0 = h(0) = \inf\{h(x) : x \in X\}.$$

*Proof.* We can assume that $f$ is centrally symmetric and u.c. by replacing $f$ with $\frac{f(x)+f(-x)}{2}$. Observe then that $f(0) = \min\{f(x) : x \in X\}$ and thus, the function $h(x) = g(x) - g(0)$ satisfies our requirements. $\square$

**Theorem 2.5.** *Let $X$ be a Banach space. Then the following are equivalent.*
*(a) There exists a l.s.c. uniformly convex function $f : X \to (-\infty, +\infty]$ such that the interior of the domain of $f$ is not empty.*
*(b) $X$ admits an equivalent uniformly convex norm.*
*(c) There exist $p \ge 2$ and an equivalent norm $\|\cdot\|$ on $X$ so that $f(x) = \|\cdot\|^p$ is uniformly convex.*

*Proof.* (a) $\Rightarrow$ (b): Shift $f$ so that $0 \in \operatorname{int}(\operatorname{dom} f)$ now by Lemma 2.4 and its proof there is a l.s.c. uniformly convex function $g$ that is is centrally symmetric, $g(x) \ge g(0) = 0$ for all $x$, and $0 \in \operatorname{int} \operatorname{dom} g$. Fix $r > 0$ so that $rB_{(X,\|\cdot\|)} \subset \operatorname{dom} g$. Then for $\|h\| = r$ we have

$$\frac{1}{2}g(h) + \frac{1}{2}g(0) - g\left(\frac{h}{2}\right) \ge \delta_g(r) > 0.$$

Thus $g(h) \geq 2\delta_g(r)$ for all $h$ such that $\|h\| = r$. Let us consider the norm $|||\cdot|||$ whose unit ball is $B = \{x : g(x) \leq \delta_g(r)\}$. We have shown that $B \subset rB_{(X,\|\cdot\|)}$ and since 0 is a point of continuity of $g$ (see [18, Corollary 2.2.13]) then $0 \in \mathrm{int}B$ and so $|||\cdot|||$ is an equivalent norm on $X$.

If $|||x_n||| = |||y_n||| = 1$ and

$$\frac{1}{2}|||x_n||| + \frac{1}{2}|||y_n||| - \left|\left|\left|\frac{x_n + y_n}{2}\right|\right|\right| \to 0,$$

then $d\left(\dfrac{x_n + y_n}{2}, S_{|||\cdot|||}\right) \to 0$ where $S_{|||\cdot|||} = \{x : |||x||| = 1\}$. Because $g$ is Lipschitz on $B$ by [18, Corollary 2.2.12], this means $g\left(\frac{x_n+y_n}{2}\right) \to \delta_g(r)$. Consequently $\frac{1}{2}g(x_n) + \frac{1}{2}g(y_n) - g\left(\frac{x_n+y_n}{2}\right) \to 0$, so the uniform convexity of $g$ ensures that $\|x_n - y_n\| \to 0$ and hence $|||x_n - y_n||| \to 0$

(b) $\Rightarrow$ (c): According to the well-known Enflo-Pisier theorem ([9, 14]), there exist $p \geq 2$ and an equivalent norm $\|\cdot\|$ whose modulus of convexity is of power type $p$. Consequently, Theorem 2.3 shows the function $\|\cdot\|^p$ is uniformly convex.

(c) $\Rightarrow$ (a): This is trivial. $\square$

Note that the indicator function of a single point in any Banach space is trivially uniformly convex. Thus, some domain interiority condition is required in Theorem 2.5(a).

## 3. Growth Rates of Uniformly Convex Functions and Renorming

In this section we will construct a uniformly convex norm whose modulus of convexity is related to the growth rate of a given uniformly convex function on the Banach space thus sharpening Theorem 2.5.

**Lemma 3.1.** *Let $\|\cdot\|$ be a norm on a Banach space $X$. Suppose $\|x\| = \|y\| \geq 1$, and $\|x - y\| \geq \delta$ where $0 < \delta \leq 2\|x\|$. Then $\inf_{t\geq 0}\|x - ty\| \geq \delta/2$.*

*Proof.* Assume that $\|x - t_0 y\| < \delta/2$ for some $t_0 \geq 0$. Then $|1 - t_0|\,\|y\| < \delta/2$ and so

$$\|x - y\| \leq \|x - t_0 y\| + |1 - t_0|\,\|y\| < \delta.$$

which is a contradiction. $\square$

**Lemma 3.2.** *Suppose $F : \mathbb{R} \to [0, \infty)$ is continuous and convex with $F(0) = 0$ and $F(t) > 0$ for $t > 0$. Suppose for all $n \geq N$ that $\{\|\cdot\|_n\}_{n\geq N}$ are norms on $X$ with*

$$\frac{K}{\sqrt{F(2^n)}}\|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n}\|\cdot\|,$$

*for some $K > 0$. Assume that $\|x\|_n = \|y\|_n = 1$ with $\|x - y\| \geq 1$ implies that*

$$\left\|\frac{x + y}{2}\right\|_n \leq 1 - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}$$

*for some $C > 0$. Then the modulus of convexity of the equivalent norm $|\cdot| = \sum_{n\geq N}\|\cdot\|_n$ satisfies*

$$\delta_{|\cdot|}(t) \geq \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)},$$

*for some positive constants $R$ and $M$.*

*Proof.* It is clear the norm $|\cdot|$ is equivalent to $\|\cdot\|$. Moreover, we can find another scalar $k > 0$ so that the norm $|\!|\!|\cdot|\!|\!| = k\,|\cdot|$ satisfies

$$K'\,|\!|\!|\cdot|\!|\!| \leq \|\cdot\| \leq |\!|\!|\cdot|\!|\!| \text{ and } \frac{K'}{\sqrt{F(2^n)}}\,|\!|\!|\cdot|\!|\!| \leq \|\cdot\|_n \leq \frac{1}{2^n}\,\|\cdot\|$$

for some $0 < K' < \frac{1}{2^{N-1}}$ and for all $n \geq N$. Now suppose that $|\!|\!|x|\!|\!| = |\!|\!|y|\!|\!| = 1$ and $x \neq y$. We can choose and fix $n \geq N$ so that

$$\frac{1}{K'2^{n-1}} \leq |\!|\!|x-y|\!|\!| < \frac{1}{K'2^{n-2}}. \tag{3.1}$$

We may without loss of generality assume that $\|x\|_n \leq \|y\|_n$. Now let us denote $a = \|x\|_n^{-1}$ and $b = \|y\|_n^{-1}$. Then $2^n \leq b \leq a \leq \frac{\sqrt{F(2^n)}}{K'}$, and therefore

$$|\!|\!|ax-ay|\!|\!| \geq \frac{a}{K'2^{n-1}} \geq \frac{2}{K'}.$$

According to Lemma 3.1 $|\!|\!|ax-by|\!|\!| \geq \frac{1}{K'}$, which in turn implies $\|ax-by\| \geq 1$. Thus we compute

$$\begin{aligned}
\left\|\frac{ax+ay}{2}\right\|_n &\leq \left\|\frac{ax+by}{2}\right\|_n + \frac{1}{2}(a-b)\,\|y\|_n \\
&\leq \frac{1}{2}\,\|ax\|_n + \frac{1}{2}\,\|by\|_n + \frac{1}{2}(a-b)\,\|y\|_n - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)} \\
&= \frac{a}{2}(\|x\|_n + \|y\|_n) - \frac{C}{F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}.
\end{aligned}$$

This inequality implies

$$\left\|\frac{x+y}{2}\right\|_n \leq \frac{1}{2}\,\|x\|_n + \frac{1}{2}\,\|y\|_n - \frac{C}{aF\left(\frac{2}{K}\sqrt{F(2^n)}\right)}. \tag{3.2}$$

Thus, using (3.2), and the triangle inequality on $\|\cdot\|_j$ for $j \neq n$, we get

$$\left|\!\left|\!\left|\frac{x+y}{2}\right|\!\right|\!\right| \leq \frac{k}{2}\sum_{j\geq N}\|x\|_j + \frac{k}{2}\sum_{j\geq N}\|y\|_j - \frac{K'kC}{\sqrt{F(2^n)}F\left(\frac{2}{K}\sqrt{F(2^n)}\right)}.$$

Let us denote $M = 4/K'$ and $R = K' \cdot k \cdot C$. Let $t = |\!|\!|x-y|\!|\!|$, according to (3.1), $t \leq \frac{1}{K'2^{n-2}}$ and thus $F(2^n) \leq F\left(\frac{4}{tK'}\right)$. Therefore

$$\left|\!\left|\!\left|\frac{x+y}{2}\right|\!\right|\!\right| \leq 1 - \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)},$$

which finishes the proof, since $\delta_{|\!|\!|\cdot|\!|\!|} = \delta_{|\cdot|}$. □

**Lemma 3.3.** *Suppose $f : X \to (-\infty, +\infty]$ is a l.s.c. convex function.*

(a) *If $f$ is uniformly convex, then* $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} > 0$.
(b) *Suppose $f(x) \leq F(\|x\|)$ for all $x$ and $F : [0, +\infty) \to [0, +\infty]$ is non-decreasing. If $x_0 \neq 0$, and $f(x_0) \geq 0$, then*

$$\sup_{\|h\|=1} f'(x_0, h) \leq F(2\,\|x_0\|)/\,\|x_0\|\,.$$

*Proof.* (a) This is shown in [18, Proposition 3.5.8].

(b) Given that $x_0 \neq 0$, $f(x_0) \geq 0$ and $\|h\| = 1$, we have

$$f'(x_0, h) = \lim_{t \to 0^+} \frac{f(x_0 + th) - f(x_0)}{t} \leq \frac{f(x_0 + \|x_0\| h) - f(x_0)}{\|x_0\|} \leq \frac{F(2\|x_0\|)}{\|x_0\|},$$

where the first inequality above follows from the convexity of $f$. □

**Theorem 3.4.** *Let $X$ be a Banach space and let $f : X \to \mathbb{R}$ be a continuous uniformly convex function satisfying $f(x) \leq F(\|x\|)$ for all $x \in X$ for some non-negative real function $F$ with $F(0) = 0$. Then $X$ admits an equivalent norm $|\cdot|$ so that*

$$\delta_{|\cdot|}(t) \geq \frac{R}{\sqrt{F(Mt^{-1})}F\left(S\sqrt{F(Mt^{-1})}\right)},$$

*for some positive constants $R$,$M$ and $S$.*

*Proof.* First of all, replacing $f$ with $\frac{f(x)+f(-x)}{2}$ we can assume that $f$ is centrally symmetric. Using Fenchel conjugation we obtain $f(x) \leq F^{**}(\|x\|)$ for all $x \in X$. According to Lemma 3.3 we choose $N \in \mathbb{N}$ and $K > 0$ so that $f(x) \geq K^2 \|x\|^2$ whenever $\|x\| \geq N$. Thus we have

$$K^2 \|x\|^2 \leq f(x) \leq F^{**}(\|x\|) \text{ whenever } \|x\| \geq N.$$

For $n \geq N$, let $|\cdot|_n$ have unit ball $B_n = \{x : f(x) \leq F^{**}(2^n)\}$. For any $x \in X \setminus \{0\}$, $f(x/|x|_n) = F^{**}(2^n)$. Hence $F^{**}(\|x\|/|x|_n) \geq F^{**}(2^n)$. Since $F(0) = 0$, $F^{**}$ and $F^{**}(s)/s$ are non-decreasing. This implies that $\|x\| \geq 2^n |x|_n$. Analogously, using that $K^2 \|x/|x|_n\|^2 \leq f(x/|x|_n)$ one obtains $\sqrt{F^{**}(2^n)}\,|x|_n \geq K\|x\|$. Consequently,

$$\frac{K}{\sqrt{F^{**}(2^n)}}\|x\| \leq |x|_n \leq \frac{1}{2^n}\|x\|.$$

Now suppose $|x|_n = |y|_n = 1$, and $\|x - y\| \geq 1$. Letting $\delta_f$ denote the modulus of convexity of $f$ with respect to $\|\cdot\|$, the uniform convexity of $f$ ensures $\delta_f(1) > 0$. Then denoting $z = \frac{x+y}{2}$ and $z' = z/|z|_n$ we obtain

$$\begin{aligned} \delta_f(1) &\leq f(z') - f(z) = \frac{f(z) - f(z')}{-1} \leq f'(z', z' - z) \\ &= \|z' - z\| f'\left(z, \frac{z' - z}{\|z' - z\|}\right) \leq M_n \|z' - z\|, \end{aligned} \tag{3.3}$$

where $M_n = \sup\{f'(u, v) : |u|_n = 1, \|v\| = 1\}$. Using the fact that $F^{**}(r)/r$ is nondecreasing for $r > 0$ (see [15]) along with Lemma 3.3 while noting $f(u) \geq 0$ when $|u|_n = 1$, we obtain

$$M_n \leq F^{**}\left(2\frac{\sqrt{F^{**}(2^n)}}{K}\right) / \frac{\sqrt{F^{**}(2^n)}}{K} \tag{3.4}$$

Consequently, using $|\cdot|_n \geq \frac{K}{\sqrt{F^{**}(2^n)}}\|\cdot\|$, (3.3) and then (3.4), we obtain

$$\begin{aligned} \left|\frac{x+y}{2}\right|_n &\leq 1 - \|z' - z\| \frac{K}{\sqrt{F^{**}(2^n)}} \leq 1 - \frac{\delta_f(1)}{M_n} \cdot \frac{K}{\sqrt{F^{**}(2^n)}} \\ &\leq 1 - \frac{\delta_f(1)}{F^{**}\left(\frac{2}{K}\sqrt{F^{**}(2^n)}\right)}. \end{aligned} \tag{3.5}$$

Applying Lemma 3.2, and noting $F^{**} \leq F$, we obtain that

$$\delta_{|\cdot|}(t) \geq \frac{R}{\sqrt{F^{**}(Mt^{-1})}F^{**}\left(\frac{2}{K}\sqrt{F^{**}(Mt^{-1})}\right)} \geq \frac{R}{\sqrt{F(Mt^{-1})}F\left(\frac{2}{K}\sqrt{F(Mt^{-1})}\right)}.$$

This completes the proof. □

**Corollary 3.5.** *Let $X$ be a Banach space and $f : X \to \mathbb{R}$ a continuous uniformly convex function satisfying that $f(x) \leq \|x\|^p$ for some $p \geq 2$ and for all $x \in X$. Then $X$ admits a norm with modulus of convexity of power type $\frac{p}{2}(p+1)$.*

*Proof.* Applying Theorem 3.4 for $F(t) = t^p$ we obtain an equivalent norm $|\cdot|$ and positive constants $R$, $M$ and $S$ such that

$$\begin{aligned}\delta_{|\cdot|}(t) &\geq \frac{R}{\sqrt{(Mt^{-1})^p}\left(S\sqrt{(Mt^{-1})^p}\right)^p} = \frac{R}{\left(\frac{M}{t}\right)^{\frac{p}{2}} S^p \left(\frac{M}{t}\right)^{\frac{p^2}{2}}}\\ &= \frac{R}{S^p M^{\frac{p}{2}(p+1)}} t^{\frac{p}{2}(p+1)},\end{aligned}$$

i.e., there exists a positive constant $K$ such that $\delta_{|\cdot|}(t) \geq K t^{\frac{p}{2}(p+1)}$. □

## 4. A Sharp Result for $p = 2$

In this section, we will sharpen the result from Corollary 3.5 in the case $p = 2$ to obtain the optimal result that if $X$ has a uniformly convex function bounded above by $\|\cdot\|^2$ then there is an equivalent norm on $X$ with a modulus of convexity of power type 2. We refer to [8] for some related information on this case. We begin with some preliminary results.

Let $X$ be a Banach space. We can associate to $X$ the following modulus

$$\widetilde{\delta}_X(\varepsilon) = \sup_{\tau \geq 0}\left\{\frac{1}{2}\tau\varepsilon - \rho_{X^*}(\tau)\right\},$$

where $\varepsilon \in [0, 2]$. By Lindenstrauss' formula $\widetilde{\delta}_X \leq \delta_X$ while $\widetilde{\delta}_X(\varepsilon) \geq \delta_X(\varepsilon/2)$, see [10].

The modulus of smoothness associated with $X$ satisfies the following property which characterizes those functions being a modulus of smoothness of a Banach space (see [11] for a dual result).

**Proposition 4.1.** [10, Proposition 10] *Let $(X, \|\cdot\|)$ be a Banach space. If $0 < \tau \leq \sigma$, then*

$$\rho_X(\sigma)/\sigma^2 \leq L\rho_X(\tau)/\tau^2,$$

*where $L$ is a constant smaller than $2\prod_{n=0}^{\infty}(1 + 2^{-n}/3) \approx 3.6591297\ldots$.*

**Lemma 4.2.** *Let $X$ be a Banach space. Suppose $\{\|\cdot\|_n\}_{n \geq N}$ are norms on $(X, \|\cdot\|)$ so that for some $K > 0$ and all $n \geq N$, one has*

$$K\|\cdot\| \leq \|\cdot\|_n \leq \|\cdot\|.$$

*Then there exists an equivalent norm $|\cdot|$ such that for all $n \geq N$*

$$\delta_{|\cdot|}(t) \geq R_0 \delta_{\|\cdot\|_n}(t),$$

*where $R_0 > 0$ is a universal constant.*

*Proof.* A norm with the required property can be defined by the formula

$$|x|^2 = \sum_{n\geq N} a_n \|x\|_n^2 ,$$

where $a_n$ satisfies $\sum_{n\geq N} a_n = \left(\frac{K}{2K_0L}\right)^2$ and where $L$ is as in Proposition 4.1.

Let us fix $n \geq N$ and denote $Y = \ell_2(X, \|\cdot\|_n)$. Applying [10, Prop. 19] with $M(t) = t^2$ and $X_i = (X, \|\cdot\|_n)^*$ one has that

$$\rho_{Y^*}(\tau) \leq K_0 \sup_{\tau\leq u\leq 1} \rho_{(X,\|\cdot\|_n)^*}(\tau/u)u^2,$$

where $K_0$ depends neither on $X$ nor on $\|\cdot\|_n$. Now, applying Proposition 4.1 we obtain

$$\rho_{Y^*}(\tau) \leq K_0L\rho_{(X,\|\cdot\|_n)^*}(\tau),$$

and by duality

$$\delta_Y(\varepsilon) \geq \widetilde{\delta}_Y(\varepsilon) \geq K_0L\widetilde{\delta}_{(X,\|\cdot\|_n)}(\varepsilon/K_0L) \geq K_0L\delta_{(X,\|\cdot\|_n)}(\varepsilon/2K_0L) ,$$

for all $0 \leq \varepsilon < 2$.

From the proof of [10, Prop. 18] one has that $\delta_{(X,|\cdot|)}(\varepsilon) \geq \frac{1}{2}\delta_Y(c\varepsilon)$, where $c = 2K_0L$. Therefore

$$\delta_{(X,|\cdot|)}(\varepsilon) \geq \frac{K_0L}{2}\delta_{(X,\|\cdot\|_n)}(\varepsilon) ,$$

which finishes the proof. □

We can now complete our final result.

**Theorem 4.3.** *Let $X$ be a Banach space and $f : X \to \mathbb{R}$ a continuous uniformly convex function satisfying $f(x) \leq \|x\|^2$ for all $x \in X$. Then $X$ admits a norm with modulus of convexity of power type 2.*

*Proof.* Again replacing $f$ with $\frac{f(x)+f(-x)}{2}$ clearly preserves the uniform convexity of $f$ and allows us to assume $f(-x) = f(x)$ for all $x \in X$. According to Lemma 3.3 we may choose $N \in \mathbb{N}$ and $K > 0$ so that $f(x) \geq K^2 \|x\|^2$ whenever $\|x\| \geq N$. Thus we have

$$K^2 \|x\|^2 \leq f(x) \leq \|x\|^2 \text{ whenever } \|x\| \geq N.$$

For $n \geq N$, let $|\cdot|_n$ have unit ball $B_n = \{x : f(x) \leq 2^{2n}\}$.

For any $x \in X \setminus \{0\}$, $f(x/|x|_n) = 2^{2n}$. Hence, using $f(x) \leq \|x\|^2$, we obtain $\|x\| \geq 2^n |x|_n$. Analogously, using that $K^2 \|x/|x|_n\|^2 \leq f(x/|x|_n)$ one obtains $2^n |x|_n \geq K \|x\|$. Consequently,

$$\frac{K}{2^n} \|x\| \leq |x|_n \leq \frac{1}{2^n} \|x\| .$$

Let us consider $|x|_n = |y|_n = 1$, this is $f(x) = f(y) = 2^{2n}$, with $|x - y|_n \geq 1/2^n$. Then $\|x - y\| \geq 1$, and letting $z = \frac{x+y}{2}$, $z' = z/|z|_n$ and $M_n = \sup\{f'(u, v) : |u|_n = 1, \|v\| = 1\}$, as in (3.3) in the proof of Theorem 3.4 one has

$$0 < \delta_f(1) \leq M_n \|z' - z\| \tag{4.1}$$

Because, $f(x) \leq \|x\|^2$, and proceeding as in (3.4) we obtain

$$M_n \leq \left(2 \cdot \frac{2^n}{K}\right)^2 / \frac{2^n}{K} = \frac{4(2^n)}{K}. \tag{4.2}$$

Using $|\cdot|_n \geq \frac{K}{2^n}\|\cdot\|$, (4.1), (4.2) and proceeding as in (3.5), we obtain

$$\left|\frac{x+y}{2}\right|_n \leq 1-\delta_f(1)\frac{K}{2^{n+2}}\cdot\frac{K}{2^n}=1-\frac{C}{2^{2n}}.$$

where $C=\delta_f(1)K^2/4$. This implies that

$$\delta_{|\cdot|_n}\left(\frac{1}{2^n}\right)\geq C\left(\frac{1}{2^n}\right)^2.$$

For a fixed $n \geq N$ let us consider $k=1,2,\dots,2^n$ and the constant $R=\frac{CR_0}{4L}$ where $L$ is the Figiel's constant of Proposition 4.1. Then

$$C=\frac{C}{2^{2n}}\cdot\frac{1}{2^{-2n}}\leq\frac{\delta_{|\cdot|_n}(2^{-n})}{2^{-2n}}\leq 4L\frac{\delta_{|\cdot|_n}(k2^{-n})}{k^2 2^{-2n}}.$$

This implies

$$\delta_{|\cdot|_n}\left(\frac{k}{2^n}\right)\geq\frac{R}{R_0}\left(\frac{k}{2^n}\right)^2.$$

For each $n \geq N$, let us consider the new norm $\|\cdot\|_n = 2^n|\cdot|_n$. These new norms satisfy

$$K\|\cdot\|\leq\|\cdot\|_n\leq\|\cdot\| \text{ and } \delta_{|\cdot|_n}(\cdot)=\delta_{\|\cdot\|_n}(\cdot).$$

Applying Lemma 4.2 we obtain an equivalent norm $|\cdot|$ on $X$ such that $\delta_{|\cdot|}(t)\geq R_0\delta_{\|\cdot\|_n}(t)$ for $n \geq N$.

Finally, let us fix $n_0$ and $k \leq 2^{n_0}$. For any $n \geq n_0$ we have that $\frac{k}{2^{n_0}}=\frac{k2^{n-n_0}}{2^n}$. Therefore $\delta_{\|\cdot\|_n}\left(\frac{k}{2^{n_0}}\right)=\delta_{\|\cdot\|_n}\left(\frac{k2^{n-n_0}}{2^n}\right)\geq\frac{R}{R_0}\left(\frac{k}{2^{n_0}}\right)^2$, which implies that

$$\delta_{|\cdot|}\left(\frac{k}{2^{n_0}}\right)\geq R\left(\frac{k}{2^{n_0}}\right)^2.$$

In the previous paragraph we have shown that $\delta_{|\cdot|}(t)\geq Rt^2$ for all $t$ lying in $\mathcal{D}=\left\{\frac{k}{2^n}: n\in\mathbb{N}, 1\leq k\leq 2^n\right\}$. Since $\mathcal{D}$ is dense in $[0,1]$ and since $\delta_{|\cdot|}(\cdot)$ is continuous [12], we have $\delta_{|\cdot|}(t)\geq Rt^2$, which finishes the proof. □

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Computer Science Faculty, 325, Dalhousie University, Halifax, NS, Canada, B3H 1W5
*E-mail address*: jborwein@cs.dal.ca

Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
*E-mail address*: ajguirao@um.es

Mathematical Institute, AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic
*E-mail address*: hajek@math.cas.cz

Department of Mathematics, La Sierra University, Riverside, CA
*E-mail address*: jvanderw@lasierra.edu