DERIVATIVES AND FAST EVALUATION OF THE WITTEN ZETA FUNCTION

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Abstract. We study analytic properties of the Witten zeta function $\mathcal{W}(r,s,t)$, which is also named after Mordell and Tornheim. In particular, we evaluate the function $\mathcal{W}(s,s,\tau s)$ $(\tau>0)$ at $s=0$ and, as our main result, find the derivative of this function at $s=0$. Our principal tool is an identity due to Crandall that involves a free parameter and provides an analytic continuation. Furthermore, we derive special values of a permutation sum. Throughout this paper we show by way of examples that Crandall's identity can be used for efficient and high-precision evaluations of the Witten zeta function.

1. Introduction

The double series

$$\mathcal{W}(r,s,t) := \sum_{m,n\geq 1} \frac{1}{m^r}\frac{1}{n^s}\frac{1}{(m+n)^t} \tag{1.1}$$

has attracted considerable attention in recent years. It converges for all complex r,s,t with $\mathrm{Re}(r+t)>1$, $\mathrm{Re}(s+t)>1$, and $\mathrm{Re}(r+s+t)>2$. This series was first investigated for positive integers r,s,t by Tornheim [22] in 1950, and independently by Mordell [17] in 1958 for the special case $r=s=t$. It is therefore often called a Tornheim (double) sum or Mordell-Tornheim (double) sum or series. Matsumoto [16] showed that $\mathcal{W}(r,s,t)$ can be meromorphically continued, separately in each of the three variables, to all of $\mathbb{C}^3$, with poles given by $r+s+t=2$ and by $r+t=1-\ell$ and $s+t=1-\ell$, where ℓ is a nonnegative integer. He also mentioned that this was first established in unpublished work by Akiyama and independently by Egami, both in 1999.

Recently Romik [19] made a detailed study of the analytic properties of $\mathcal{W}(r,s,t)$ and of the function $\omega_3(s) := \mathcal{W}(s,s,s)$, with special emphasis on the values $\omega_3(0)$ and $\omega_3'(0)$. It is the main purpose of the present paper to use a different method to re-derive the value of $\omega_3(0)$ in a more general setting, and to obtain an explicit and very simple value of $\omega_3'(0)$; this will also be done in greater generality.

This paper is structured as follows. In Section 2 we derive various integral representations for the Witten zeta function, involving polylogarithms and the incomplete gamma function. This is then used to prove a crucial identity due to Crandall, which is subsequently applied to evaluating $\omega_3(0)$ and its generalization. Section 3 is then devoted to evaluating $\omega_3'(0)$ and its generalization. Section 4 is

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independent of the earlier sections and contains some special values, including those of a symmetric sum of Witten zeta functions. We conclude this paper with some further remarks in Section 5.

2. Some analytic representations

In this section we state some identities that were already mentioned in [5], and present their proofs. In all of this, the *polylogarithm* plays an important role. It is defined by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{2.1}$$

where s is often called the *order*. For each fixed $s \in \mathbb{C}$, the series (2.1) defines an analytic function of z for $|z| < 1$; in particular, $\mathrm{Li}_0(z) = z/(1-z)$ and $\mathrm{Li}_1(z) = -\log(1-z)$. The series also converges when $|z| = 1$, provided that $\mathrm{Re}(s) > 1$; for instance, $\mathrm{Li}_s(1) = \zeta(s)$, the Riemann zeta function. Among numerous other properties (see, e.g., [18, Sect. 25.12]), we require the following representations.

Lemma 2.1. *(a) For any $s \in \mathbb{C}$ not a positive integer, and for $|\log z| < 2\pi$, we have*

$$\mathrm{Li}_s(z) = \sum_{m=0}^{\infty} \zeta(s-m)\frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}. \tag{2.2}$$

(b) When $s = n$ is a positive integer then, again for $|\log z| < 2\pi$,

$$\mathrm{Li}_n(z) = \sum_{\substack{m=0\\ m\neq n-1}}^{\infty} \zeta(n-m)\frac{\log^m z}{m!} + \frac{\log^{n-1} z}{(n-1)!}\left(H_{n-1} - \log(-\log z)\right), \tag{2.3}$$

where $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the nth harmonic number, with $H_0 := 1$.

These identities follow from slightly more general identities that are derived in [13, pp. 27–30].

The Witten zeta function enters through the following integral representation which was given, without proof, in [12] as identity (6.2).

Lemma 2.2. *For $t > 0$ and $r, s > 1$ we have*

$$\Gamma(t)\,\mathcal{W}(r,s,t) = \int_0^{\infty} x^{t-1}\mathrm{Li}_r(e^{-x})\mathrm{Li}_s(e^{-x})\mathrm{d}x. \tag{2.4}$$

Proof. We use the Euler integral for $\Gamma(s)$,

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}\mathrm{d}t \qquad (\mathrm{Re}(s) > 0) \tag{2.5}$$

and substitute $t = nx$. Then

$$\Gamma(s) = \int_0^{\infty} e^{-nx}(nx)^{s-1}n\mathrm{d}x = n^s\int_0^{\infty} e^{-nx}x^{s-1}\mathrm{d}x,$$

and with s replaced by t and n replaced by $n+m$,

$$\frac{1}{(n+m)^t} = \frac{1}{\Gamma(t)}\int_0^{\infty} x^{t-1}e^{-(n+m)x}\mathrm{d}x \qquad (\mathrm{Re}(t) > 0). \tag{2.6}$$

Substituting this into (1.1) and changing the order of summation and integration (which is legitimate since all terms are positive), we get

$$\begin{aligned}\mathcal{W}(r,s,t) &= \frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\left(\sum_{n=1}^\infty \frac{e^{-nx}}{n^r}\right)\left(\sum_{m=1}^\infty \frac{e^{-mx}}{m^s}\right)\mathrm{d}x\\ &= \frac{1}{\Gamma(t)}\int_0^\infty x^{t-1}\mathrm{Li}_r(e^{-x})\mathrm{Li}_s(e^{-x})\mathrm{d}x,\end{aligned}$$

where we have used (2.1). This proves (2.4). □

With the substitution $\sigma = e^{-x}$, the integral in (2.4) immediately leads to the following identity.

Corollary 2.3. *For $t > 0$ and $r, s > 1$ we have*

$$\Gamma(t)\,\mathcal{W}(r,s,t) = \int_0^1 \mathrm{Li}_r(\sigma)\mathrm{Li}_s(\sigma)\frac{(-\log\sigma)^{t-1}}{\sigma}\mathrm{d}\sigma. \tag{2.7}$$

For some applications of this integral, see [5].

Although not needed in the remainder of this paper, the next identity is similar in nature to (2.4) and (2.7) and can be found, without proof, in [12] as first part of the identity (6.2).

Corollary 2.4. *For $t > 0$ and $r, s > 1$ we have*

$$\mathcal{W}(r,s,t) = \int_0^1 \mathrm{Li}_r(e^{2\pi i x})\mathrm{Li}_s(e^{2\pi i x})\mathrm{Li}_t(e^{-2\pi i x})\mathrm{d}x. \tag{2.8}$$

Proof. For the given r, s and t, the interchange of summation and integration is justified, and with the definition (2.1), the right-hand side of (2.8) becomes

$$\sum_{n,m,k\geq 1}\frac{1}{n^r m^s k^t}\int_0^1 e^{2\pi i(n+m-k)x}\mathrm{d}x.$$

This last integral vanishes unless $n + m = k$, in which case it is 1, and we obtain the right-hand side of (1.1). □

We now use Lemmas 2.1 and 2.2 to obtain the main tool for the remainder of this paper, namely an expansion of $\mathcal{W}(r,s,t)$ with a free parameter $\theta > 0$. This was first obtained by Crandall and communicated to the first author; see also [5, p. 133]. Similar expansions can also be found in [12]. In what follows, we need the *incomplete Gamma function*, defined by

$$\Gamma(a,z) := \int_z^\infty y^{a-1}e^{-y}\mathrm{d}y. \tag{2.9}$$

Theorem 2.5 (Crandall)**.** *Let r, s, t be complex variables with $r \notin \mathbb{N}$ and $s \notin \mathbb{N}$. Then for any real $\theta > 0$ we have*

(2.10)
$$\begin{aligned}\Gamma(t)\mathcal{W}(r,s,t) &= \sum_{m,n\geq 1}\frac{\Gamma(t,(m+n)\theta)}{m^r n^s (m+n)^t} + \sum_{u,v\geq 0}(-1)^{u+v}\frac{\zeta(r-u)\zeta(s-v)\theta^{u+v+t}}{u!v!(u+v+t)}\\ &+ \Gamma(1-r)\sum_{q\geq 0}(-1)^q\frac{\zeta(s-q)\theta^{r+q+t-1}}{q!(r+q+t-1)}\\ &+ \Gamma(1-s)\sum_{q\geq 0}(-1)^q\frac{\zeta(r-q)\theta^{s+q+t-1}}{q!(s+q+t-1)}\\ &+ \Gamma(1-r)\Gamma(1-s)\frac{\theta^{r+s+t-2}}{r+s+t-2}.\end{aligned}$$

Proof. From the definition (2.9) we get, for any $\theta > 0$,

$$\Gamma(t,(m+n)\theta) = \int_{(m+n)\theta}^{\infty} y^{t-1}e^{-y}\mathrm{d}y,$$

and the simple substitution $y = (m+n)x$ gives

$$\begin{aligned}\Gamma(t,(m+n)\theta) &= \int_{\theta}^{\infty}((m+n)x)^{t-1}e^{-(m+n)x}(m+n)\mathrm{d}x\\ &= (m+n)^t\int_{\theta}^{\infty}x^{t-1}e^{-(m+n)x}\mathrm{d}x,\end{aligned}$$

and thus

$$\int_{\theta}^{\infty}x^{t-1}e^{-(m+n)x}\mathrm{d}x = \frac{\Gamma(t,(m+n)\theta)}{(m+n)^t}. \tag{2.11}$$

Now, using (2.6) and the definition (1.1), breaking up the integral and using (2.11), we get

$$\begin{aligned}\Gamma(t)\mathcal{W}(r,s,t) &= \sum_{m,n\geq 1}\frac{1}{m^r n^s}\left(\int_0^{\theta}+\int_{\theta}^{\infty}\right)x^{t-1}e^{-(m+n)x}\mathrm{d}x\\ &= \sum_{m,n\geq 1}\frac{\Gamma(t,(m+n)\theta)}{m^r n^s(m+n)^t} + \int_0^{\theta}x^{t-1}\mathrm{Li}_r(e^{-x})\mathrm{Li}_s(e^{-x})\mathrm{d}x,\end{aligned} \tag{2.12}$$

where for the integral on the right we have used the same argument as at the end of the proof of Lemma 2.2.

Next, in view of this last term, we use (2.2) with $z = e^{-ix}$ to obtain

$$\begin{aligned}x^{t-1}\mathrm{Li}_r(e^{-x})\mathrm{Li}_s(e^{-x}) &= \sum_{u,v\geq 0}\zeta(r-u)\zeta(s-v)\frac{(-1)^{u+v}}{u!v!}x^{u+v+t-1}\\ &+ \Gamma(1-r)\sum_{u\geq 0}(-1)^u\frac{\zeta(s-u)}{u!}x^{u+r+t-2}\\ &+ \Gamma(1-s)\sum_{v\geq 0}(-1)^v\frac{\zeta(r-v)}{v!}x^{v+s+t-2}\\ &+ \Gamma(1-r)\Gamma(1-s)x^{r+s+t-3}.\end{aligned} \tag{2.13}$$

Now integrate (2.13) from 0 to θ, and substitute the result into (2.12). □

As mentioned in the Introduction, $\mathcal{W}(r,s,t)$ has a singularity when $r+s+t=2$. We now see from (2.10) that this is a simple pole with residue $\Gamma(1-r)\Gamma(1-s)/\Gamma(t)$ when r and s are not positive integers.

Since θ is a free parameter, any convenient value of θ can be used to compute values of $\mathcal{W}(r,s,t)$. Crandall [11] gave a few examples, and since the paper [11] may not be easy to obtain, we quote them here:

(a) A typical numerical value for three nonintegers r,s,t is

$$\mathcal{W}(\pi,\pi,\pi) = 0.121784932649073172392415831466446\ldots$$

(b) A typical evaluation near a pole is, for $d := 200001/300000$,

$$\mathcal{W}(d,d,d) = 529982.9016524962105\ldots$$

(c) We can also obtain values for the analytic continuation outside the original domain of convergence; e.g.,

$$\mathcal{W}(-\tfrac{1}{2},-\tfrac{1}{2},1) = 0.6378331771492328160229422319062\ldots$$

We independently verified these computations, using the parameter $\theta = 4/5$ which turns out to be a convenient choice.

The identity (2.10) has singularities when $r \in \mathbb{N}$ or $s \in \mathbb{N}$, due to the occurrence of $\zeta(1)$ and of $\Gamma(z)$ at negative integers. However, these singularities cancel, and using the identity (2.3) instead of (2.2) in the proof of (2.10), we get the following more complicated identity, which is also valid when r or s is a positive integer. We leave the proof to the reader.

Theorem 2.6 (Crandall)**.** *For any complex r,s,t and real parameter $\theta > 0$,*

$$\begin{aligned}
(2.14)\quad \Gamma(t)\mathcal{W}(r,s,t) &= \sum_{m,n\geq 1} \frac{\Gamma(t,(m+n)\theta)}{m^r n^s (m+n)^t} \\
&+ \sum_{u,v\geq 0}{}' (-1)^{u+v} \frac{\zeta(r-u)\zeta(s-v)\theta^{u+v+t}}{u!v!(u+v+t)} \\
&+ \sum_{q\geq 0}{}' (-1)^q \frac{\zeta(r-q)\theta^{s+q+t-1}}{q!} \left(\frac{A_s + B_s \log\theta}{(s+q+t-1)} - \frac{B_s}{(s+q+t-1)^2} \right) \\
&+ \sum_{q\geq 0}{}' (-1)^q \frac{\zeta(s-q)\theta^{r+q+t-1}}{q!} \left(\frac{A_r + B_r \log\theta}{(r+q+t-1)} - \frac{B_r}{(r+q+t-1)^2} \right) \\
&+ \theta^{r+s+t-2} \left(\frac{(A_r + B_r\log\theta)(A_s + B_s\log\theta)}{r+s+t-2} \right. \\
&\qquad \left. - \frac{A_r B_s + A_s B_r + 2B_r B_s \log\theta}{(r+s+t-2)^2} + \frac{2B_r B_s}{(r+s+t-2)^3} \right),
\end{aligned}$$

where the notation $\sum'$ means that any term that would lead to $\zeta(1)$ is avoided, and

$$A_p := \begin{cases} \Gamma(1-p), & p \notin \mathbb{N}, \\ \frac{(-1)^{p-1}}{\Gamma(p)} H_{p-1}, & p \in \mathbb{N}, \end{cases} \qquad B_p := \begin{cases} 0, & p \notin \mathbb{N}, \\ \frac{(-1)^{p}}{\Gamma(p)} H_{p-1}, & p \in \mathbb{N}. \end{cases}$$

Theorems 2.5 and 2.6 together show that $\mathcal{W}(r,s,t)$ can be meromorphically continued to all of $\mathbb{C}^3$, separately for each of the three variables. See [19] for a detailed study of the singularities and corresponding residues. Since $B_p = 0$ when

$p \notin \mathbb{N}$, we immediately see that (2.14) reduces to (2.10) when neither r nor s is a positive integer.

Once again taking $\theta = 4/5$, and using a limit of 100 on every summation index, we find the following evaluation:

$$\mathcal{W}(2,2,1) = 0.8438254351644824574000744235991486399930\ldots$$

This agrees to 40 places with the formula

$$\mathcal{W}(2,2,1) = 2\zeta(2)\zeta(3) - 3\zeta(5),$$

given in [5]. A second example of the numerical evaluation of the analytic continuation is given by

$$\mathcal{W}(-3,-3,\tfrac{1}{2}) = 0.005111240608000682705038955 19\ldots$$

For another application of Theorem 2.5 we introduce the function

$$\omega_3(s;\tau) := \mathcal{W}(s,s,\tau s) \qquad (s \in \mathbb{C}, \tau > 0), \tag{2.15}$$

which extends the function $\omega_3(s) := \mathcal{W}(s,s,s)$ introduced in [19].

In what follows, we require the first few terms of the Laurent expansion of the gamma function about the origin, which can be written as

$$s\Gamma(s) = 1 - \gamma\, s + O(s^2), \tag{2.16}$$

where γ is the Euler-Mascheroni constant. Some special values of the Riemann zeta function will also be required, in particular (see, e.g., [18, Sect. 25.6(i)])

$$\zeta(0) = -\frac{1}{2}, \qquad \zeta(-1) = -\frac{1}{12}. \tag{2.17}$$

We are now ready to state and prove the following consequence of (2.10).

Theorem 2.7. *For any $\tau > 0$ we have*

$$\omega_3(0;\tau) = \zeta(0)^2 - \frac{2\,\tau}{\tau+1}\zeta(-1) = \frac{1}{12}\,\frac{5\,\tau+3}{\tau+1}, \tag{2.18}$$

and in particular,

$$\omega_3(0) = \frac{1}{3}. \tag{2.19}$$

The identity (2.18) shows that $\mathcal{W}(0,0,0)$ is not well defined, and makes sense only in the context of how we approach $(0,0,0) \in \mathbb{C}^3$. The evaluation (2.19) was earlier obtained by Ronik [19] who, among other results, showed that $\omega_3(s)$ vanishes for negative integers s. We first discovered (2.18), and earlier (2.19), numerically.

Proof of Theorem 2.7. We set $r = s$ and $t = \tau s$ in (2.10), obtaining

$$\begin{aligned} \Gamma(\tau s)\omega_3(s;\tau) &= \sum_{m,n\geq 1} \frac{\Gamma(\tau s,(m+n)\theta)}{(mn(m+n)^{\tau})^s} \\ &\quad + \sum_{u,v\geq 0} (-1)^{u+v} \frac{\zeta(s-u)\zeta(s-v)\theta^{u+v+\tau s}}{u!v!(u+v+\tau s)} \\ &\quad + 2\Gamma(1-s)\sum_{q\geq 0}(-1)^q \frac{\zeta(s-q)\theta^{(1+\tau)s+q-1}}{q!((1+\tau)s+q-1)} \\ &\quad + \Gamma(1-s)^2 \frac{\theta^{(2+\tau)s-2}}{(2+\tau)s-2}. \end{aligned} \tag{2.20}$$

For a fixed $\theta > 0$ we now multiply both sides by τs and let $s \to 0$. Then the left-hand side of (2.20), with (2.16), becomes $\omega_3(0;\tau)$ while on the right-hand side only the following two terms remain:
– second row, for $u = v = 0$, we get $\zeta(0)^2$;
– third row, for $q = 1$, we get $-2\zeta(-1)\tau/(1+\tau)$.
Combining everything, we get the first equation in (2.18). The second equation follows immediately from (2.17). □

3. Derivatives at $(0, 0, 0)$

In this section we will see that the identity (2.20) not only gives an evaluation of $\omega_3(s;\tau)$ at $s = 0$, but also enables us to find the derivative at $s = 0$.

Theorem 3.1. *For any fixed $\tau > 0$ we have*

$$\omega_3'(0;\tau) = \frac{\tau+1}{2}\log(2\pi) + \frac{(\tau-1)\tau}{\tau+1}\zeta'(-1), \tag{3.1}$$

and in particular,

$$\omega_3'(0) = \log(2\pi). \tag{3.2}$$

This result, as well, was first obtained conjecturally, based on numerical computations and the integer relation method PSLQ (see, e.g., [6]).

1. The proof of this result is based on Theorem 2.5, and the main idea is to once again multiply both sides of (2.20) by τs, isolate some critical terms, and let both s and θ approach 0. With this in mind, we rewrite (2.20) as

$$\begin{aligned}\tau s\Gamma(\tau s)\omega_3(s;\tau) - \zeta(s)^2\theta^{\tau s} + \frac{2\tau}{1+\tau}\Gamma(1-s)\zeta(s-1)\theta^{(1+\tau)s} \\ = \tau s\Bigg[\sum_{m,n\ge 1}\frac{\Gamma(\tau s,(m+n)\theta)}{(mn(m+n)^\tau)^s} \\ +2\Gamma(1-s)\zeta(s)\frac{\theta^{(1+\tau)s-1}}{(1+\tau)s-1} + \Gamma(1-s)^2\frac{\theta^{(2+\tau)s-2}}{(2+\tau)s-2} \\ + \sum_{\substack{u,v\ge 0\\(u,v)\ne(0,0)}}(-1)^{u+v}\frac{\zeta(s-u)\zeta(s-v)\theta^{u+v+\tau s}}{u!v!(u+v+\tau s)} \\ +2\Gamma(1-s)\sum_{q\ge 2}(-1)^q\frac{\zeta(s-q)\theta^{(1+\tau)s+q-1}}{q!((1+\tau)s+q-1)}\Bigg].\end{aligned} \tag{3.3}$$

2. In order to take the derivative of the left-hand side of (3.3), we begin with the first term and use (2.16) and (2.18) to obtain

$$\begin{aligned}\frac{d}{ds}\left[\tau s\Gamma(\tau s)\omega_3(s;\tau)\right]_{s=0} &= \lim_{s\to 0}(\tau s\Gamma(\tau s))\omega_3'(0;\tau) + \frac{d}{ds}(\tau s\Gamma(\tau s))\bigg|_{s=0}\omega_3(0;\tau) \\ &= \omega_3'(0;\tau) - \frac{\gamma}{12}\cdot\frac{5\tau+3}{\tau+1}.\end{aligned} \tag{3.4}$$

Next we use the well-known identity (see, e.g., [18, (25.6.11)])

$$\zeta'(0) = -\frac{1}{2}\log(2\pi) \tag{3.5}$$

to obtain, again with (2.17),

$$\frac{d}{ds}\left[\zeta(s)^2\theta^{\tau s}\right]_{s=0} = 2\zeta(0)\zeta'(0) + \zeta(0)^2\tau\log\theta = \frac{1}{2}\log(2\pi) + \frac{1}{4}\tau\log\theta. \tag{3.6}$$

For the derivative of the third term we use another well-known special value, namely

$$\Gamma'(1) = -\gamma \tag{3.7}$$

(see, e.g., [18, (5.4.11)]), together with (2.17), to obtain

$$\begin{aligned}\frac{d}{ds}&\left[\Gamma(1-s)\zeta(s-1)\theta^{(1+\tau)s}\right]_{s=0}\\ &= -\Gamma'(1)\zeta(-1) + \Gamma(1)\zeta'(-1) + \Gamma(1)\zeta(-1)(1+\tau)\log\theta\\ &= -\frac{1}{12}\gamma + \zeta'(-1) - \frac{1+\tau}{12}\log\theta.\end{aligned} \tag{3.8}$$

Finally, combining (3.4), (3.6) and (3.8), we see that the derivative of the left-hand side, $L_\theta(s)$, of (3.3) at $s=0$ is

$$L'_\theta(0) = \omega'_3(0;\tau) - \frac{5\tau}{12}\gamma - \frac{1}{2}\log(2\pi) - \frac{5\tau}{12}\log\theta + \frac{2\tau}{1+\tau}\zeta'(-1). \tag{3.9}$$

3. Next we note that the derivative of the right-hand side of (3.3), $\tau s R_\theta(s)$, at $s=0$, amounts to evaluating $\tau R_\theta(0)$, where $R_\theta(s)$ is the expression in large brackets. Keeping in mind that eventually we wish to take the limit as $\theta\to 0$, we see, with (2.17), that

$$R_\theta(0) = \sum_{m,n\ge 1}\Gamma(0,(m+n)\theta) + \frac{1}{\theta} - \frac{1}{2\theta^2} + O(\theta). \tag{3.10}$$

The key now is to evaluate the double sum in (3.10), which will be the object of the remainder of this section. The main result in this respect is as follows.

Lemma 3.2. *For any $\theta > 0$ we have*

$$\sum_{m,n\ge 1}\Gamma(0,(m+n)\theta) = \frac{1}{2}\log(2\pi) - \frac{5\gamma}{12} + \zeta'(-1) - \frac{1}{\theta} + \frac{1}{2\theta^2} - \frac{5}{12}\log\theta + O(\theta). \tag{3.11}$$

With this lemma we immediately get (3.1). Indeed, by substituting (3.11) into (3.10) we see that the pole in θ cancels. Then, by setting $L'_\theta(0) = \tau R_\theta(0)$, the logarithmic singularity also cancels, and the remaining terms combine to give (3.1) as we let $\theta\to 0$. This completes the proof of Theorem 3.1, given Lemma 3.2 which we prove next.

4. As a first step, we reduce the double sum in (3.10) to a single integral that depends on the parameter $\theta > 0$.

Lemma 3.3. *For any $\theta > 0$ we have*

$$\sum_{m,n\ge 1}\Gamma(0,(m+n)\theta) = \int_1^\infty \frac{du}{(e^{\theta u}-1)^2 u}. \tag{3.12}$$

Proof. We use the identity

$$\Gamma(0,x) = E_1(x) := \int_x^\infty \frac{e^{-t}}{t}dt \qquad (x>0), \tag{3.13}$$

where $E_1(x)$ is the *exponential integral*; see, e.g., [18, (6.2.1), (6.11.1)]. A simple substitution gives

$$\Gamma(0,(m+n)\theta)=\int_1^\infty \frac{e^{-(m+n)\theta u}}{u}du,$$

and upon interchanging the order of summation and integration, we get

$$\sum_{m,n\geq 1}\Gamma(0,(m+n)\theta)=\int_1^\infty\left(\sum_{m,n\geq 1}e^{-(m+n)\theta u}\right)\frac{du}{u}.$$

Now

$$\begin{aligned}\sum_{m,n\geq 1}e^{-(m+n)\theta u}&=\left(\sum_{m\geq 1}e^{-m\theta u}\right)\left(\sum_{n\geq 1}e^{-n\theta u}\right)=\left(\sum_{n\geq 1}\left(e^{-\theta u}\right)^n\right)^2\\&=\left(\frac{e^{-\theta u}}{1-e^{-\theta u}}\right)^2=\frac{1}{(e^{\theta u}-1)^2},\end{aligned}$$

and this immediately gives (3.12) □

To get an idea of the behaviour of the integral in (3.12) for small $\theta>0$, we note that $e^{\theta u}-1\geq\theta u$, and evaluate

$$0<\int_1^\infty\frac{du}{(e^{\theta u}-1)^2u}\leq\theta^{-2}\int_1^\infty u^{-3}du=\frac{1}{2\theta^2}.$$

While this correctly predicts a pole of order 2 at $\theta=0$, a more careful analysis is needed. We do this with the following few lemmas. As a tool we use the *second-order Bernoulli numbers* $B_n^{(2)}$ which are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^2=\sum_{n=0}^\infty\frac{B_n^{(2)}}{n!}t^n\qquad(|t|<2\pi).\tag{3.14}$$

They could also be written as convolution sums of ordinary Bernoulli numbers. The values of $B_n^{(2)}/n!$ for $n=0,1,2$ are 1, −1, and 5/12, respectively.

Lemma 3.4. *Let $0<R<2\pi$ be fixed. Then for $0<\theta\leq R$ we have*

$$\begin{aligned}\int_1^\infty\frac{du}{(e^{\theta u}-1)^2u}&=\sum_{n=3}^\infty\frac{B_n^{(2)}}{n!(n-2)}+\int_1^\infty\frac{dt}{t(e^t-1)^2}\\&\quad+\frac12-\frac1\theta+\frac{1}{2\theta^2}-\frac{5}{12}\log\theta+O(\theta).\end{aligned}\tag{3.15}$$

Proof. We substitute $t=\theta u$ and split the resulting integral into two parts:

$$\int_1^\infty\frac{du}{(e^{\theta u}-1)^2u}=\int_\theta^1\frac{1}{t^3}\left(\frac{t}{e^t-1}\right)^2dt+\int_1^\infty\frac{dt}{t(e^t-1)^2},\tag{3.16}$$

and we denote the first integral on the right by $I_1(\theta)$. We now use (3.14), and with the first few values of $B_n^{(2)}/n!$ and upon changing the order of summation and integration, we get

$$I_1(\theta) = \int_\theta^1 \left(\frac{1}{t^3} \sum_{n=0}^{\infty} \frac{B_n^{(2)}}{n!} t^n \right) dt \tag{3.17}$$
$$= \int_\theta^1 \frac{dt}{t^3} - \int_\theta^1 \frac{dt}{t^2} + \frac{5}{12} \int_\theta^1 \frac{dt}{t} + \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!} \int_\theta^1 t^{n-3} dt$$
$$= \frac{-1}{2} + \frac{1}{2\theta^2} + 1 - \frac{1}{\theta} - \frac{5}{12} \log\theta + \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} - \sum_{n=3}^{\infty} \frac{B_n^{(2)} \theta^{n-2}}{n!(n-2)}.$$

The final series on the right can be rewritten as

$$\theta \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} \theta^{n-3} = O(\theta),$$

since by (3.14) this last series is bounded for all θ with $0 < \theta \le R < 2\pi$. This, together with (3.17) and (3.16), gives (3.15). □

5. To complete the proof of Lemma 3.2, and thus of Theorem 3.1, we evaluate a certain integral in two different ways. As an auxiliary function we require the exponential integral $E_1(x)$ defined in (3.13), and first note that

$$\int_1^\infty \frac{e^{-x}}{x} \mathrm{d}x = E_1(1), \qquad \int_1^\infty \frac{e^{-x}}{x^3} \mathrm{d}x = \tfrac{1}{2} E_1(1). \tag{3.18}$$

The first identity follows from the definition, while the second one is obtained from the first by two successive integrations by part. We also need the following series expansions.

Lemma 3.5. *We have*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!n} = -\gamma - E_1(1), \tag{3.19}$$

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n!(n-2)} = \tfrac{1}{4} - \tfrac{1}{2}\gamma - \tfrac{1}{2} E_1(1). \tag{3.20}$$

Proof. The identity (3.19) is a special case of a power series expansion of $E_1(x)$ which can be found, e.g., in [18, (6.6.1)]. The identity (3.20) follows from a special case of a Laurent expansion for the incomplete Gamma function (see [18, (8.4.15)]) together with the fact that the second integral in (3.18) can also be written as the incomplete Gamma function $\Gamma(-2, 1)$. □

In the final two lemmas of this section we evaluate the integral

$$I_2 := \int_0^\infty \left(\frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} + \frac{1}{12} \right) \frac{dt}{te^t}. \tag{3.21}$$

Since we have the Laurent expansion

$$\frac{e^t}{(e^t-1)^2} = \frac{1}{t^2} - \frac{1}{12} + \frac{1}{240} t^2 + O(t^4),$$

the integral in (3.21) converges, and can be evaluated as follows.

Lemma 3.6. *With I_2 as defined in* (3.21), *we have*

$$I_2 = -\frac{1}{4} + \frac{5}{12}\gamma + \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} + \int_1^{\infty} \frac{dt}{t(e^t-1)^2}. \tag{3.22}$$

Proof. We split the integral I_2 into two and first note that with (3.18) we have

$$\begin{aligned} \int_1^{\infty} \left(\frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} + \frac{1}{12}\right)\frac{dt}{te^t} &= \int_1^{\infty} \frac{dt}{t(e^t-1)^2} - \int_1^{\infty} \frac{e^{-t}}{t^3}dt + \frac{1}{12}\int_1^{\infty} \frac{e^{-t}}{t}dt \\ &= \int_1^{\infty} \frac{dt}{t(e^t-1)^2} - \frac{5}{12}E_1(1). \end{aligned} \tag{3.23}$$

Next, it is straightforward to verify that

$$\frac{t^2}{(e^t-1)^2} - \left(1 - \frac{t^2}{12}\right)e^{-t} = \sum_{n=3}^{\infty} \left(B_n^{(2)} + (-1)^n\left(\tfrac{(n-1)n}{12} - 1\right)\right)\frac{t^n}{n!},$$

where $B_n^{(2)}$ is as defined in (3.14). We divide both sides of this by t^3, and by absolute and uniform convergence of the series on the right, we may interchange summation and integration, obtaining

$$\begin{aligned} \int_0^1 \left(\frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} + \frac{1}{12}\right)\frac{dt}{te^t} &= \sum_{n=3}^{\infty} \frac{B_n^{(2)} + (-1)^n\left(\frac{(n-1)n}{12} - 1\right)}{n!(n-2)} \\ &= \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} + \frac{1}{12}\sum_{n=1}^{\infty} \frac{(-1)^n}{n!n} - \sum_{n=3}^{\infty} \frac{(-1)^n}{n!(n-2)} \\ &= \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} + \frac{1}{12}\left(-\gamma - E_1(1)\right) - \left(\frac{1}{4} - \frac{1}{2}\gamma - \frac{1}{2}E_1(1)\right) \\ &= \sum_{n=3}^{\infty} \frac{B_n^{(2)}}{n!(n-2)} + \frac{5}{12}\gamma + \frac{5}{12}E_1(1) - \frac{1}{4}, \end{aligned} \tag{3.24}$$

where we have used (3.19) and (3.20). Finally, adding (3.23) and (3.24), we get (3.22). ☐

The next, and final, lemma gives a different evaluation of the integral I_2.

Lemma 3.7. *With I_2 as defined in* (3.21), *we have*

$$I_2 = \zeta'(-1) + \frac{1}{2}\log(2\pi) - \frac{3}{4}. \tag{3.25}$$

Proof. For $\alpha \in \mathbb{C}$ with, initially, $\mathrm{Re}(\alpha) > 2$, we define

$$I_2(\alpha) := \int_0^{\infty} \left(\frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} + \frac{1}{12}\right)\frac{t^{\alpha-1}}{e^t}dt, \tag{3.26}$$

and rewrite

$$\begin{aligned} I_2(\alpha) &= \int_0^{\infty} \frac{t^{\alpha-1}}{(e^t-1)^2}dt - \int_0^{\infty} t^{\alpha-3}e^{-t}dt + \frac{1}{12}\int_0^{\infty} t^{\alpha-1}e^{-t}dt \\ &= \Gamma(\alpha)(\zeta(\alpha-1) - \zeta(\alpha)) - \Gamma(\alpha-2) + \frac{1}{12}\Gamma(\alpha), \end{aligned} \tag{3.27}$$

where the evaluation of the first integral is valid for $\mathrm{Re}(\alpha) > 2$ and can be found, e.g., in [15, (3.423.1)], and the other two are instances of Euler's integral (2.5). The right-hand side of of (3.27) shows that $I_2(\alpha)$ can be analytically continued to all of $\mathbb{C}$, with the exception of $\alpha = 2, 1, 0$, and all negative integers. However, $\alpha = 0$ must be a removable singularity, and we compute the limit as follows, where in doing so we use the fact that $\zeta(0) = 1/2$ and $\zeta(-1) = -1/12$ (see (2.17)). Rewriting (3.27), we get

$$\begin{aligned} I_2(\alpha) &= \Gamma(\alpha)\left(\zeta(\alpha-1) - \zeta(\alpha)) - \frac{1}{(\alpha-1)(\alpha-2)} + \frac{1}{12}\right) \\ &= \alpha\Gamma(\alpha)\left(\frac{\zeta(\alpha-1)+\frac{1}{12}}{\alpha} - \frac{\zeta(\alpha)+\frac{1}{2}}{\alpha} + \frac{1}{\alpha}\left(\frac{1}{2} - \frac{1}{(\alpha-1)(\alpha-2)}\right)\right). \end{aligned} \tag{3.28}$$

Using a simple power series expansion, we find that

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\left(\frac{1}{2} - \frac{1}{(\alpha-1)(\alpha-2)}\right) = -\frac{3}{4},$$

while the limits of the other two fractions on the right of (3.28) can be seen as $\zeta'(-1)$ and $\zeta'(0)$, respectively. Finally, using (2.16), the identity (3.28) yields

$$I_2 = \lim_{\alpha\to 0} I_2(\alpha) = \zeta'(-1) - \zeta'(0) - \frac{3}{4},$$

which gives the desired identity (3.25), having used (3.5). □

To complete the proof of Lemma 3.2, we equate (3.22) and (3.25), and combine the result with (3.15), and then with (3.12).

4. A permutation sum, and special values

Given that the Witten zeta function $\mathcal{W}(r, s, t)$ is symmetric in r and s, but not in all three variables, it makes sense to consider the permutation sum

$$\mathcal{P}(r,s,t) := \mathcal{W}(r,s,t) + \mathcal{W}(r,t,s) + \mathcal{W}(s,r,t) + \mathcal{W}(s,t,r) + \mathcal{W}(t,r,s) + \mathcal{W}(t,s,r).$$

This was previously done by other authors. We first found experimentally

$$\mathcal{P}(2,4,6) = \frac{43\pi^{12}}{58046625},$$

correct to 60 decimals digits. This led to the following conjectured identity which is, in essence, contained in a paper by Espinosa and Moll [14]:

Theorem 4.1 (Espinosa and Moll)**.** *For positive integers r, s, t we have*

$$\mathcal{P}(2r,2s,2t) = (-1)^{r+s+t+1}\frac{(2\pi)^{2r+2s+2t}}{(2r)!(2s)!(2t)!}\int_0^1 B_{2r}(x)B_{2s}(x)B_{2t}(x)\,dx, \tag{4.1}$$

where $B_m(x)$ is the standard Bernoulli polynomial of degree m.

Obviously the right-hand side is a rational multiple of $\pi^{2r+2s+2t}$. This result is Corollary 2.2 in [14], in different notation. To obtain our identity (4.1), the identities (2.5), (1.2), (1.6) and (2.9) in [14], which define various functions used there, will also be needed.

While the identity (4.1) is pleasing in its symmetry, the right-hand side can be rewritten as a sum involving products of two Bernoulli numbers. Carlitz [10] was apparently the first to evaluate the integral on the right of (4.1); his identity was also used by Espinosa and Moll [14]. Here, however, an equivalent integral evaluation obtained in [1] as a consequence of a more general result leads to a somewhat simpler identity. Indeed, using Corollary 1 in [1], we immediately get the following identity.

(4.2)

$$\mathcal{P}(2r,2s,2t) = (-1)^{r+s+t}(2\pi)^{2r+2s+2t} \times \sum_{j=0}^{2r+2s} \left(\binom{j}{2r-1} + \binom{j}{2s-1} \right) \frac{B_{2t+j+1}}{(2t+j+1)!} \frac{B_{2r+2s-j-1}}{(2r+2s-j-1)!},$$

where B_m is the mth Bernoulli number.

The evaluation (4.1), including an identity equivalent to (4.2), was in fact earlier obtained in a more direct way by Subbarao and Sitaramachandra Rao; see [20], Theorem 4.1 and its proof.

The identities (4.1) and (4.2) now lead to the question as to whether there are similar identities in the case where one of r, s, t is zero. This is in fact easy to settle. First we note that

$$\mathcal{W}(r,s,0) = \sum_{m,n\geq 1} \frac{1}{m^r} \frac{1}{n^s} = \zeta(r)\zeta(s). \tag{4.3}$$

Next, the double sum

$$\mathcal{W}(r,0,t) = \sum_{m,n\geq 1} \frac{1}{m^r} \frac{1}{(m+n)^t}$$

is a well-known object that has been studied before. For instance (see, e.g., [9]), Euler showed that

$$\mathcal{W}(r,0,t) + \mathcal{W}(t,0,r) = \zeta(r)\zeta(t) - \zeta(r+t). \tag{4.4}$$

Combining (4.3) and (4.4), we therefore get

$$\mathcal{P}(r,0,t) = 4\zeta(r)\zeta(t) - 2\zeta(r+t), \tag{4.5}$$

valid for complex r, t with $\mathrm{Re}(r) > 1$ and $\mathrm{Re}(t) > 1$. When r and t are positive even integers, we can use Euler's formula, and upon replacing r, t by $2r, 2t$, respectively, we get the identity

$$\mathcal{P}(2r,0,2t) = (-1)^{r+t}(2\pi)^{2r+2t} \left(\frac{B_{2r}}{(2r)!} \frac{B_{2t}}{(2t)!} + \frac{B_{2r+2t}}{(2r+2t)!} \right), \tag{4.6}$$

valid for positive integers r and t. We note that (4.6) cannot be obtained from (4.1) by setting $s = 0$. However, interestingly we do have

$$\mathcal{P}(2r,0,2t) = (-1)^{r+t+1} \frac{(2\pi)^{2r+2t}}{(2r)!(2t)!} \int_0^1 \left(B_{2r}(x) - B_{2r}\right)\left(B_{2t}(x) + B_{2t}\right) dx.$$

This follows from another well-known integral identity for Bernoulli polynomials; see, e.g., [18, (24.13.6)].

Finally in this section, we briefly consider $\mathcal{W}(0,0,t)$, which makes sense only in the context of analytic continuation. Using (2.10) and an analysis similar to (but easier than) the proof of Theorem 3.1, we get for t with $\mathrm{Re}(t) > 2$,

$$\mathcal{W}(0,0,t) = \frac{1}{\Gamma(t)} \int_0^\infty \frac{x^{t-1}}{(e^x-1)^2} dx.$$

The integral on the right has already been used in (3.27); it can be found in [15, (3.423.1)]. As a consequence, we obtain

$$\mathcal{W}(0,0,t) = \zeta(t-1) - \zeta(t) \qquad (\mathrm{Re(s)} > 2). \tag{4.7}$$

This identity means that $\mathcal{W}(0,0,t)$ can be analytically continued to $\mathbb{C} \setminus \{1,2\}$. In particular, we find

$$\lim_{t\to 0} \mathcal{W}(0,0,t) = \zeta(-1) - \zeta(0) = -\frac{1}{12} - \frac{-1}{2} = \frac{5}{12}, \tag{4.8}$$

where we have used (2.17). It is interesting to compare this with (4.4) for $r = t$, namely

$$\mathcal{W}(t,0,t) = \frac{1}{2}\left(\zeta(t)^2 - \zeta(2t)\right)$$

and

$$\lim_{t\to 0} \mathcal{W}(t,0,t) = \frac{1}{2}\left(\zeta(0)^2 - \zeta(0)\right) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\right) = \frac{3}{8}, \tag{4.9}$$

and to compare both (4.8) and (4.9) with (2.18). Finally, (4.3) immediately gives

$$\lim_{t\to 0} \mathcal{W}(t,t,0) = \frac{1}{4},$$

which is in fact consistent with (2.18), if we let $\tau \to 0$.

5. Further remarks

The ideas presented in this paper can be extended in at least two directions.

1. Character analogues of the Witten zeta function were introduced and studied in [4]. While a general investigation of character analogues of Theorems 2.7 and 3.1 will be the subject of separate paper [7], we briefly state some results, without proofs, of the special case of an alternating analogue of $\mathcal{W}(r,s,t)$.

For complex variables r, s, t with (initially) positive real parts we define

$$\mathcal{A}(r,s,t) := \sum_{m,n\ge 1} \frac{(-1)^m}{m^r}\frac{(-1)^n}{n^s}\frac{1}{(m+n)^t}, \tag{5.1}$$

and in analogy to $\omega_3(s)$ after (2.15), we set $\alpha_3(s) := \mathcal{A}(s,s,s)$. The functions $\mathcal{A}(r,s,t)$ and $\alpha_3(s)$ have much simpler analytic continuations than $\mathcal{W}(r,s,t)$ and $\omega_3(s)$, which is the case for all sums with non-principal characters. For instance, we obtain

$$\begin{aligned} \Gamma(s)\alpha_3(s) = &\sum_{m,n\ge 1} \frac{(-1)^m}{m^r}\frac{(-1)^n}{n^s}\frac{\Gamma(s,(m+n)\theta)}{(m+n)^s} \\ &+ \sum_{u,v\ge 0} (-1)^{u+v}\frac{\eta(s-u)\eta(s-v)\theta^{u+v+s}}{u!v!(u+v+s)}, \end{aligned} \tag{5.2}$$

where

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1-2^{1-s})\zeta(s)$$

is the alternating zeta function with $\eta(1) = -\log 2$ and $\eta'(0) = \frac{1}{2}\log\frac{\pi}{2}$. Following the proof of Theorem 2.7, we multiply both sides of (5.2) by s and take the limit at $s \to 0$. Then we immediately obtain

$$\alpha_3(0) = \eta(0)^2 = \tfrac{1}{4}. \tag{5.3}$$

Finally, using methods similar to those in Section 3, we found

$$\alpha_3'(0) = 2\eta'(0) - \eta'(-1) - \tfrac{1}{4}\gamma = \log(2\pi) - \tfrac{5}{3}\log 2 - \tfrac{1}{4}\gamma + 3\zeta'(-1). \tag{5.4}$$

Once again, we confirmed the numerical value $\alpha_3'(0) = 0.04206441814936729840 5\ldots$ by computation.

2. A multi-dimensional analogue of the Witten zeta function (1.1) has been studied by several authors; see, e.g., [3] or [16]. It can be defined, for $n \geq 2$, by

$$\mathcal{W}(r_1, \ldots, r_n, t) := \sum_{m_1,\ldots,m_n \geq 1} \frac{1}{m_1^{r_1} \ldots m_n^{r_n} (m_1 + \ldots m_n)^t},$$

where $r_1, \ldots, r_n$ and t are complex variables with (initially) $\mathrm{Re}(r_j) > 1$ for $1 \leq j \leq n$ and $\mathrm{Re}(t) > 0$. We also define

$$\omega_{n+1}(s) := \mathcal{W}(s, \ldots, s, s).$$

Using methods from this paper, it was shown in [21] that

$$\omega_{n+1}(0) = \frac{(-1)^n}{n+1} \tag{5.5}$$

holds for $n \leq 7$, with the conjecture that it is true for all n, thus extending (2.19). It was also shown in [21] that

$$\omega_4'(0) = -\log(2\pi) + \zeta'(-2) = -\log(2\pi) - \frac{\zeta(3)}{4\pi^2}. \tag{5.6}$$

This can likely be extended to arbitrary n; this is the subject of further work in progress [8].

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