Drive
Dalhousie Distributed Research Institute and Virtual Environment

MEDIALIGHTPATHS

MLP

Atlantic Computational Excellence Network

ACEnet

WestGrid

What is HIGH PERFORMANCE (Pure) MATHEMATICS?

Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

"I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate .... The spoken word and the written word are quite different arts .... I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."

Sir Lawrence Bragg

What would he say about Ppt?

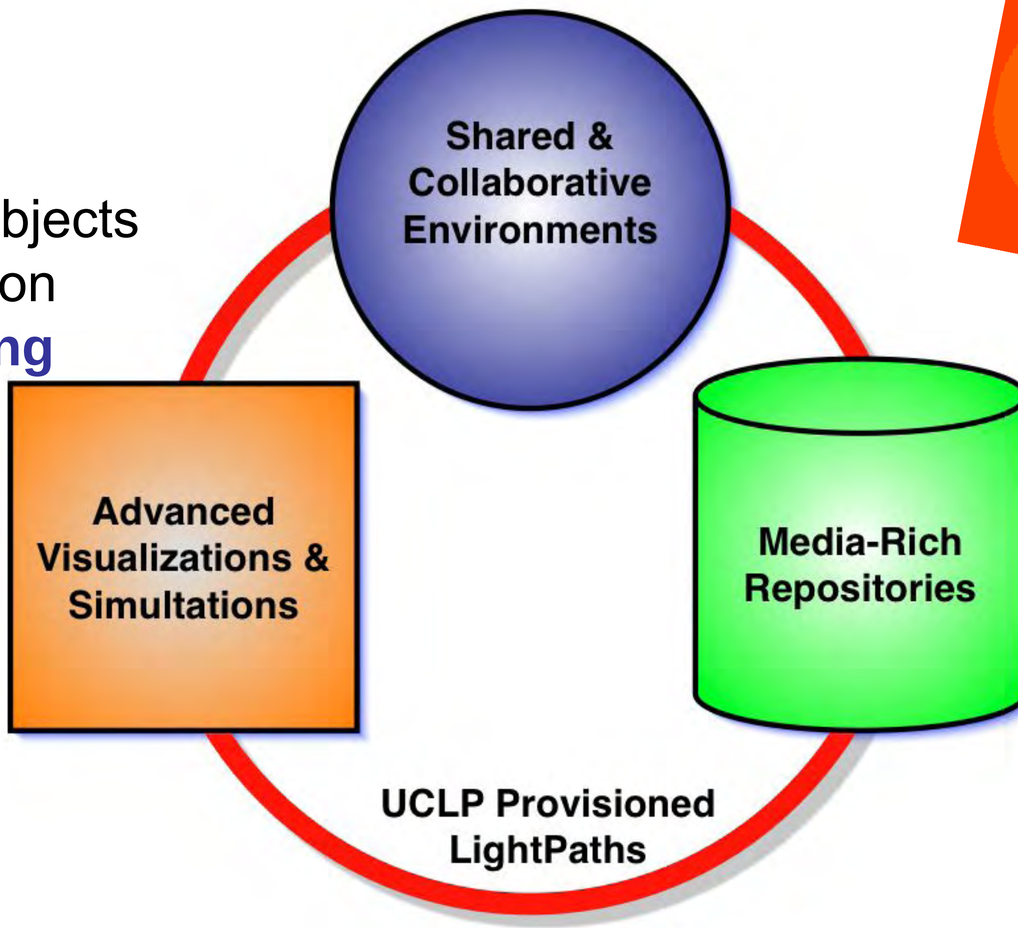
DALHOUSIE UNIVERSITY
Inspiring Minds

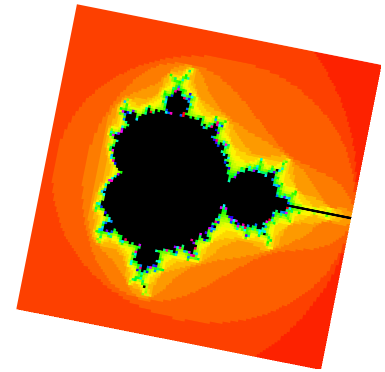
Revised 07/06/05

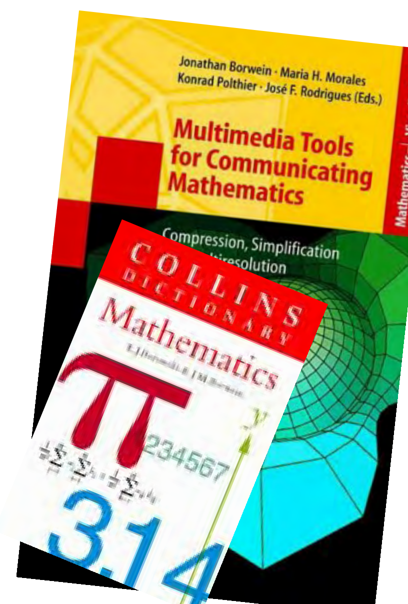
Advanced Networking …

Components include

- **AccessGrid**
- **UCLP** for
  - ✓ haptics
  - ✓ learning objects
  - ✓ visualization
- **Grid Computing**







Advanced Collaboration …

East meets West: Collaboration goes National

Welcome to D-DRIVE whose mandate is to study and develop resources specific to distributed research in the sciences with first client groups being the following communities

- High Performance Computing
- Mathematical and Computational Science Research
- Math and Science
  - Educational
  - Research

Centre seen as 'serious nirvana'

April 07, 2005 , vol. 32, no. 7

By Carol Thorbes

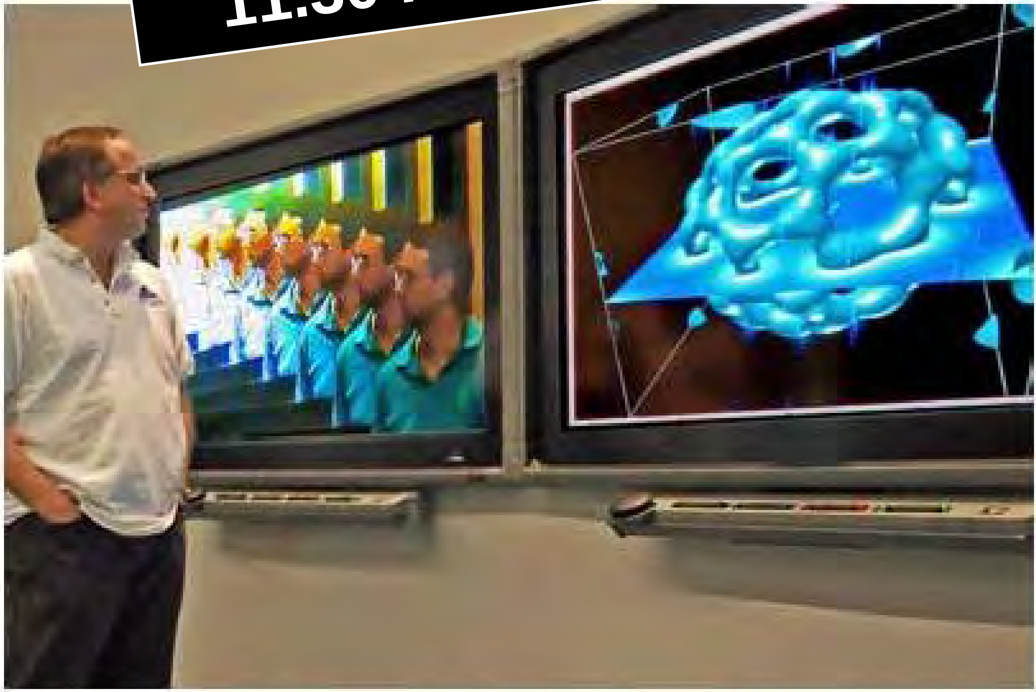
Trans Canada Seminar 11.30 PST and 3.30 AST

The 2,500 square metre IRMACS research centre

- ✓The building is a also a 190cpu G5 Grid
- ✓At the official April opening, I gave one of the four presentations from D-DRIVE

Move over creators of Max Head-room, Matrix and Metropolis. What researchers can accomplish at Simon Fraser University's IRMACS centre rivals the high tech feats of the most memorable futuristic films.

The $14 million centre's acronym stands for interdisciplinary research in the mathematical and computational sciences. The centre's expansive view of the ... from atop ... in echoes its ... al as a facility ... tering ... research ... s whose ... is the computer.

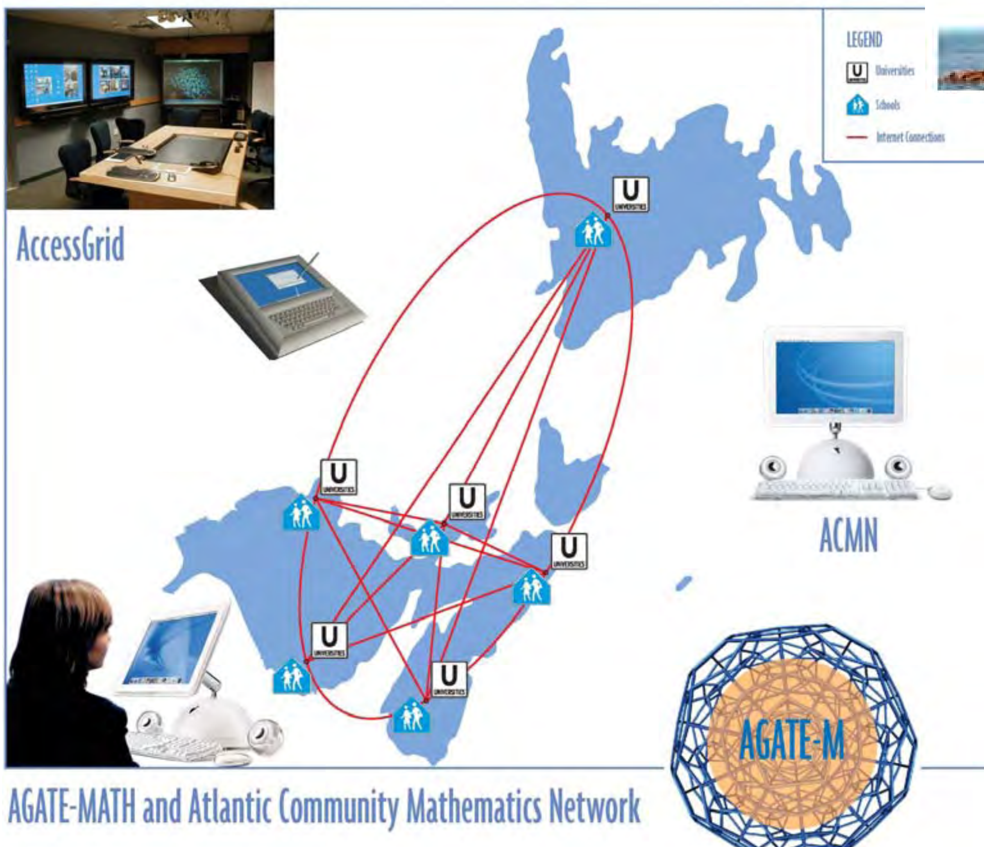


SFU mathematician and IRMACS executive director Peter Borwein (left) communicates with IRMACS collaboration and visualization coordinator Brian Corrie. To the right of them another plasma display portrays a 3D image of a molecular structure.

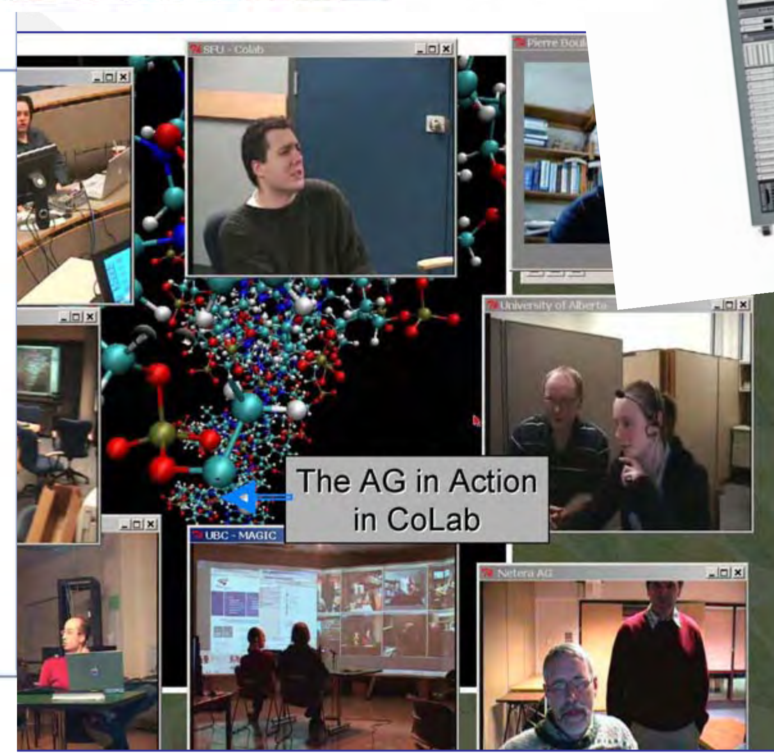
... cted 2,500 square metre space atop the applied sciences building, the centre has eight ... ng rooms and a presentation theatre, seating up to 100 people. They are equipped with ... ble computational, multimedia, internet and remote conferencing (including satellite) technology. High performance distributed computing and clustering technology, designed at SFU, and

Access Grid, AGATE and Apple

- **First 25 teachers identified**



agate
Math



Haptics in the MLP

Drive
Dalhousie Distributed
Research Institute and
Virtual Environment

Haptic Devices extend the world of I/O into the tangible and tactile

We aim to link multiple devices together such that two or more users may interact at a distance

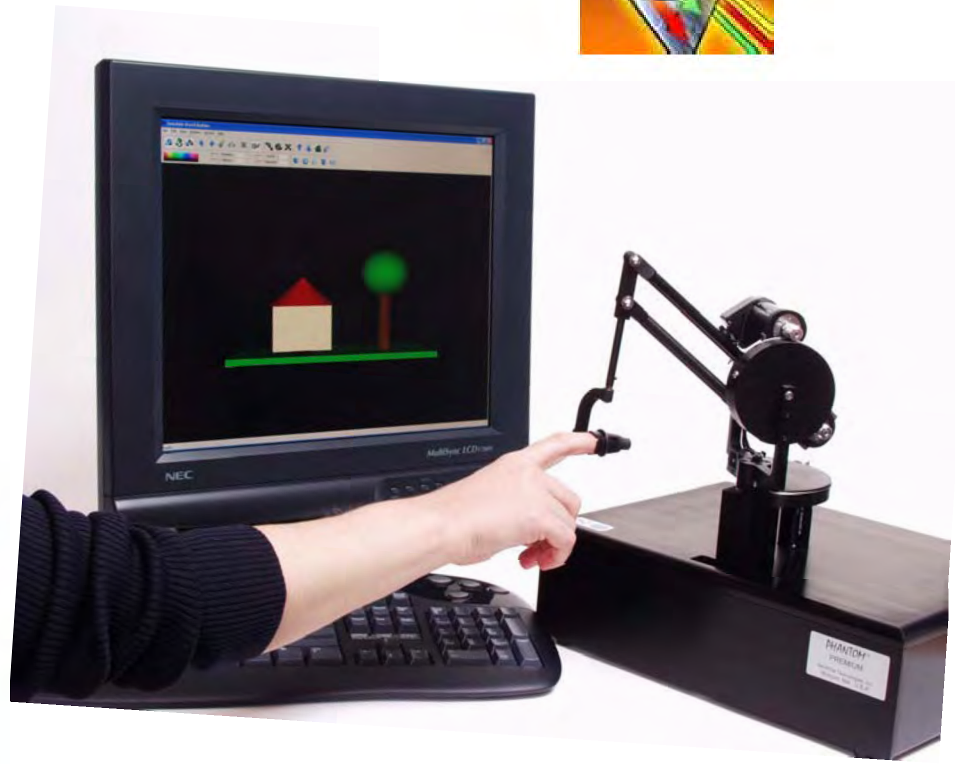
- in Museums and elsewhere
- Kinesiology, HCI ….

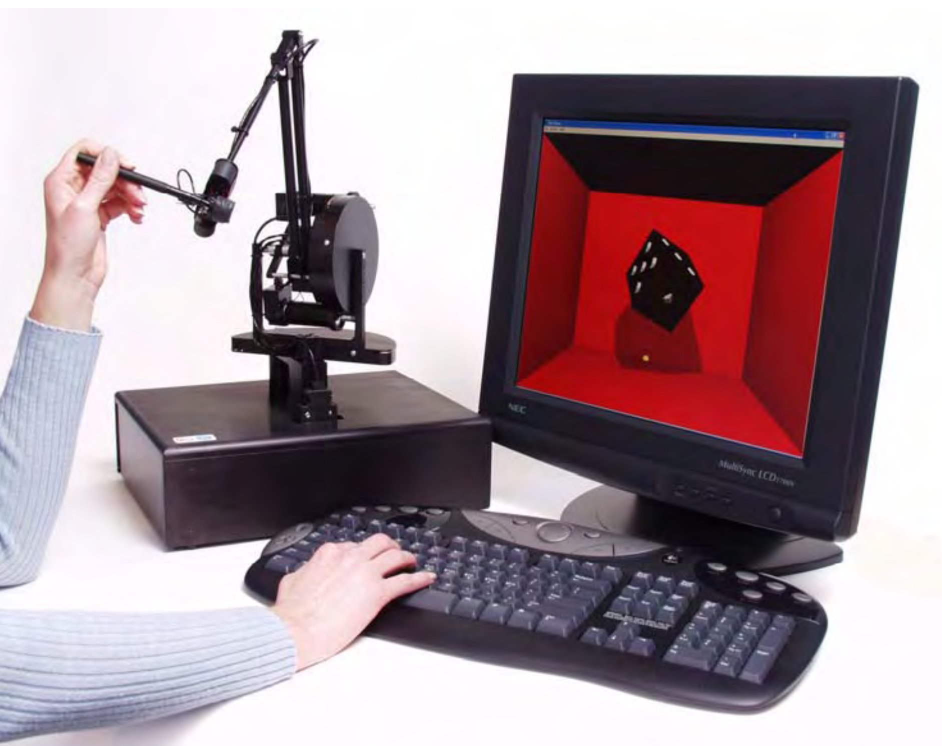
Sensable's **Phantom Omni**

MEDIALIGHTPATHS

And what they do

Force feedback informs the user of his virtual environment adding an increased depth to human computer interaction



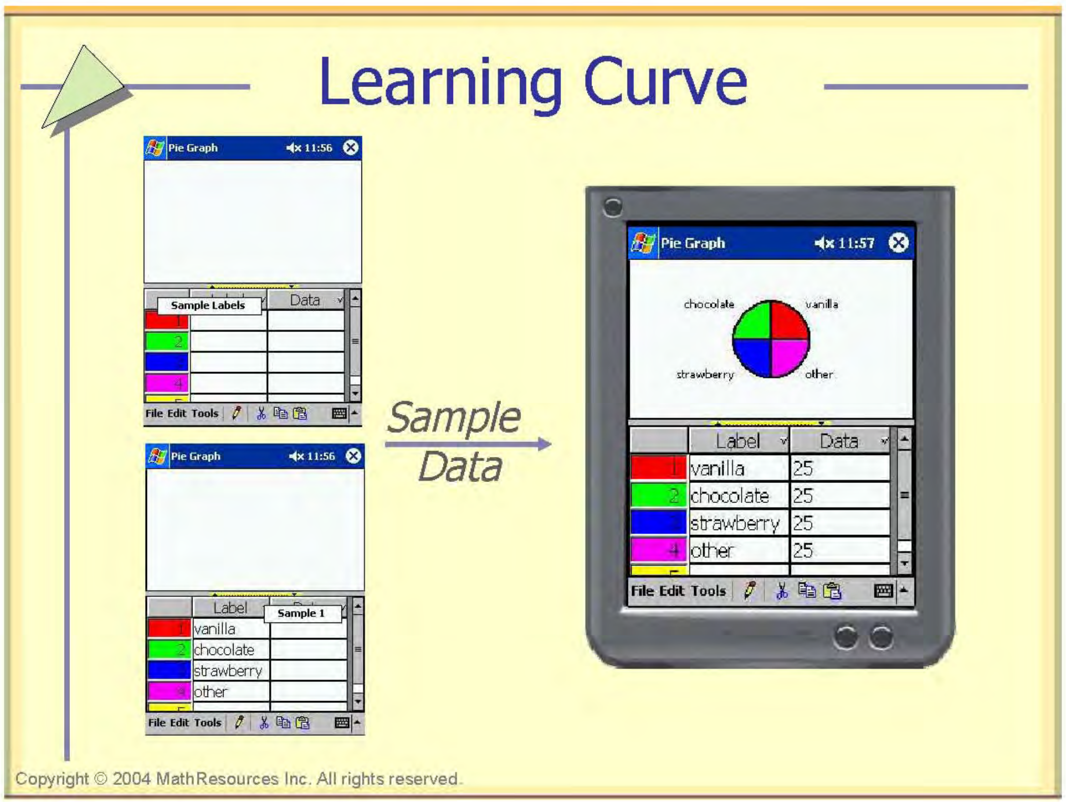


The user feels the contours of the virtual die via resistance from the arm of the device

Advanced LOR's …

MITACS – MRI

putting high end science
on a hand held



D4 Wednesday, December 15, 2004 BUSINESS

Try your hand at new math

Firm develops software to help guide kids through maze of numbers

By GREG MacVICAR

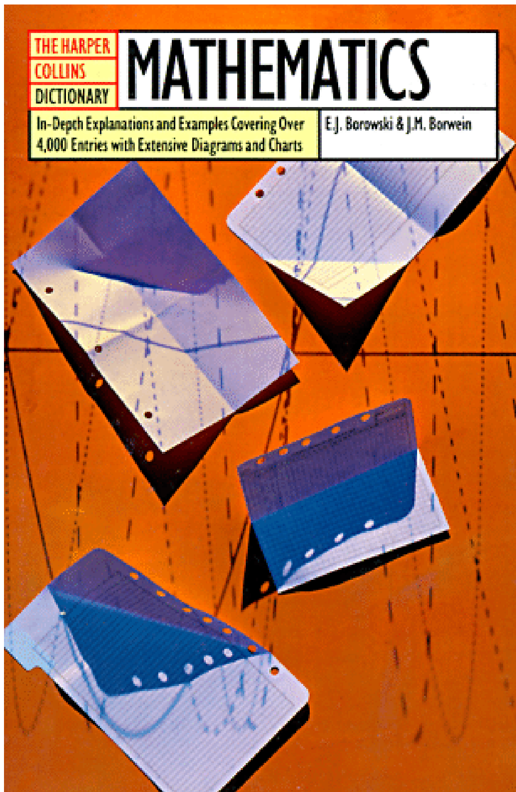
Ron Fitzgerald says math is a language — and most students are illiterate.

The president of Halifax software company **MathResources Inc.** wants to change that. That's why Mr. Fitzgerald and his wife quit their jobs as book editors in Toronto in 1994.

Ten years later, he says his company …

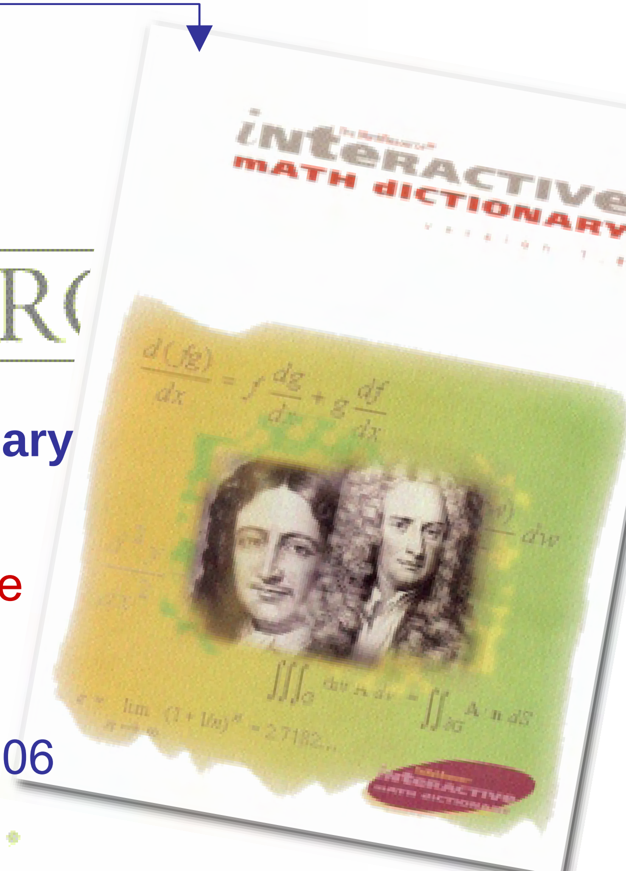
Ronald Fitzgerald, president of MathResources Inc., holds a hand-held computer capable of sh… conventional computers and running the company's mathematics programs.

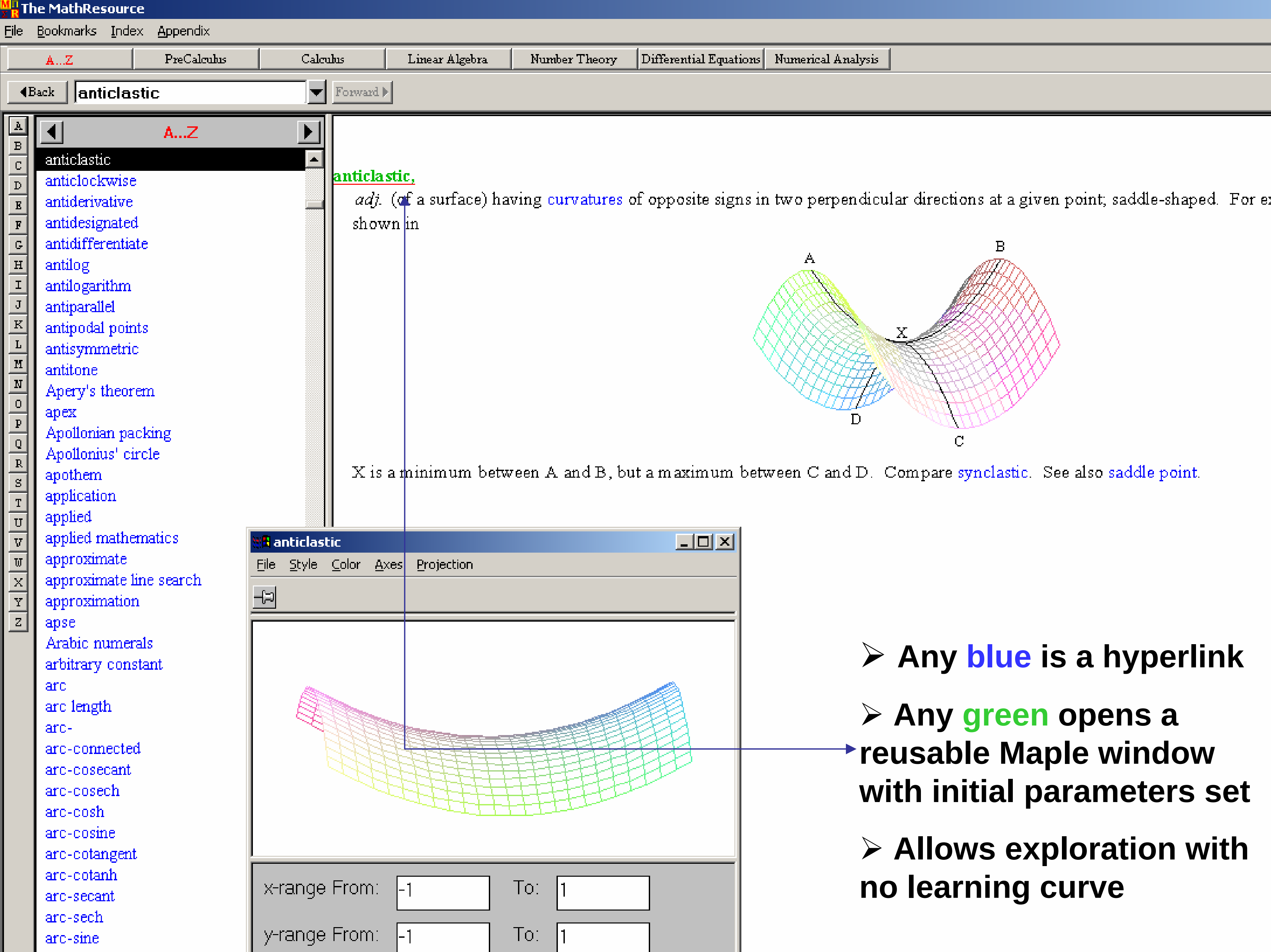
MRI's First Product in Mid-nineties



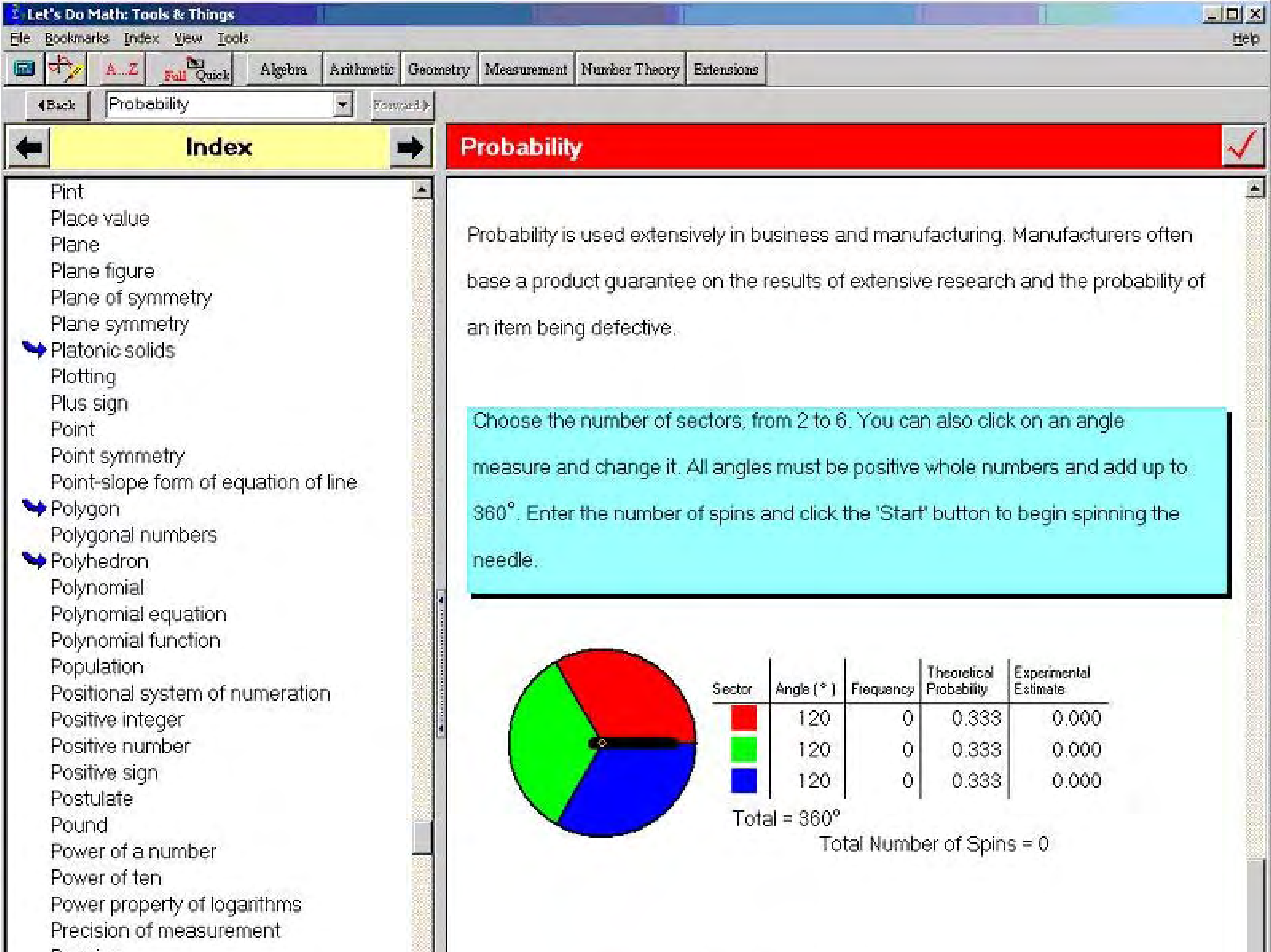
MATHRESOURC

- ▶ Built on **Harper Collins dictionary** - an IP adventure!
- ▶ **Maple** inside the MathResource
- ▶ Data base now in **Maple**
- ▶ **Smithsonian edition** March 2006




The MathResource
File Bookmarks Index Appendix
A...Z PreCalculus Calculus Linear Algebra Number Theory Differential Equations Numerical Analysis
Back anticlastic Forward
A...Z
anticlastic
anticlockwise
antiderivative
antidesignated
antidifferentiate
antilog
antilogarithm
antiparallel
antipodal points
antisymmetric
antitone
Apery's theorem
apex
Apollonian packing
Apollonius' circle
apothem
application
applied
applied mathematics
approximate
approximate line search
approximation
apse
Arabic numerals
arbitrary constant
arc
arc length
arc-
arc-connected
arc-cosecant
arc-cosech
arc-cosh
arc-cosine
arc-cotangent
arc-cotanh
arc-secant
arc-sech
arc-sine
anticlastic,
adj. (of a surface) having curvatures of opposite signs in two perpendicular directions at a given point; saddle-shaped. For ex
shown in
A
B
X
D
C
X is a minimum between A and B, but a maximum between C and D. Compare synclastic. See also saddle point.
anticlastic
File Style Color Axes Projection
x-range From: -1 To: 1
y-range From: -1 To: 1
Any blue is a hyperlink
Any green opens a reusable Maple window with initial parameters set
Allows exploration with no learning curve

A use of appropriate virtual **manipulables**

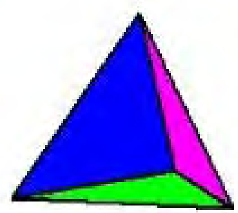

Let's Do Math: Tools & Things
File Bookmarks Index View Tools Help
A..Z
Full Quick
Algebra
Arithmetic
Geometry
Measurement
Number Theory
Extensions
Back
Probability
Forward
Index
Pint
Place value
Plane
Plane figure
Plane of symmetry
Plane symmetry
Platonic solids
Plotting
Plus sign
Point
Point symmetry
Point-slope form of equation of line
Polygon
Polygonal numbers
Polyhedron
Polynomial
Polynomial equation
Polynomial function
Population
Positional system of numeration
Positive integer
Positive number
Positive sign
Postulate
Pound
Power of a number
Power of ten
Power property of logarithms
Precision of measurement
Probability
Probability is used extensively in business and manufacturing. Manufacturers often base a product guarantee on the results of extensive research and the probability of an item being defective.
Choose the number of sectors, from 2 to 6. You can also click on an angle measure and change it. All angles must be positive whole numbers and add up to 360°. Enter the number of spins and click the 'Start' button to begin spinning the needle.
Sector
Angle (°)
Frequency
Theoretical Probability
Experimental Estimate
120 0 0.333 0.000
120 0 0.333 0.000
120 0 0.333 0.000
Total = 360°
Total Number of Spins = 0

Platonic solids

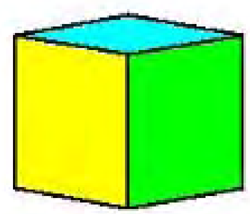
Also called regular polyhedra.

The five special polyhedra where all of the faces of each polyhedron are congruent regular polygons and the same number of polygons meet at each vertex. The ancient Greeks proved that there are only five platonic solids. They are: cube, tetrahedron, octahedron, dodecahedron, and icosahedron.

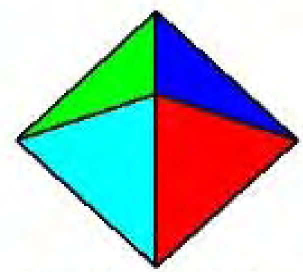
Click on one of the polyhedra below and drag the mouse to rotate it. By right clicking on one of the polyhedra you can change to a wire frame view.



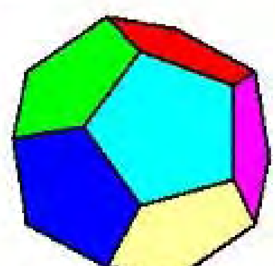
A Regular Tetrahedron



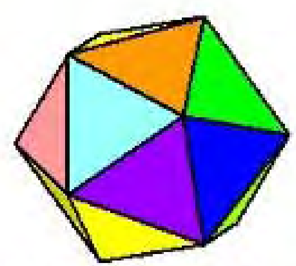
A Cube



A Regular Octahedron



A Regular Dodecahedron



A Regular Icosahedron

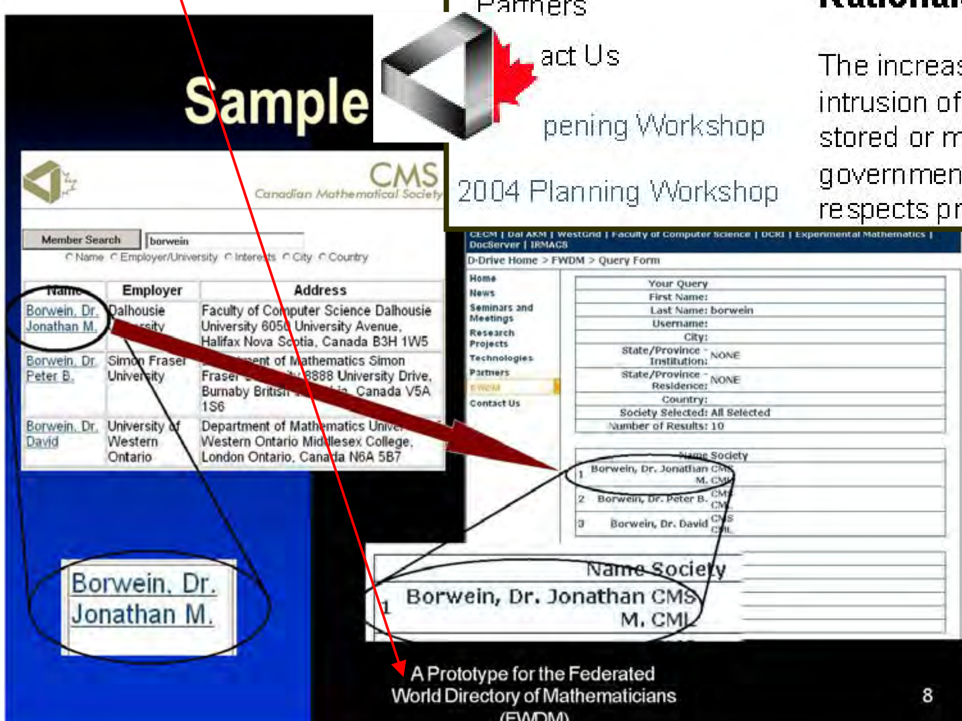
Advanced Knowledge Management

D-Drive

Projects include

- PSL
- FWDM (IMU)
- CiteSeer

(Opened June 9th)



Diverse partners include

- ✓ International Mathematical Union
- ✓ CMS
- ✓ Symantec and IBM

"It says it's sick of doing things like inventories and payrolls, and it wants to make some break-throughs in astrophysics."

REFERENCES


Enigma

J.M. Borwein and D.H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century* A.K. Peters, 2003.

J.M. Borwein, D.H. Bailey and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery,* A.K. Peters, 2004.

D.H. Bailey and J.M Borwein, "Experimental Mathematics: Examples, Methods and Implications," *Notices Amer. Math. Soc.*, **52** No. 5 (2005), 502-514.

“The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.”

• J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.

- ✓ Fractals, Polynomials, Continued Fractions, Pseudospectra

2. High Precision Mathematics.

3. Integer Relation Methods.

- ✓ Chaos, Zeta and the Riemann Hypothesis, HexPi and Normality

4. Inverse Symbolic Computation.

- ✓ A problem of Knuth, $\pi/8$, Extreme Quadrature

---

5. The Future is Here.

- ✓ D-DRIVE: Examples and Issues

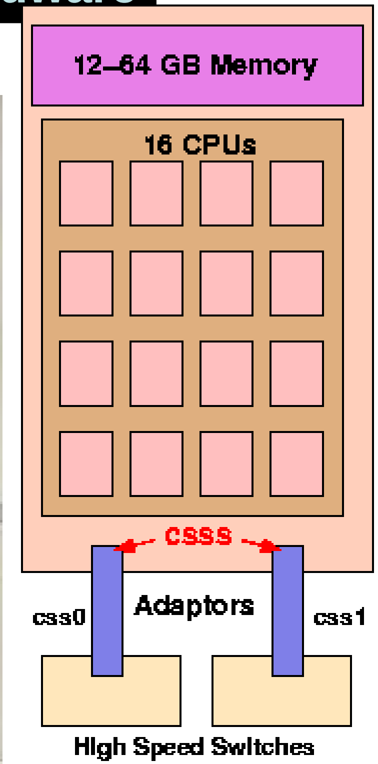
6. Conclusion.

- ✓ Engines of Discovery. The 21st Century Revolution
  - ✓ Long Range Plan for HPC in Canada

This picture is worth 100,000 ENIACs

NERSC's 6000 cpu Seaborg in 2004 (10Tflops/sec)

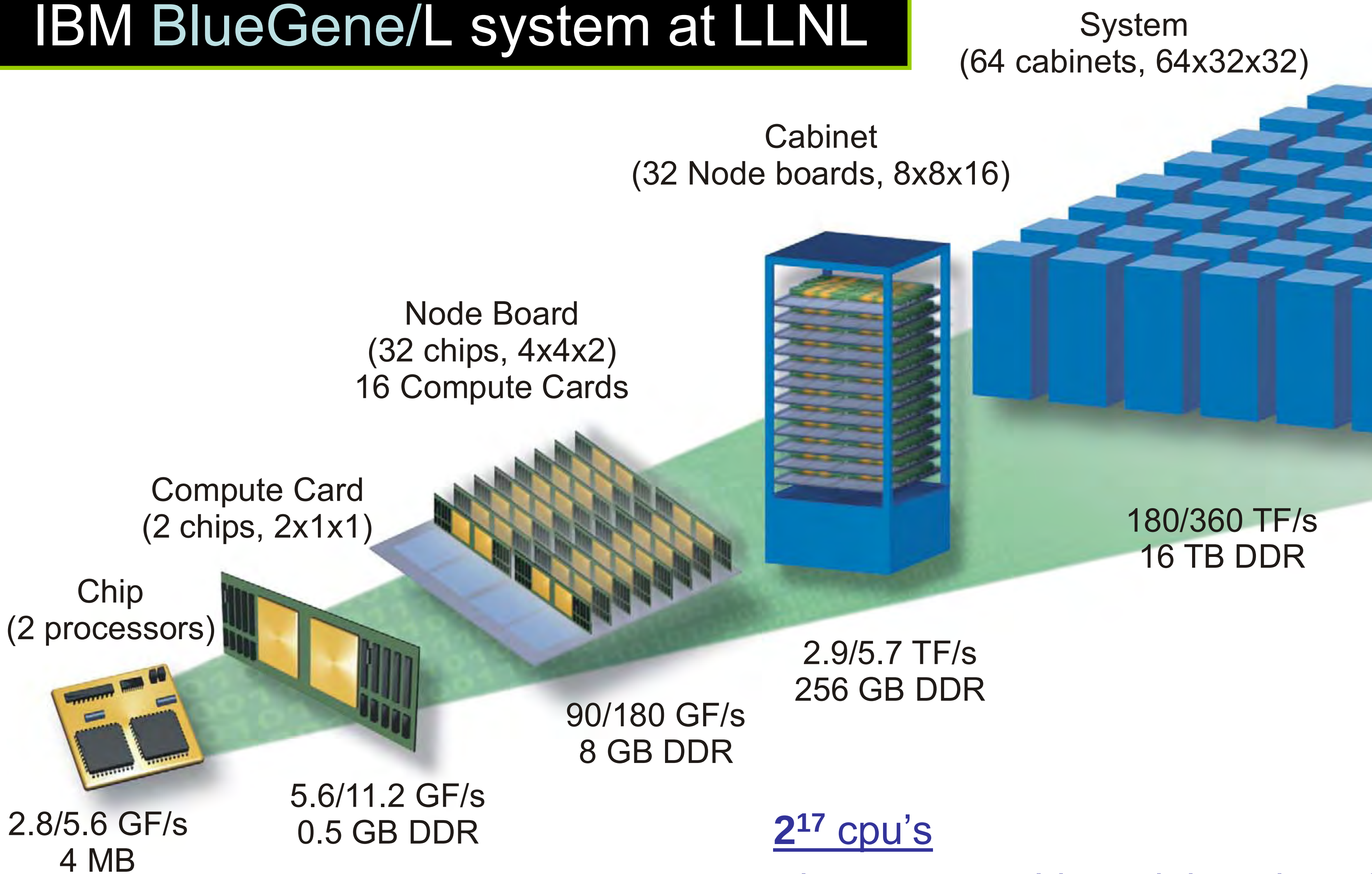
we need new software paradigms for `bigga-scale' hardware



The future

Mathematical Immersive Reality
in Vancouver

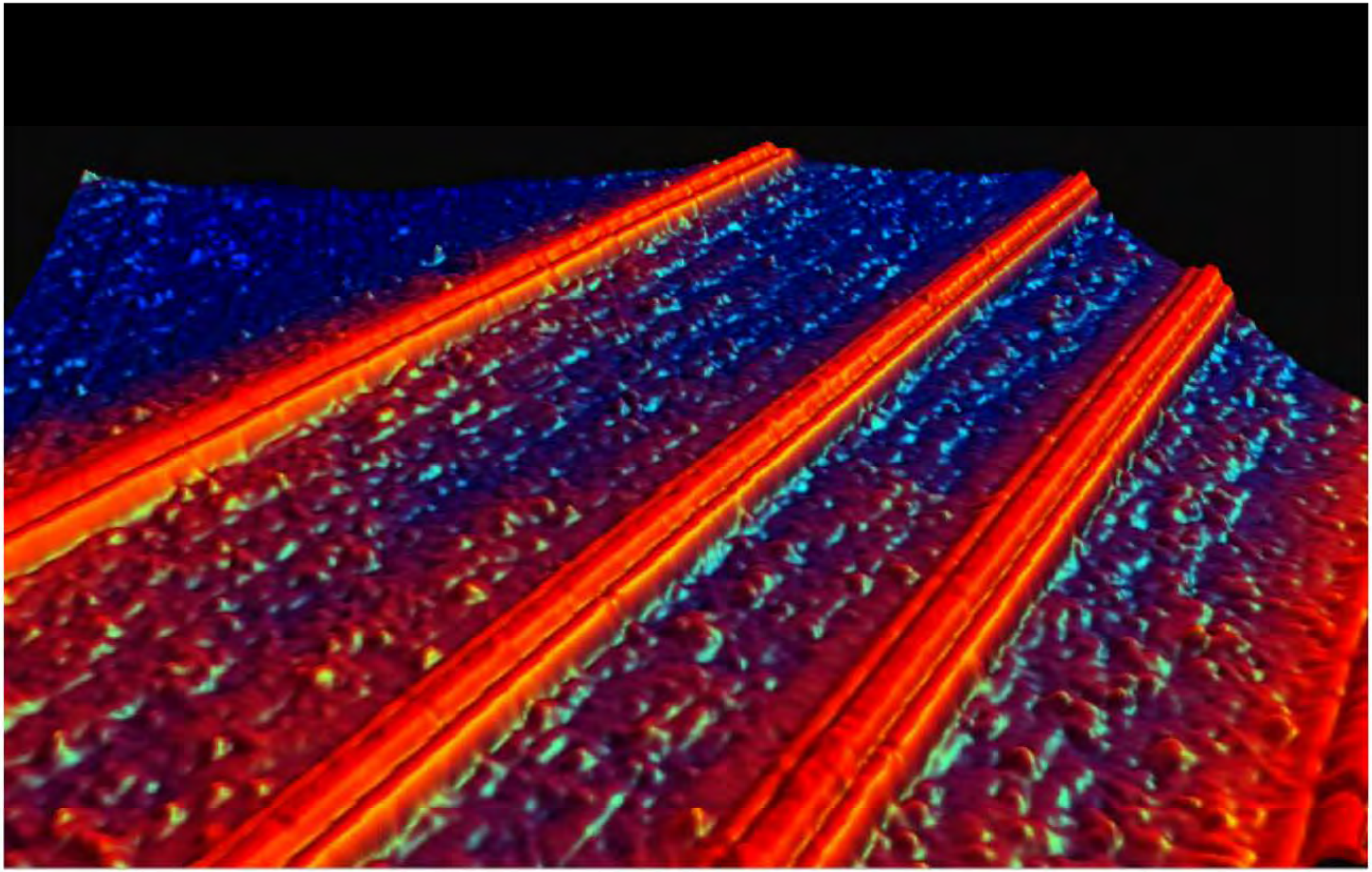
IBM BlueGene/L system at LLNL



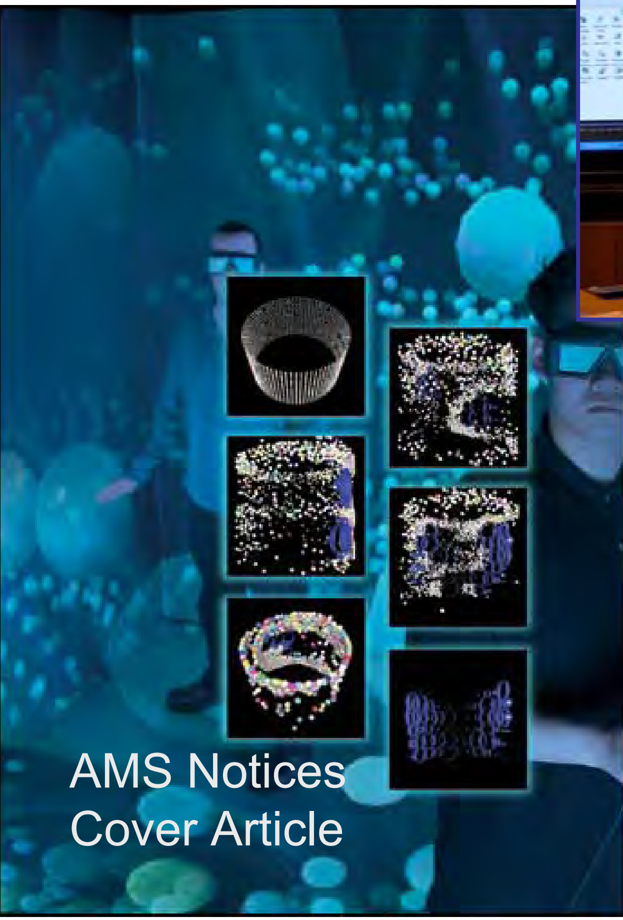
2^{17} cpu's

- has now run Linpack benchmark
- at over **120 Tflop/s**

Self-Assembled Wires 2nm Wide
[P. Kuekes, S. Williams, HP Labs]



5 SMART Touch-sensitive Interwoven Screens



AMS Notices
Cover Article

My intention is to show a variety of mathematical uses of high performance computing and communicating as part of

Experimental Inductive Mathematics

Our web site:

www.experimentalmath.info

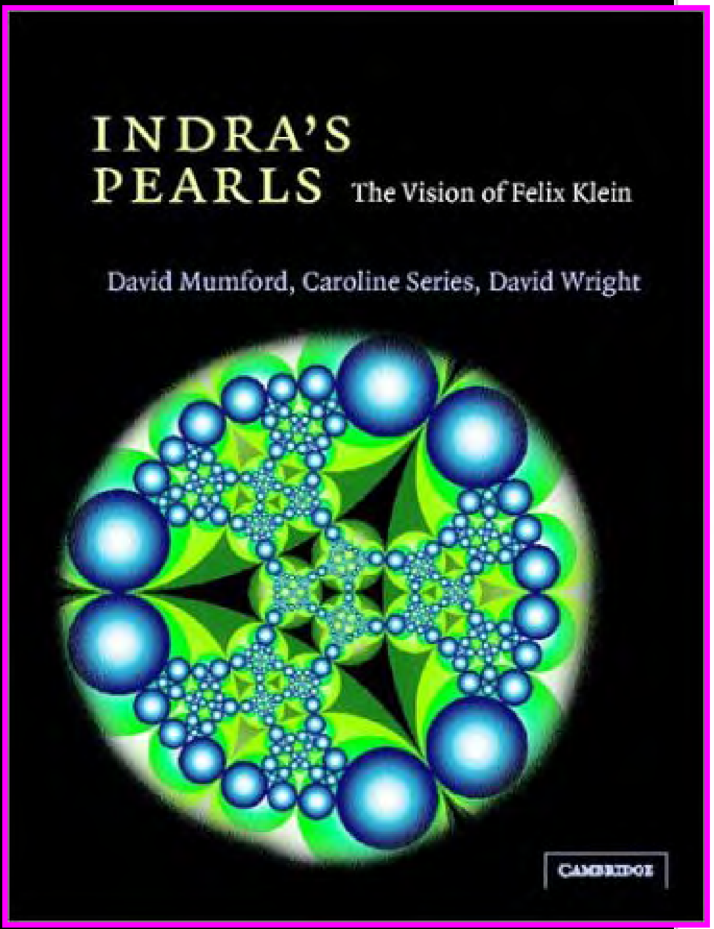
contains all links and references

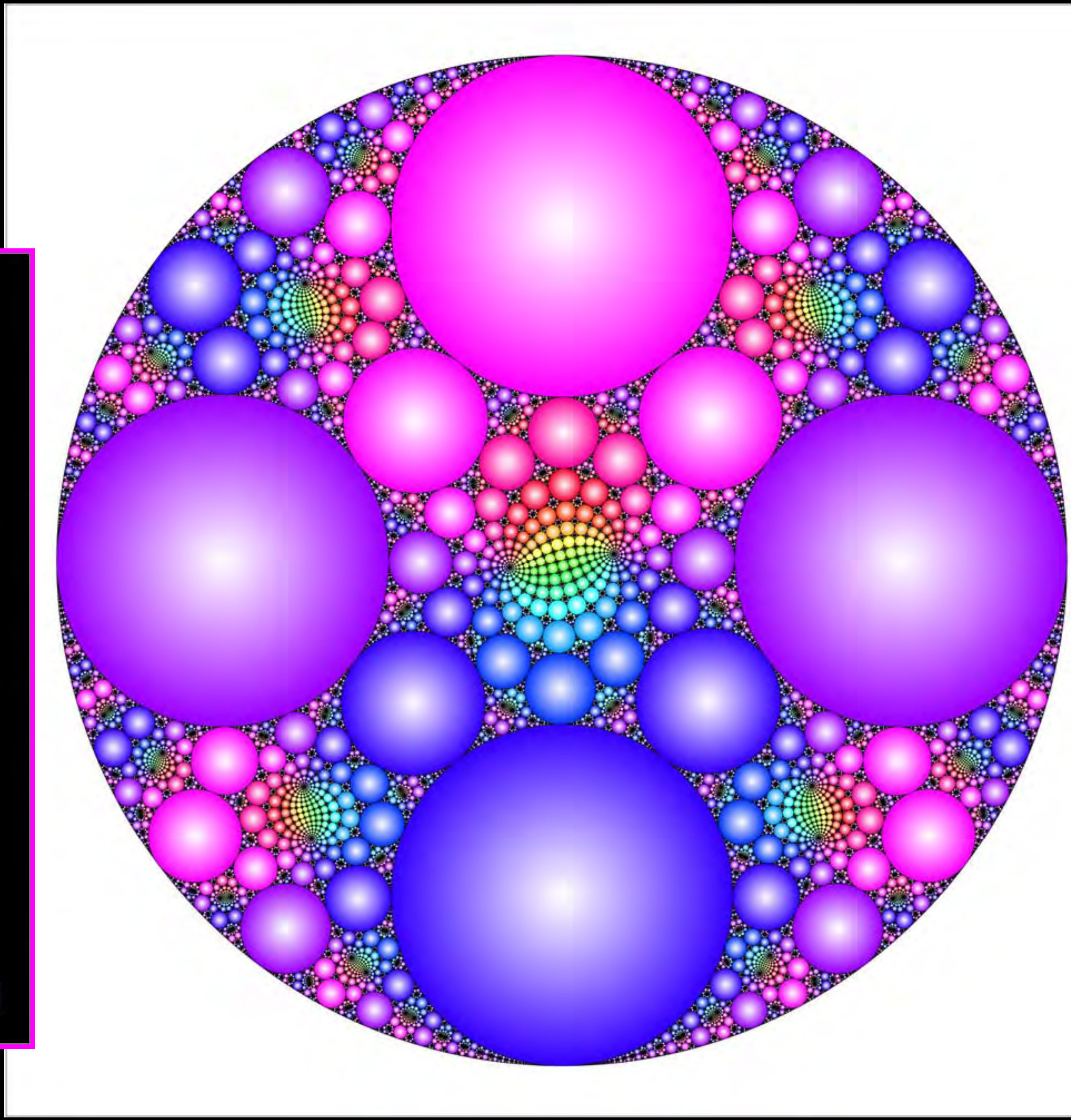
“Elsewhere Kronecker said ``In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." ... I would rather say ``computations" than ``formulas", but my view is essentially the same.”

Harold Edwards, *Essays in Constructive Mathematics*, 2004

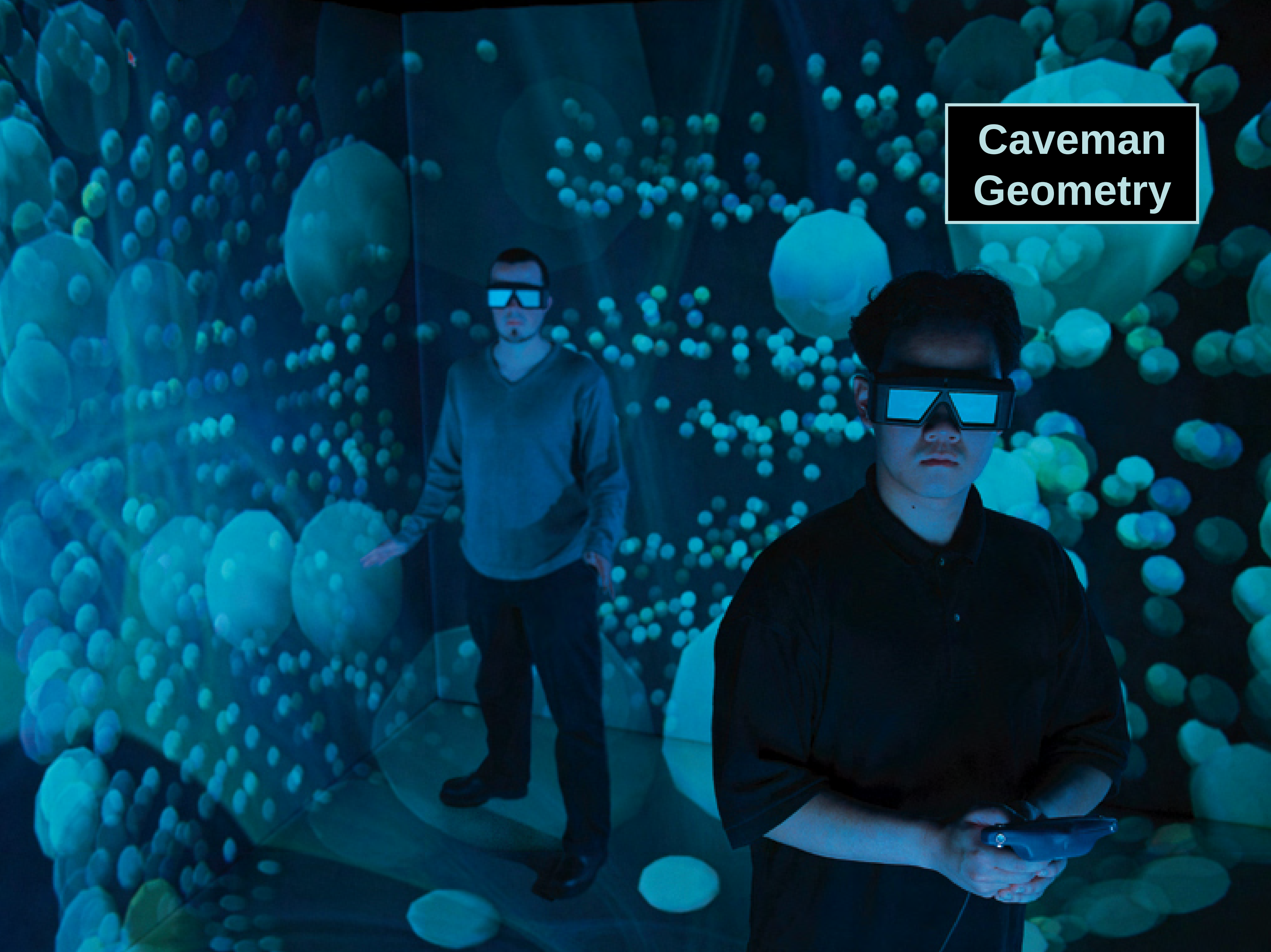
Indra's Pearls

A merging of 19th and 21st Centuries

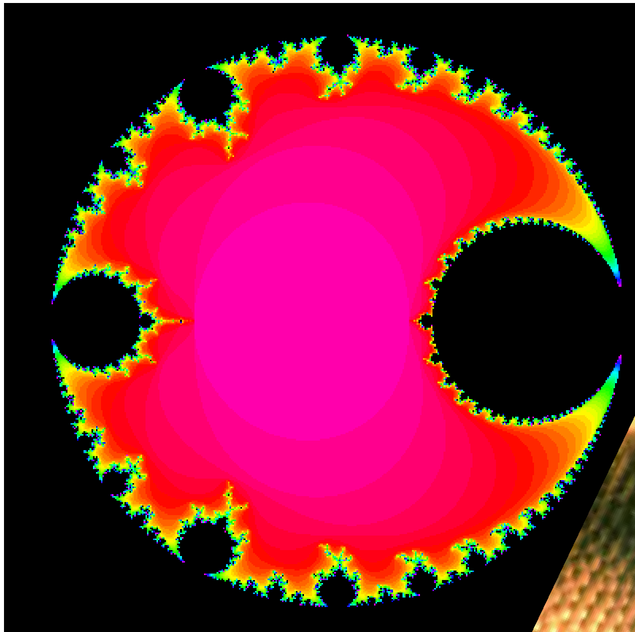




http://klein.math.okstate.edu/IndrasPearls/


Caveman
Geometry

Mathematical Data Mining



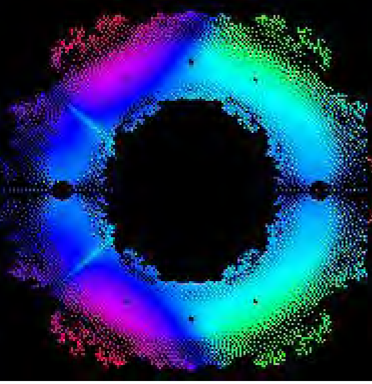
An unusual Mandelbrot parameterization

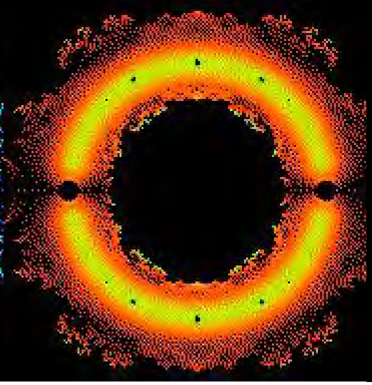
Three visual examples follow

- ✓ Roots of `1/-1’ polynomials
- ✓ Ramanujan’s fraction
- ✓ Sparsity and Pseudospectra

AK Peters, 2004

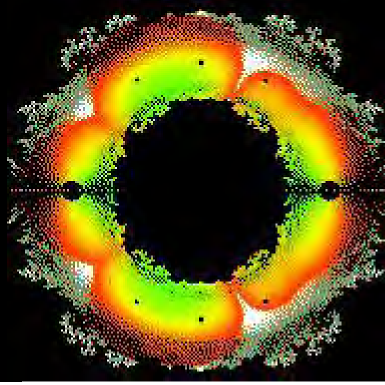
(CD in press)

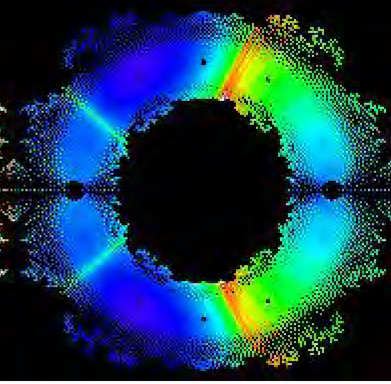




Roots of Zeros

What you draw is what you see (visible patterns in number theory)





Striking fractal patterns formed by plotting complex zeros for all polynomials in powers of x with coefficients 1 and -1 to degree 18

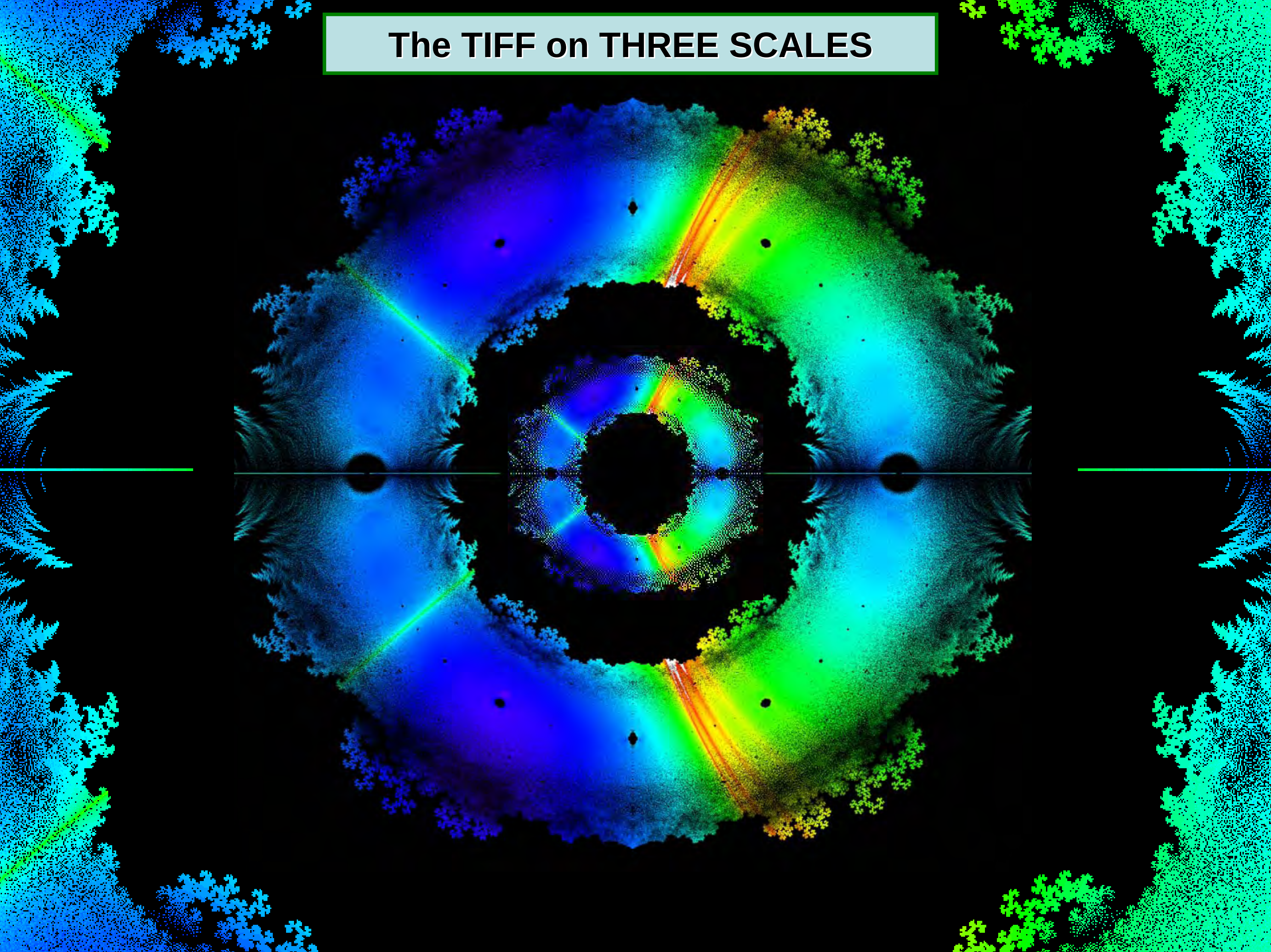
Coloration is by sensitivity of polynomials to slight variation around the values of the zeros. The color scale represents a normalized sensitivity to the range of values; red is insensitive to violet which is strongly sensitive.

- All zeros are pictured (at **3600 dpi**)
- Figure 1b is colored by their local density
- Figure 1d shows sensitivity relative to the $\mathbf{x^9}$ term
- **The white and orange striations are not understood**

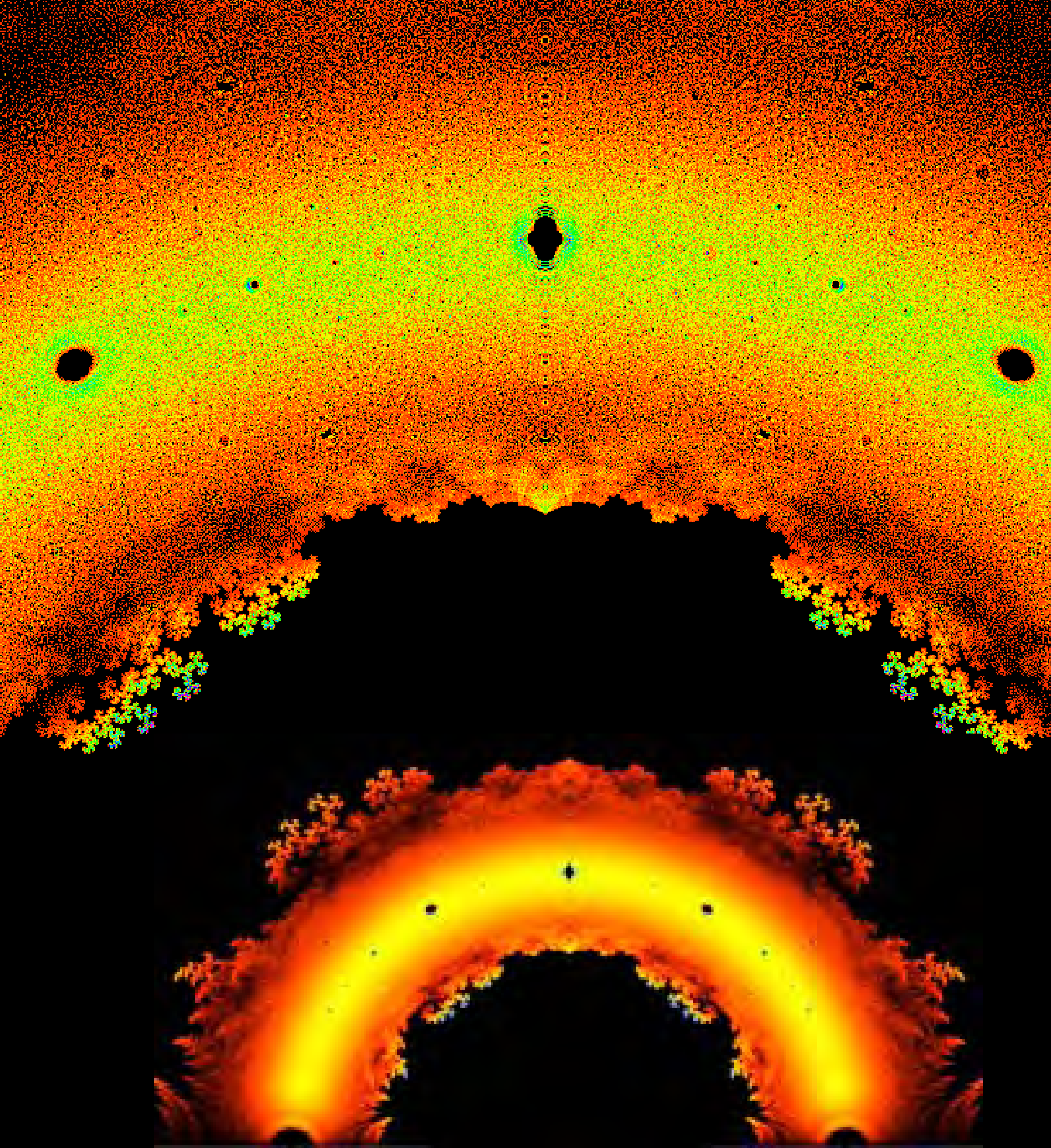
A wide variety of patterns and features become visible, leading researchers to totally unexpected mathematical results

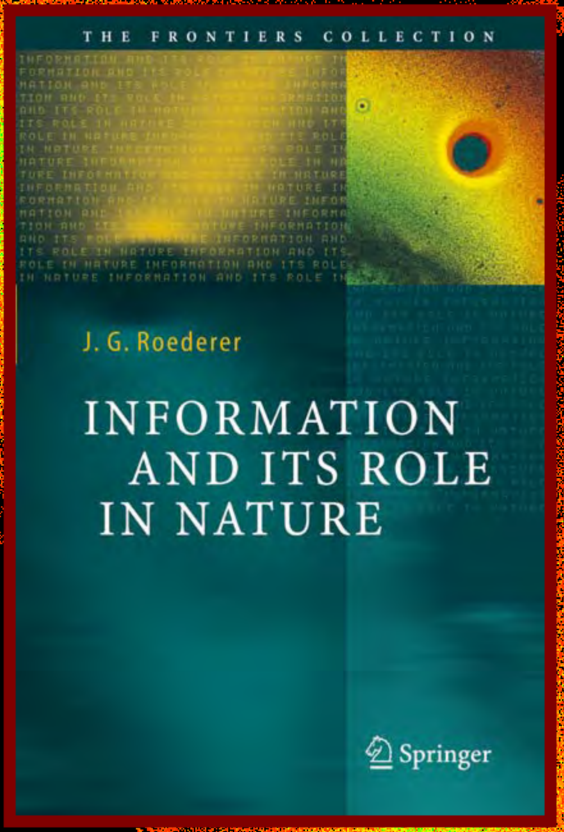
"The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological!" Greg Chaitin, Interview, 2000.

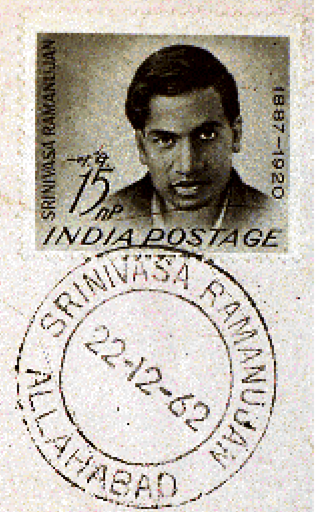
The TIFF on THREE SCALES



… and in the most stable colouring

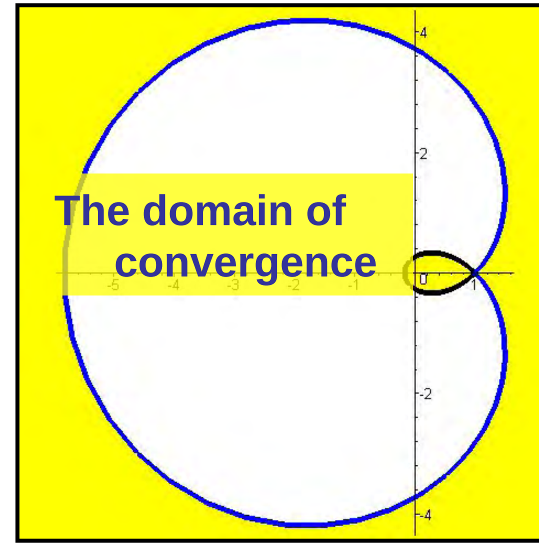






Ramanujan's Arithmetic-Geometric Continued fraction (CF)

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ldots}}}}$$



A cardioid

- For a,b>0 the CF satisfies a lovely symmetrization

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}$$

- Computing directly was too hard even just 4 places of $\mathcal{R}_1(1,1) = \log 2$

We wished to know for which a/b in **C** this all held

✓ The **scatterplot** revealed a precise **cardioid** where r=a/b.

✓ which discovery it remained to prove?

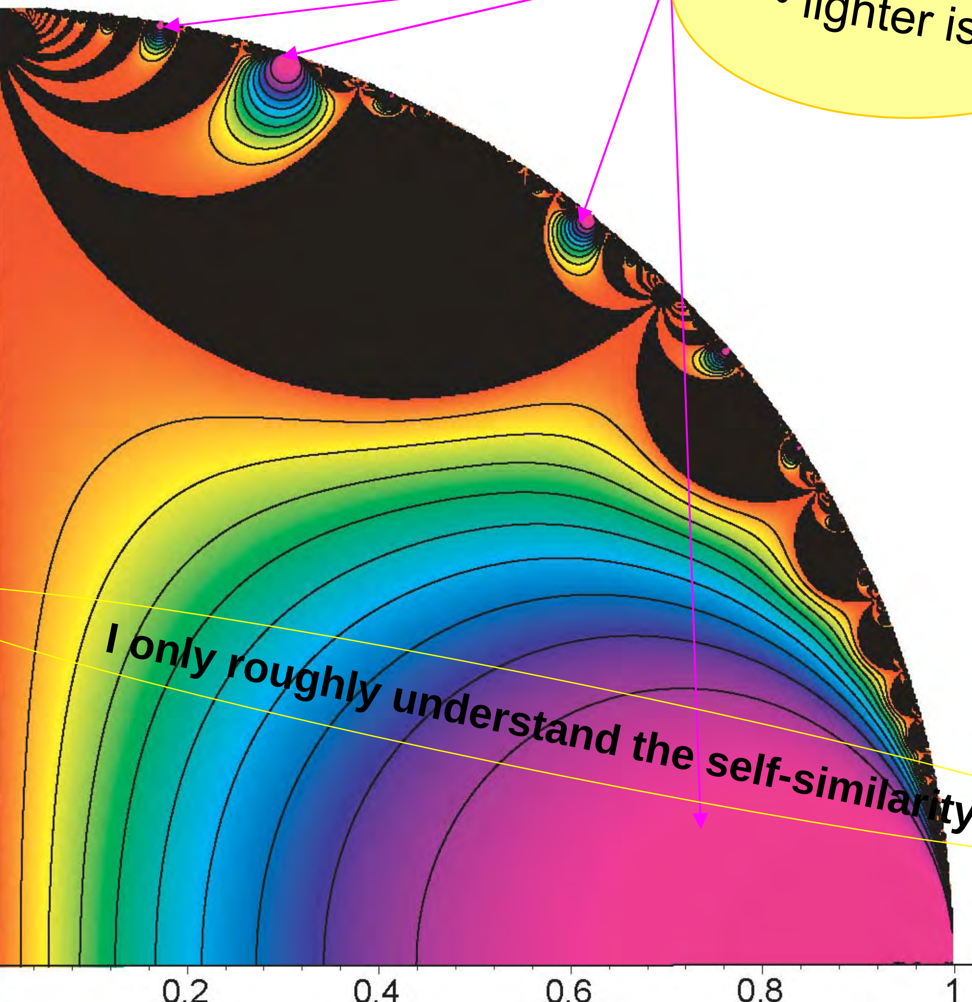
$r^2 - 2r\{2 - \cos(\theta)\} + 1 = 0$

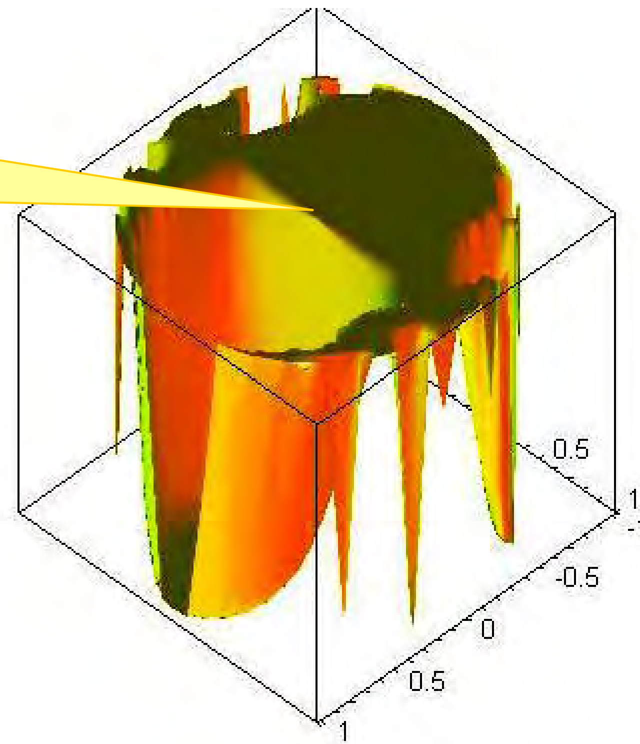
$$\left|\frac{a+b}{2}\right| \geq \sqrt{|ab|}$$

FRACTAL of a Modular Inequality

$$\mathcal{R} = \frac{\left|\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}\right|}{\left|\sum_{n\in\mathbf{Z}} q^{n^2}\right|}$$

plots $\mathcal{R}$ in disk
- black exceeds 1
- lighter is lower



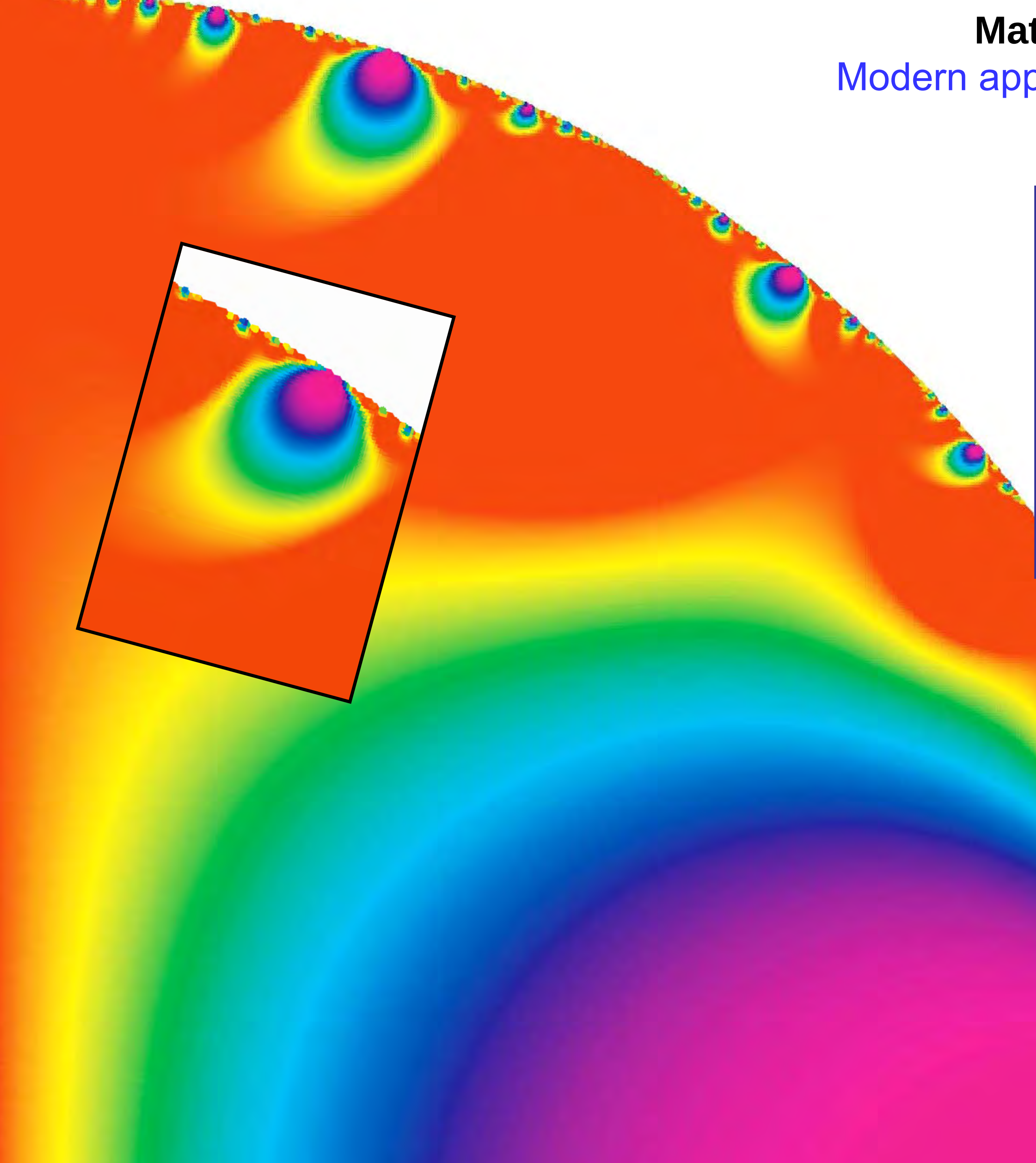


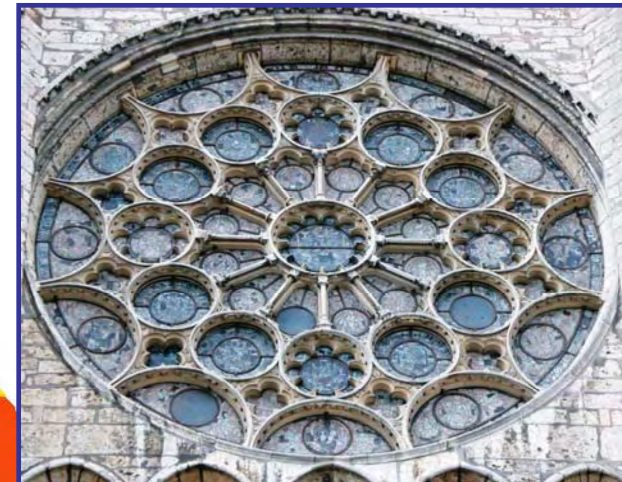
- ✓ related to Ramanujan's continued fraction
- ✓ took several hours to print
- ✓ Crandall/Apple has **parallel rendering** mode

Mathematics and the aesthetic

Modern approaches to an ancient affinity

(CMS-Springer, 2005)

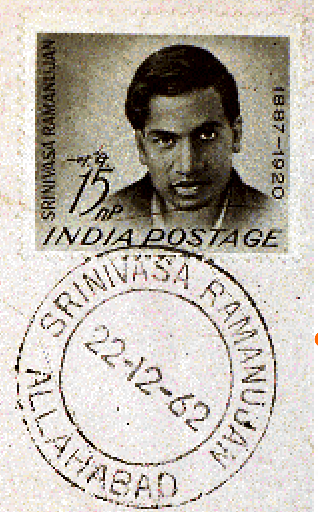




Why should I refuse a good dinner simply because I don't understand the digestive processes involved?

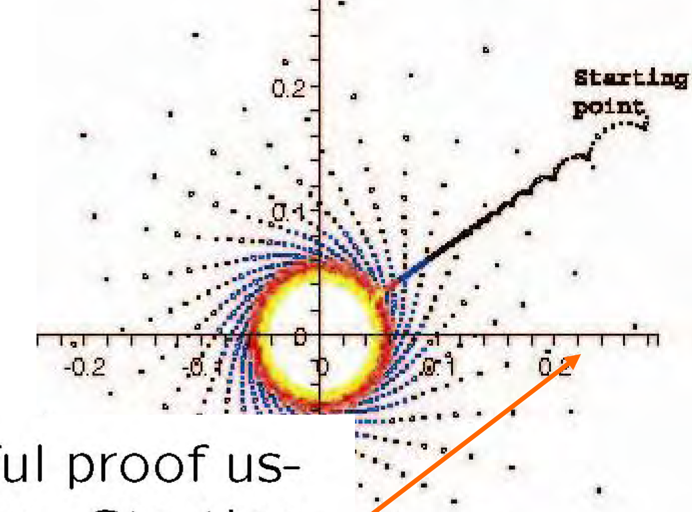
Oliver Heaviside (1850 - 1925)

✓ when criticized for his daring use of operators before they could be justified formally



Ramanujan's Arithmetic-Geometric Continued fraction

1. The Blackbox

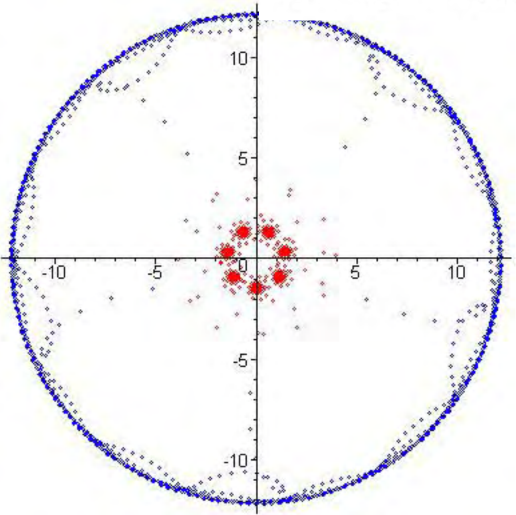


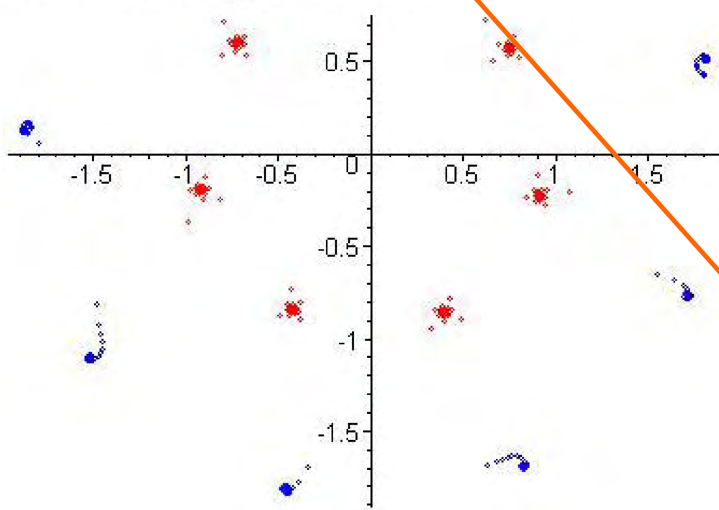
2. Seeing convergence

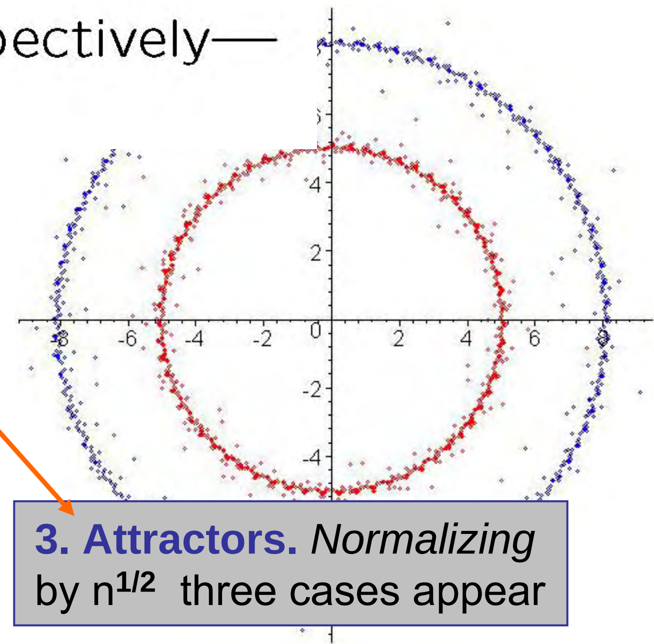
Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1}\left(1-\frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively— or is much more general.*

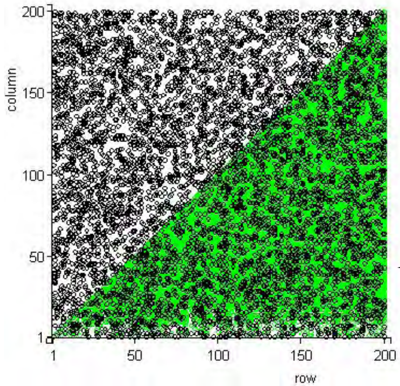




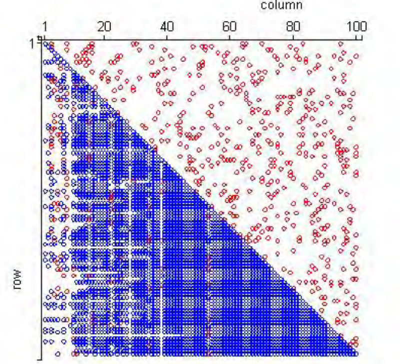


3. Attractors. *Normalizing* by $n^{1/2}$ three cases appear

Pseudospectra or Stabilizing Eigenvalues



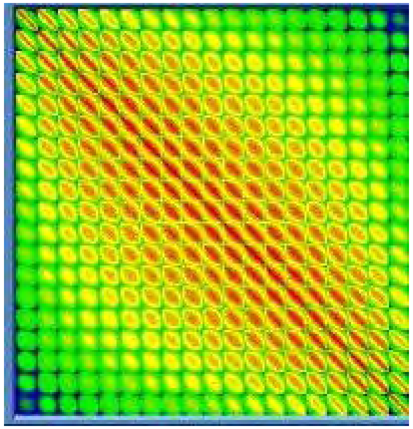
Gaussian elimination of random sparse (10%-15%) matrices



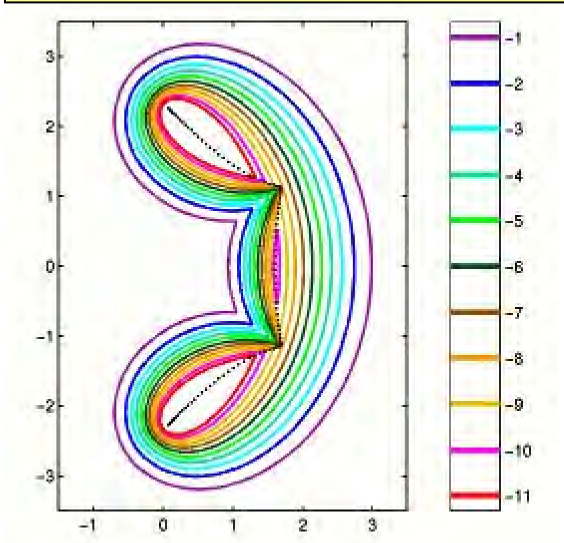
‘Large’ (10^5 to 10^8) Matrices must be seen

- ✓ sparsity and its preservation
- ✓ conditioning and ill-conditioning
- ✓ eigenvalues
- ✓ singular values (helping Google work)

A dense inverse



Pseudospectrum of a banded matrix



The ε-pseudospectrum of A

is: $\sigma_\varepsilon(A) = \{x : \exists\lambda \text{ s.t. } \|Ax - \lambda x\| \le \varepsilon\}$

- ✓ for ε = 0 we recover the eigenvalues
- ✓ full pseudospectrum carries much more information

http://web.comlab.ox.ac.uk/projects/pseudospectra

Generic Code Optimization

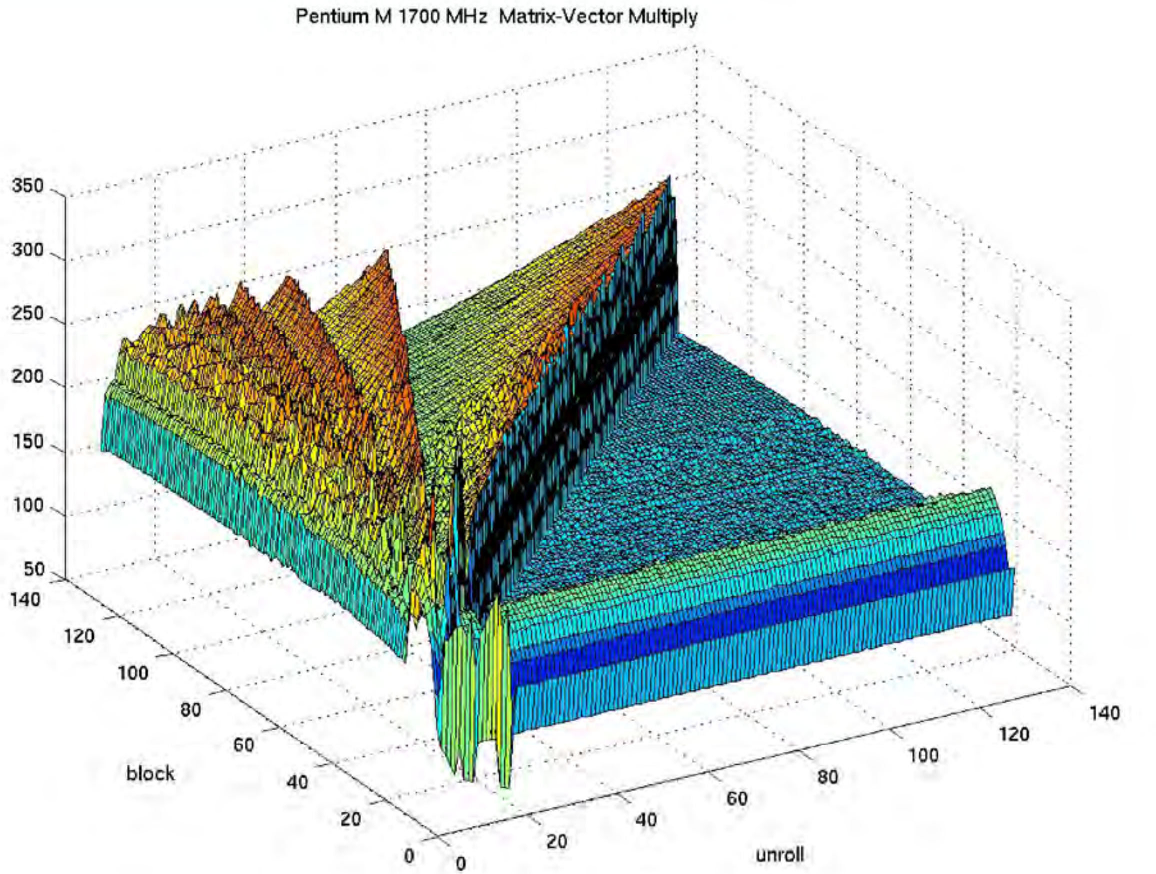
Experimentation with DGEMV (matrix-vector multiply):

128x128=16,384 cases.

Experiment took 30+ hours to run.

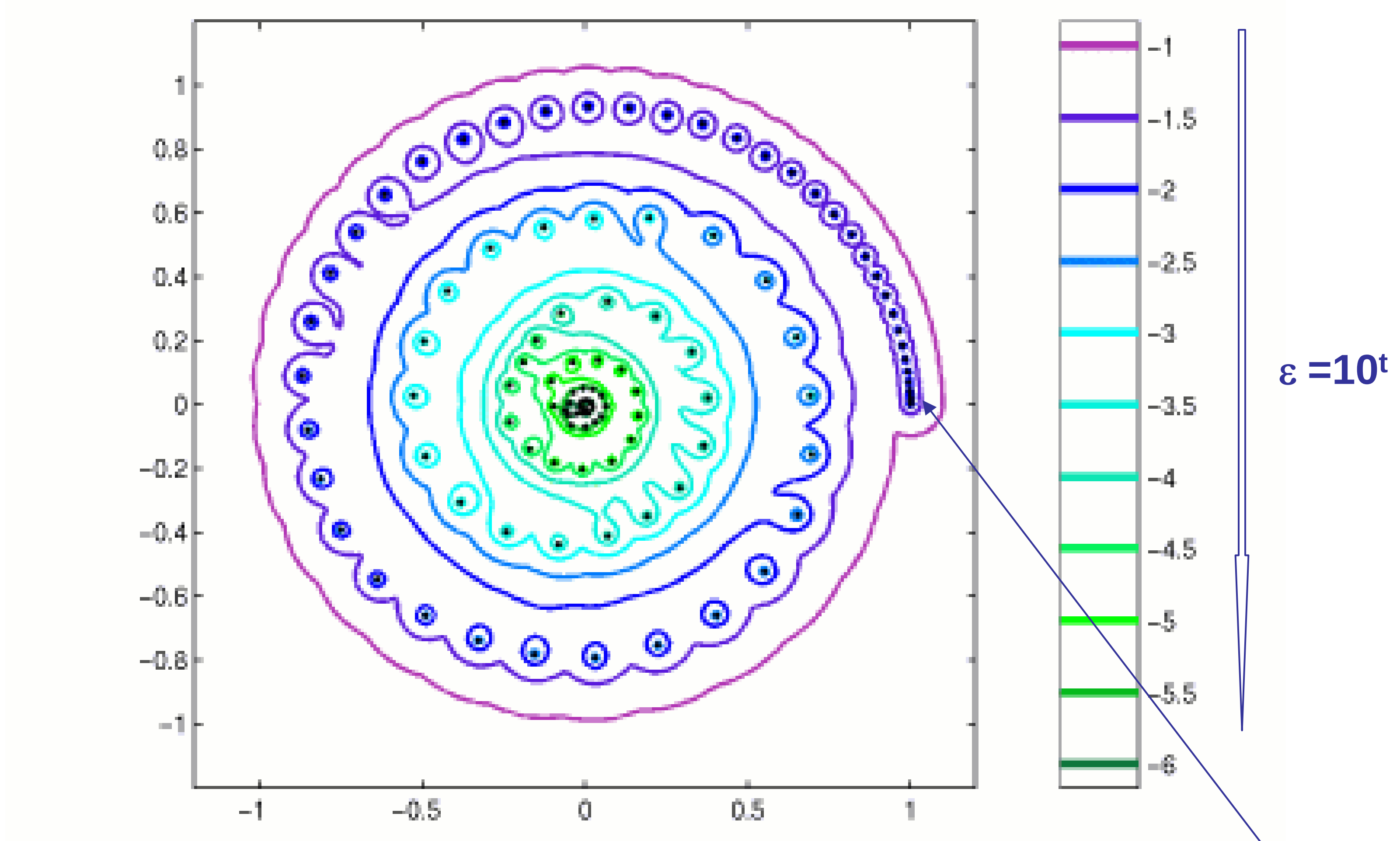
Best performance =
 338 Mflop/s with
  blocking=11
  unrolling=11

Original performance =
 232 Mflop/s



Visual Representation of Automatic Code Parallelization

An Early Use of Pseudospectra (Landau, 1977)



An infinite dimensional integral equation in laser theory

- ✓discretized to a matrix of dimension **600**
- ✓projected onto a well chosen invariant subspace of dimension **109**

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

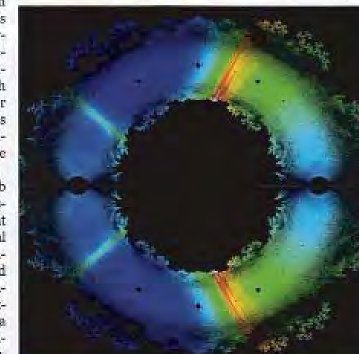
Science News 2004

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results



UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

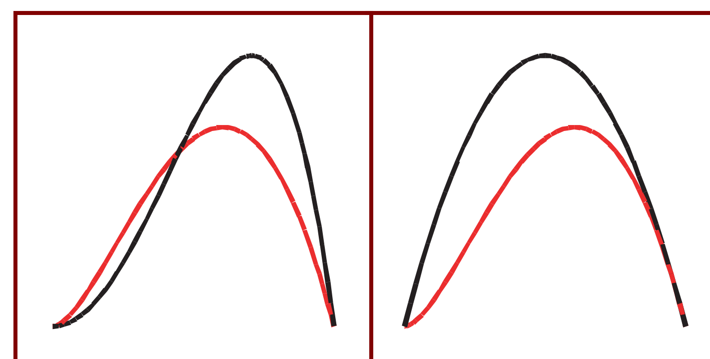
For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

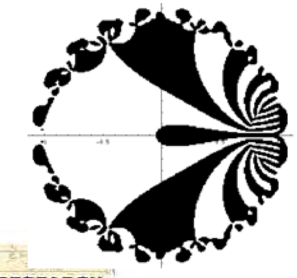
In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-



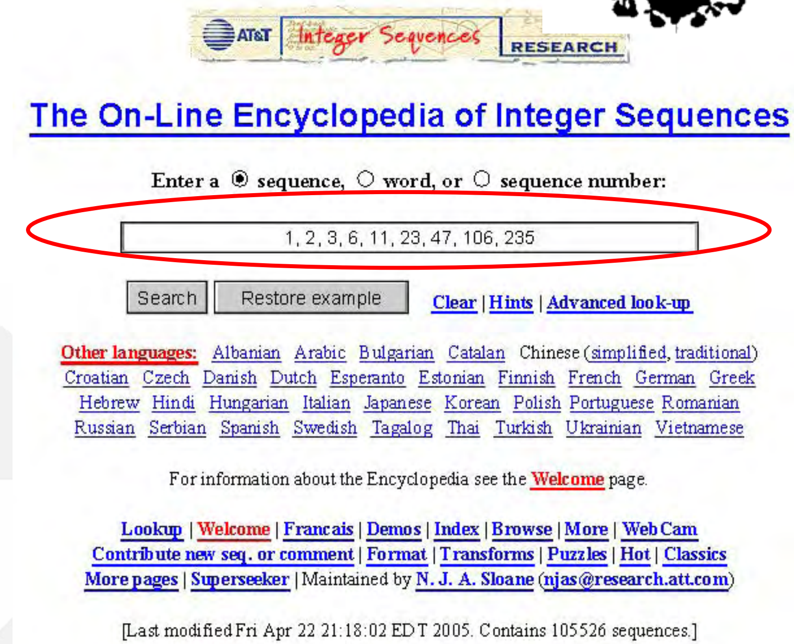
Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

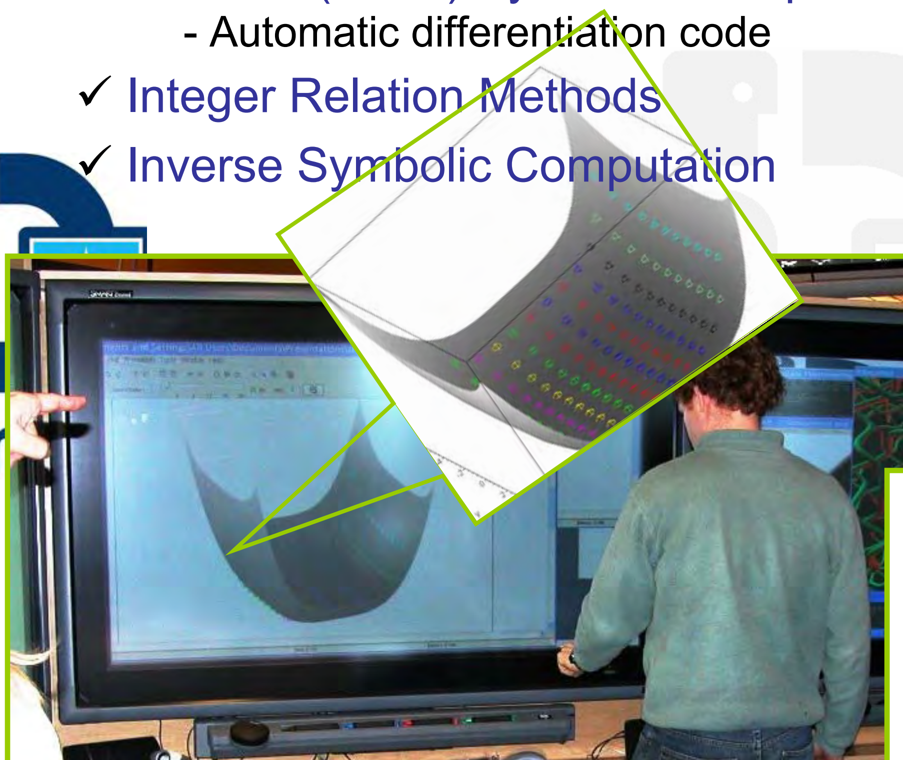
Fast High Precision Numeric Computation (and Quadrature)



❑ <u>Central to my work</u> - with Dave Bailey - meshed with visualization, randomized checks, many web interfaces and

- ✓ Massive (serial) Symbolic Computation
  - Automatic differentiation code
- ✓ Integer Relation Methods
- ✓ Inverse Symbolic Computation





*Parallel derivative free optimization in **Maple***

<u>Other useful tools</u>: Parallel Maple

- Sloane's online sequence database
- Salvy and Zimmerman's generating function package '*gfun*'
- Automatic identity proving: Wilf-Zeilberger method for hypergeometric functions

Greetings from the On-Line Encyclopedia of Integer Sequences!

Matches (up to a limit of 30) found for 1 2 3 6 11 23 47 106 235 :
[It may take a few minutes to search the whole database, depending on how many matches are found (the second and later looku
are faster)]

An Exemplary Database

ID Number: A000055 (Formerly M0791 and N0299)
URL: http://www.research.att.com/projects/OEIS?Anum=A000055
Sequence: 1,1,1,1,2,3,6,11,23,47,106,235,551,1301,3159,7741,19320,
48629,123867,317955,823065,2144505,5623756,14828074,
39299897,104636890,279793450,751065460,2023443032,
5469566585,14830871802,40330829030,109972410221
Name: Number of trees with n unlabeled nodes.
Comments: Also, number of unlabeled 2-gonal 2-trees with n 2-gons.
References F. Bergeron, G. Labelle and P. Leroux, Combinatorial Species and Tree-Like Structures, Camb. 1998, p. 279.
N. L. Biggs et al., Graph Theory 1736-1936, Oxford, 1976, p. 49.
S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 295-316.
D. D. Grant, The stability index of graphs, pp. 29-52 of Combinatorial Mathematics (Proceedings 2nd Australian Conf.), Lect. Notes Math. 403, 1974.
F. Harary, Graph Theory. Addison-Wesley, Reading, MA, 1969, p. 232.
F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, NY, 1973, p. 58 and 244.
D. E. Knuth, Fundamental Algorithms, 3d Ed. 1997, pp. 386-88.
R. C. Read and R. J. Wilson, An Atlas of Graphs, Oxford, 1998.
J. Riordan, An Introduction to Combinatorial Analysis, Wiley, 1958, p. 138.

Integrated real time use

- moderated
- 100,000 entries
- grows daily
- AP book had 5,000

Links: P. J. Cameron, Sequences realized by oligomorphic permutation groups J. Integ. Seqs. Vol
Steven Finch, Otter's Tree Enumeration Constants
E. M. Rains and N. J. A. Sloane, On Cayley's Enumeration of Alkanes (or 4-Valent Trees),.
N. J. A. Sloane, Illustration of initial terms
E. W. Weisstein, Link to a section of The World of Mathematics.
Index entries for sequences related to trees
Index entries for "core" sequences
G. Labelle, C. Lamathe and P. Leroux, Labeled and unlabeled enumeration of k-gonal 2-tree

Formula: G.f.: A(x) = 1 + T(x)-T^2(x)/2+T(x^2)/2, where T(x) = x + x^2 + 2*x^3 + ...

Fast Arithmetic
(Complexity Reduction in Action)

Multiplication

✓ Karatsuba multiplication (200 digits +) or Fast Fourier Transform (FFT)

✓ in ranges from 100 to 1,000,000,000,000 digits

- The other operations

  ✓ via Newton's method $\times, \div, \sqrt{\cdot}$

- Elementary and special functions

  ✓ via Elliptic integrals and Gauss AGM

$$O\left(n^{\log_2(3)}\right)$$

For example:

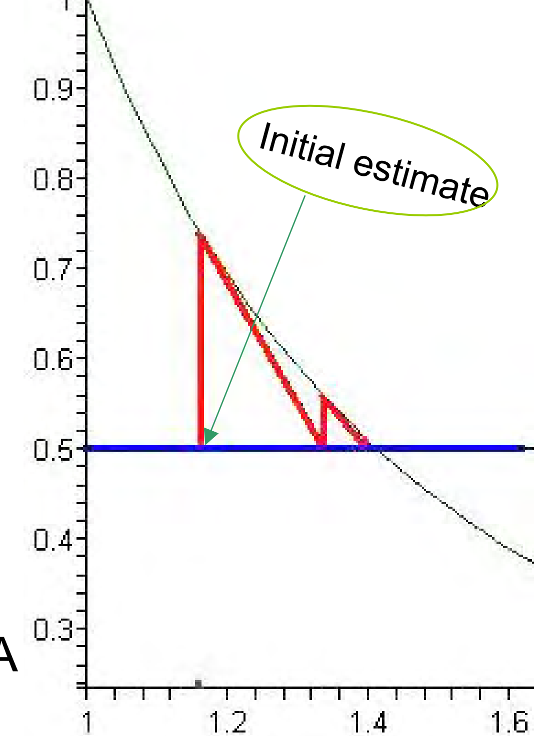
$$\begin{aligned}&\left(a + c \cdot 10^N\right) \times \left(b + d \cdot 10^N\right)\\ =\; & ab + (ad + bc) \cdot 10^N + cd \cdot 10^{2N}\\ =\; & ab + \underbrace{\{(a+c)(b+d) - ab - cd\}}_{\text{three multiplications}} \cdot 10^N + cd \cdot 10^{2N}\end{aligned}$$

Karatsuba replaces one 'times' by many 'plus'

FFT multiplication of multi-billion digit numbers reduces centuries to minutes. Trillions must be done with Karatsuba!

$$x \leftarrow x - \frac{f(x)}{\frac{d}{dx} f(x)}$$

Newton's Method for Elementary Operations and Functions



1. Doubles precision at each step
   - ✓ Newton is **self correcting** and **quadratically convergent**
2. Consequences for work needed:
   - ✓ division = **4 x mult**: $1/x = A$
   - ✓ sqrt = **6 x mult**: $1/x^2 = A$

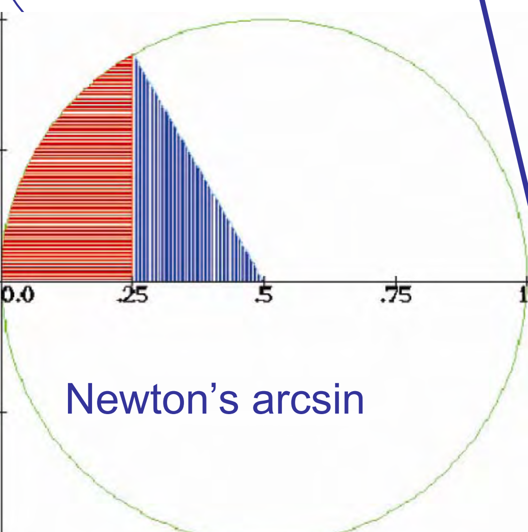
$$x \leftarrow x(2 - xA)$$

$$x \leftarrow 1/2\, x\left(3 - x^2 A\right)$$

Now multiply by A

3. For the **logarithm** we approximate by **elliptic integrals (AGM)** which admit **quadratic transformations**: near zero

$$\frac{d}{dk} K(k) \sim \log\left(\frac{4}{k}\right)$$



4. We use **Newton** to obtain the **complex exponential**

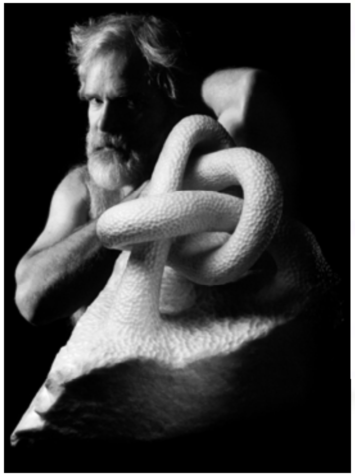
- ✓ hence ***all*** **elementary functions** are fast computable

Integer Relation Methods

The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

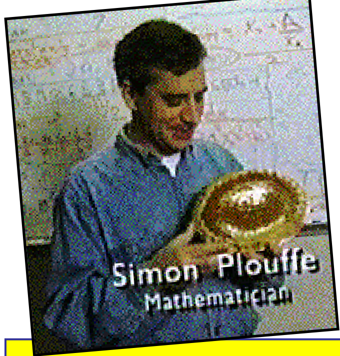
$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$



- At the present time, the PSLQ algorithm of mathematician-sculptor Helaman Ferguson is the best-known integer relation algorithm.
- PSLQ was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least d x n digits, where d is the size (in digits) of the largest of the integers a_k.

An Immediate Use

To see if α is algebraic of degree N, consider $(1,\alpha,\alpha^2,\ldots,\alpha^N)$



PSLQ and Hex Digits of Pi

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{k\, 2^k}$$

My brother made the observation that this log formula allows one to compute binary digits of log 2 *without* knowing the previous ones! (a **BBP formula**)

Bailey, Plouffe and he hunted for such a formula for Pi. Three months later **the computer** - doing **bootstrapped PSLQ** hunts - **returned**:

$$\pi = 4F(1/4, 5/4; 1; -1/4) + 2\arctan(1/2) - \log 5$$

- this reduced to

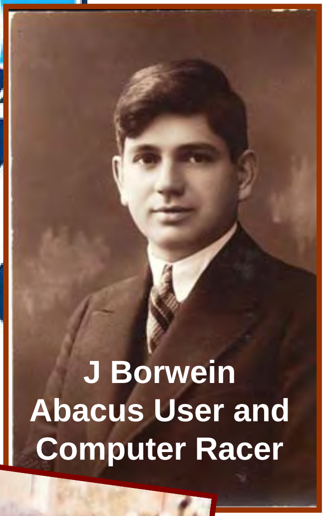
$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$

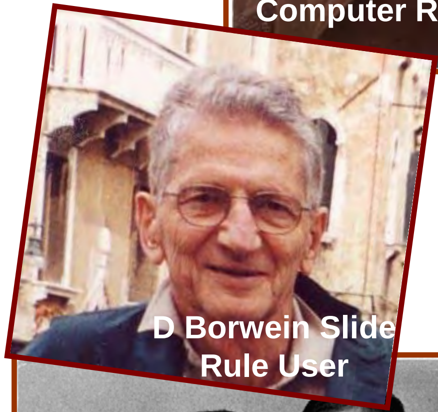
which *Maple*, *Mathematica* and humans can easily prove.

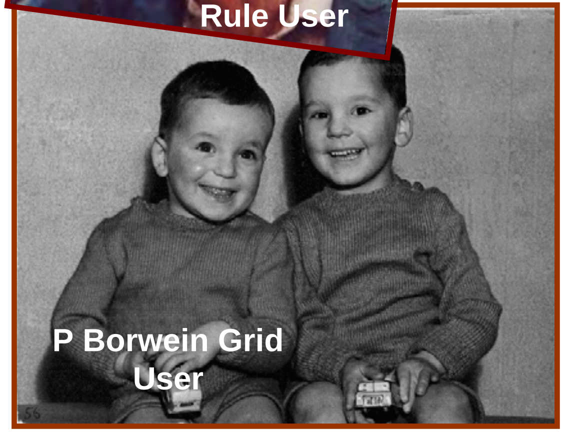
✓ A triumph for "**reverse engineered mathematics**" - algorithm design

✓ No such formula exists base-ten (provably)

The pre-designed Algorithm ran the next day







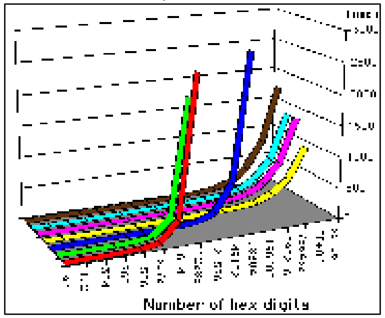
ALGORITHMIC PROPERTIES

(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and



(5) has a computational cost growing only slightly faster than the digit position.

PiHex– A distributed effort to calculate Pi

http://www.cecm.sfu.ca/projects/pihex/

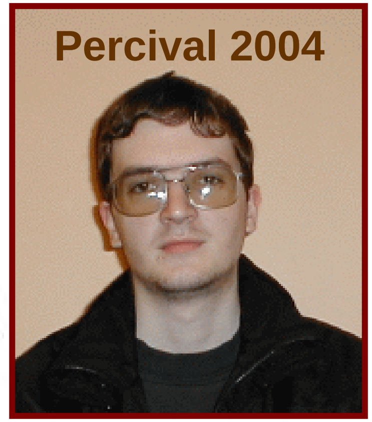
Getting Started   Latest Headlines   www.icbc.ca

- Join PiHex
  - Download
  - Source Code
- About
  - Credits
  - Status
  - Top Producers
- What's New?
- Other Projects
- Who am I?
- Email me!

174962 hits since the counter last reset.

PiHex

A distributed effort to calculate Pi.



The Quadrillionth Bit of Pi is '0'!

The Forty Trillionth Bit of Pi is '0'!

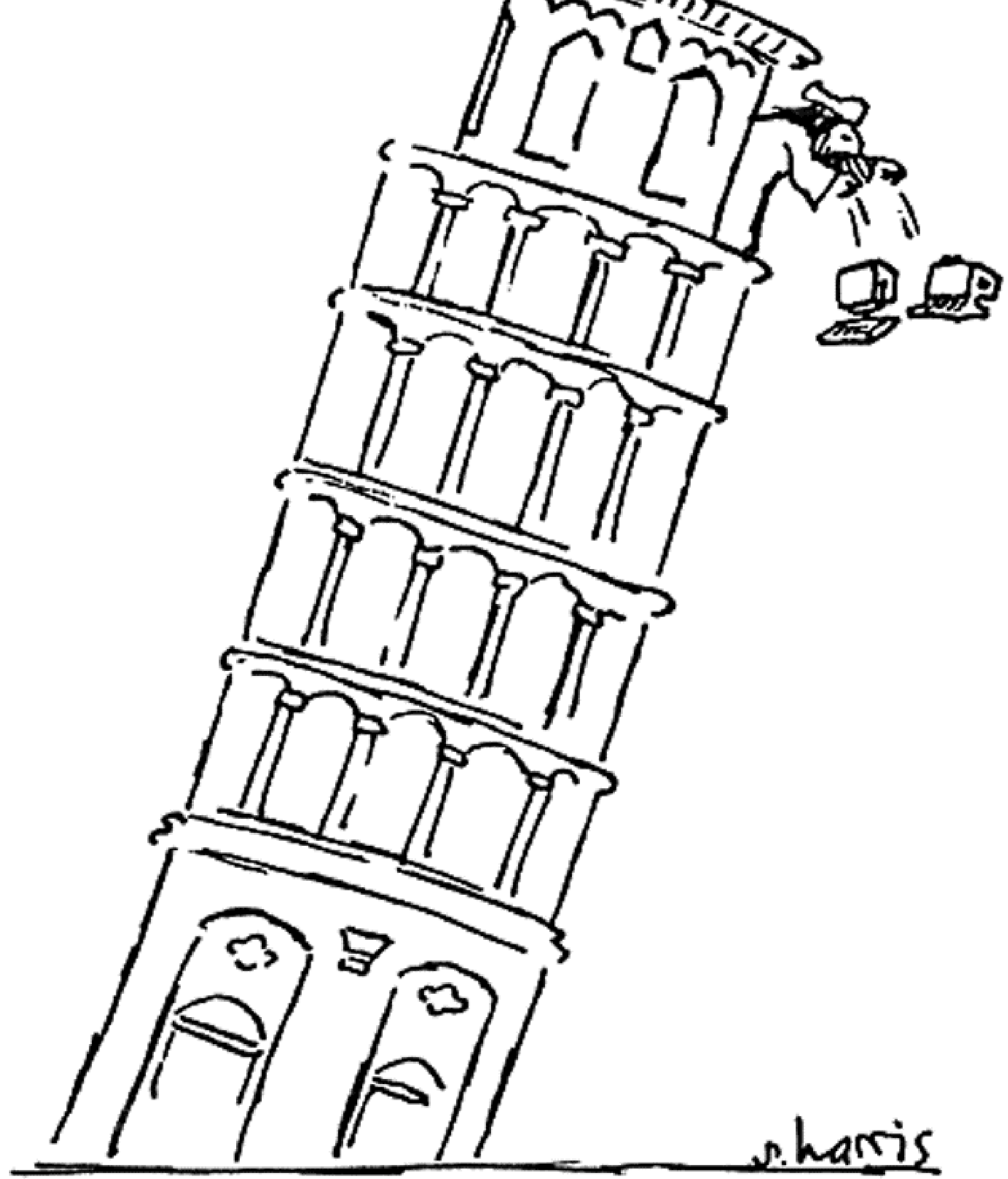
The Five Trillionth Bit of Pi is '0'!

PiHex was a distributed computing project which used idle computing power to set three records for calculating specific bits of Pi. PiHex has now finished.

Undergraduate **Colin Percival's** **Grid Computation** (**PiHex**) rivaled **Finding Nemo**

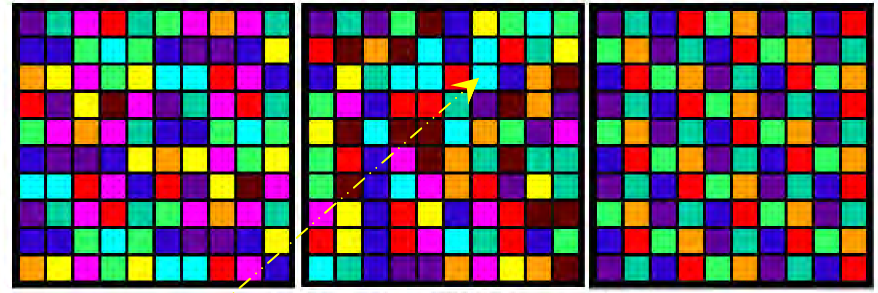
Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

1999 on 1736 PCS in 56 countries using 1.2 million Pentium2 cpu-hours



IF THERE WERE COMPUTERS IN GALILEO'S TIME

An Inverse and a Color Calculator

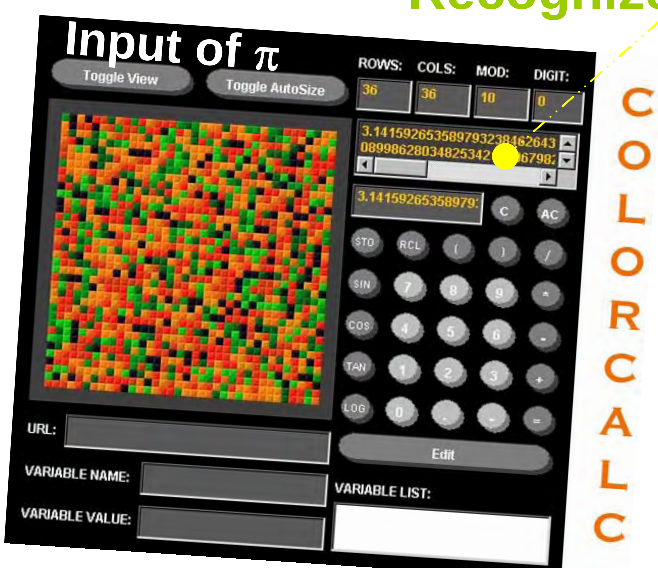


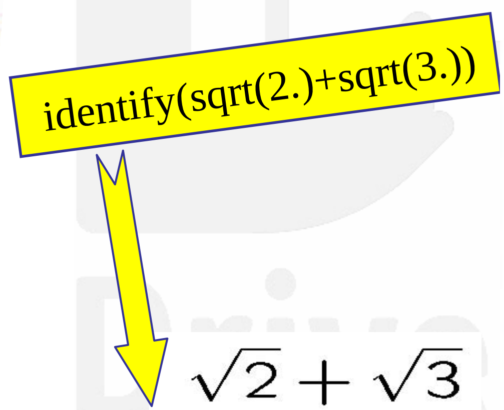
Archimedes: $223/71 < \pi < 22/7$

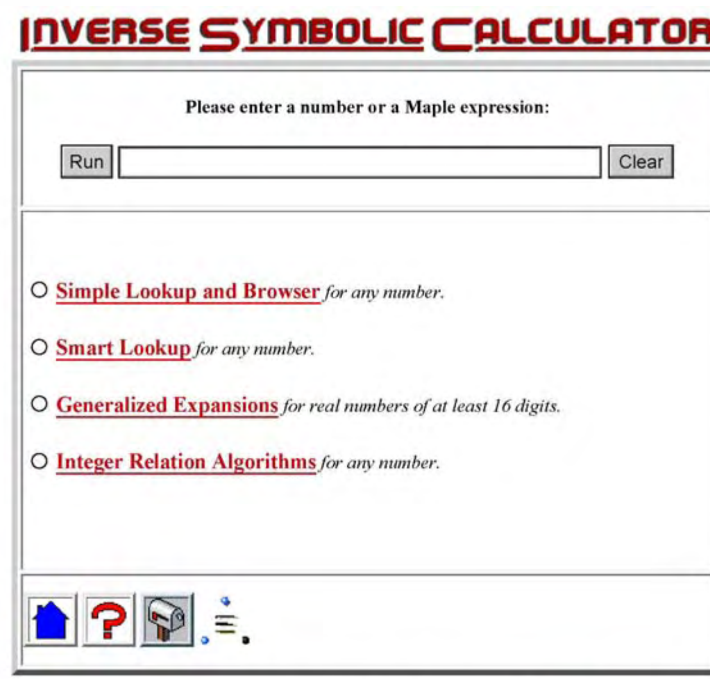
Inverse Symbolic Computation

- “**Inferring symbolic structure from numerical data**”
- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs ***micro-parallelism***
- It faces the “curse of exponentiality”

➢ Implemented as **identify** in Maple and **Recognize** in Mathematica







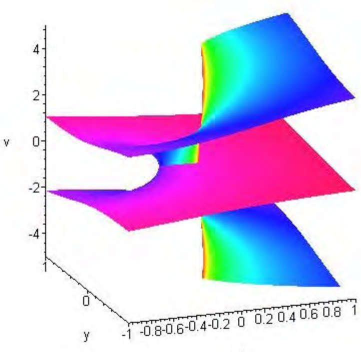
Expressions that are **not** numeric like ln(Pi*sqrt(2)) are evaluated in Maple in symbolic form first, followed by a floating point evaluation followed by a lookup.

Knuth's Problem – we can know the answer first

A guided proof followed on **asking why** Maple could compute the answer so fast.

The answer is **Lambert's W** which solves

W exp(W) = x



W's **Riemann** surface

Donald Knuth* asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty}\left\{\frac{k^k}{k!\,e^k}-\frac{1}{\sqrt{2\,\pi\,k}}\right\}=-0.084069508727655\ldots.$$

- **2000 CE.** It is easy to compute 20 or 200 digits of this sum

† ISC is shown on next slide

∡ The 'smart lookup' facility in the *Inverse Symbolic Calculator*† rapidly returns

$$0.084069508727655 \approx \frac{2}{3}+\frac{\zeta\,(1/2)}{\sqrt{2\,\pi}}.$$

We thus have a prediction which *Maple* 9.5 on a laptop confirms to 100 places in under 6 seconds and to 500 in 40 seconds.

*** ARGUABLY WE ARE DONE**

ENTERING

evalf(Sum(k^k/k!/exp(k)-1/sqrt(2*Pi*k),k=1..infinity),16)

'Simple Lookup' fails;
'Smart Look up' gives:

INVERSE SYMBOLIC CALCULATOR

The ISC is the **Inverse Symbolic Calculator**, a set of programs and specialized tables of mathematical constants dedicated to the identification of real numbers. It also serves as a way to produce identities with functions and real numbers. It is one of the main ongoing projects at the Centre for Experimental and Constructive Mathematics (CECM).

Results of the search:

Maple output:

.08406950872765600

.8406950872765600e-1

Value to be looked up: .8406950872765600e-1 = K

Performing a smart lookup on .8406950872765600e-1:

Function	Result	Precision	Matches
K-2/3	.5825971579390106666666666	16	1

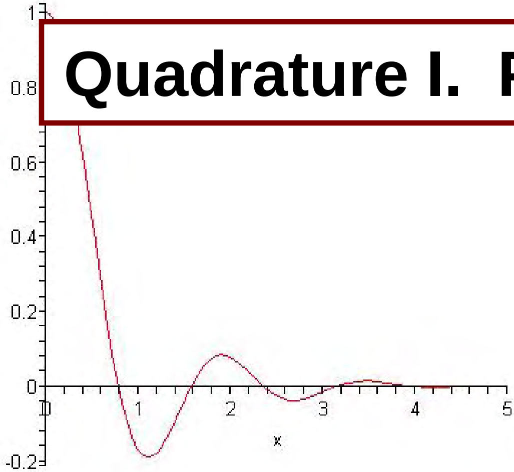
...579390106 was probably generated by one ...s or found in one of the given tables.
Answers are given from shortest to longest description

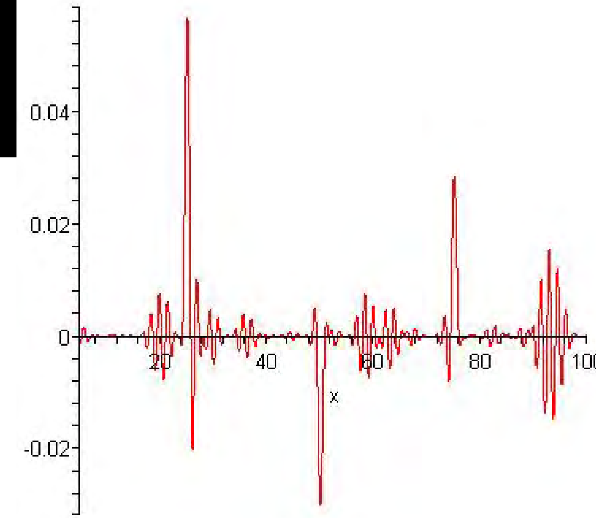
Mixed constants with 5 operations
5825971579390106 = Zeta(1/2)/sr(2)/sr(Pi)

Browse around .5825971579390106.

Quadrature I. Pi/8?

A numerically challenging integral





$$\int_0^\infty \cos(2x)\prod_{n=1}^\infty \cos\left(\frac{x}{n}\right)\,dx \stackrel{?}{=} \frac{\pi}{8}.$$

But $\pi/8$ is

0.392699081698724154807830422909937860524645434

while the integral is

0.392699081698724154807830422909937860524646174

A careful *tanh-sinh quadrature* **proves** this difference after **43 correct digits**

✓ Fourier analysis explains this as happening when a hyperplane meets a hypercube

Before and After

Quadrature II. Hyperbolic Knots

$$\frac{24}{7\sqrt{7}}\int_{\pi/3}^{\pi/2}\log\left|\frac{\tan t+\sqrt{7}}{\tan t-\sqrt{7}}\right|dt \stackrel{?}{=} L_{-7}(2) \qquad (@)$$

where

$$L_{-7}(s) = \sum_{n=0}^{\infty}\left[\frac{1}{(7n+1)^s}+\frac{1}{(7n+2)^s}-\frac{1}{(7n+3)^s}+\frac{1}{(7n+4)^s}-\frac{1}{(7n+5)^s}-\frac{1}{(7n+6)^s}\right].$$

“Identity” (@) has been verified to 20,000 places. I have *no idea* of how to prove it.

✓ Easiest of 998 empirical results linking physics/topology (LHS) to number theory (RHS). [JMB-Broadhurst]

We have certain knowledge without proof

Extreme Quadrature …
20,000 Digits on 1024 CPUs

Ш. The integral was split at the nasty interior singularity
Ш. The sum was `easy'.
Ш. All fast arithmetic & function evaluation ideas used

Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

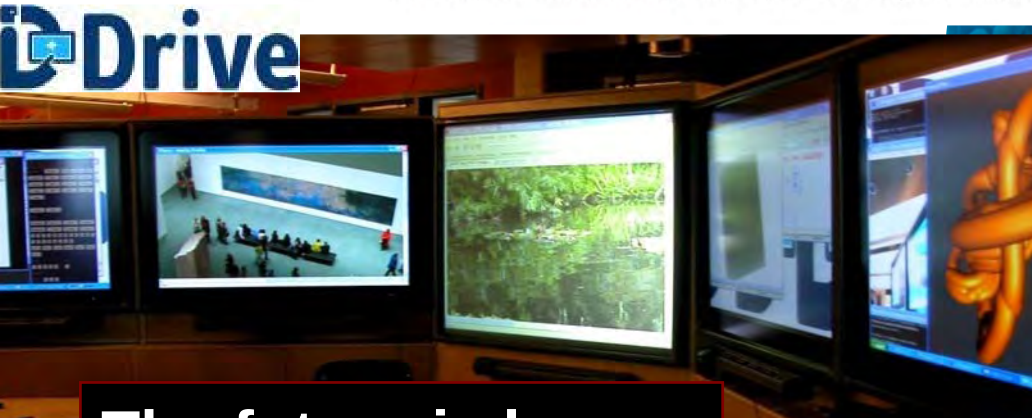
Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

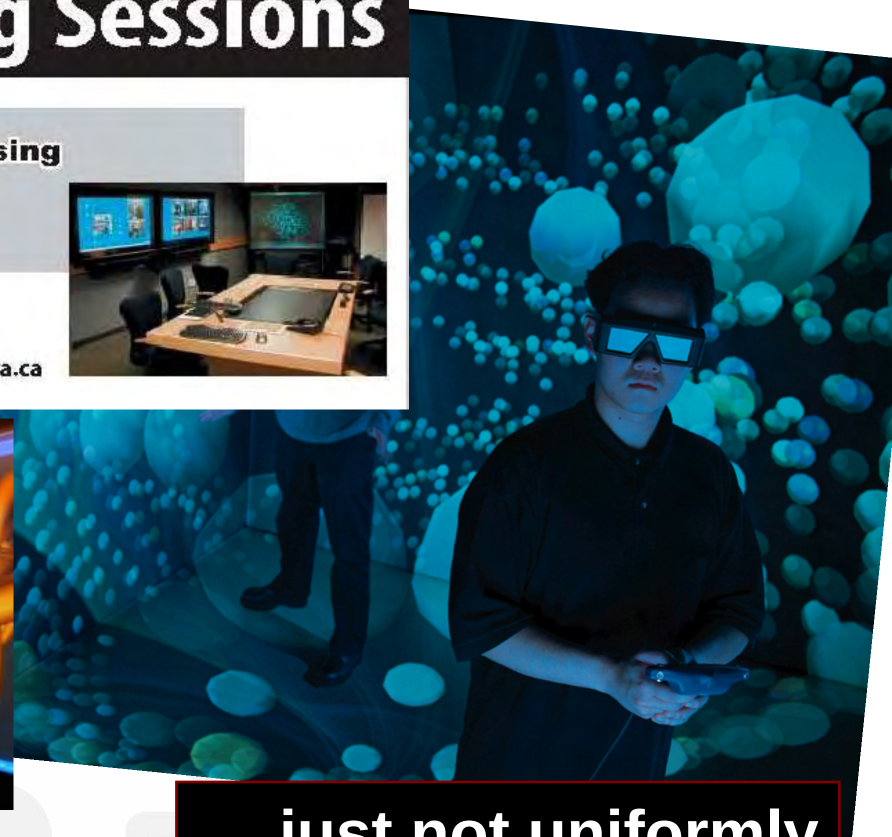
Expected and unexpected scientific spinoffs

- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **1995-** Math Resources (next overhead)


THEORIES
PROOFS

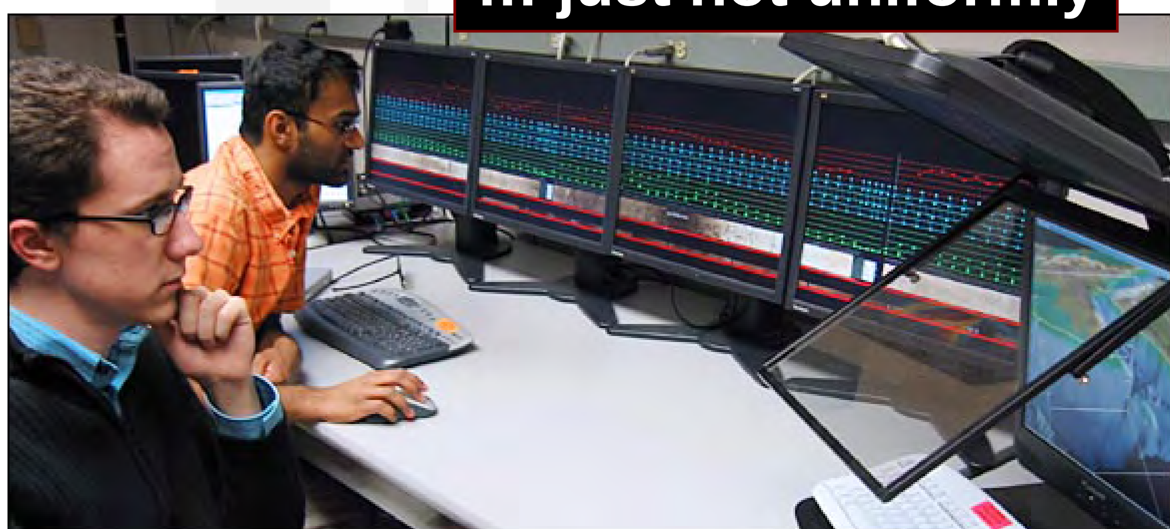
WestGrid
www.westgrid.ca





The future is here…

… just not uniformly



Remote Visualization via **Access Grid**

- The touch sensitive interactive **D-DRIVE**
- An Immersive '**Cave**' Polyhedra
- and the 3D **GeoWall**