Experimental Mathematics:

3×Lists and Computational Challenges

Jonathan M. Borwein, FRSC

Research Chair in IT
Dalhousie University

DALHOUSIE UNIVERSITY
Inspiring Minds

Halifax, Nova Scotia, Canada

Based on 2005 Clifford Lecture III

Tulane, March 31–April 2, 2005

Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution. ··· Besides it is an error to believe that rigor in the proof is the enemy of simplicity. (David Hilbert, 1900)

Drive www.cs.dal.ca/ddrive Drive

Ten Computational Challenge Problems

This lecture will look at 'lists and challenges' and discuss three lists, two of which are sets of *Ten Computational Mathematics Problems* including

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right) dx \stackrel{?}{=} \frac{\pi}{8}.$$

This problem set was stimulated by Nick Trefethen's recent more numerical *SIAM 100 Digit, 100 Dollar Challenge*.*

- We start with a general description of the Digit Challenge† and finish with an examination of some of its components and of some in our own list.

*The talk is based on an article to appear in the May 2005 *Notices of the AMS*, and related resources such as `www.cs.dal.ca/~jborwein/digits.pdf`.

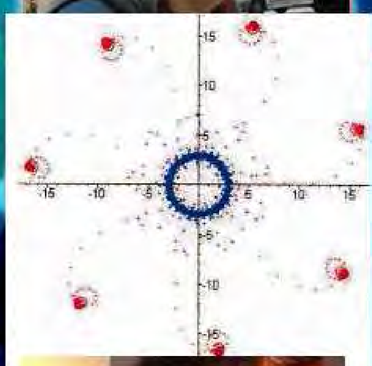
†Quite full details of which are beautifully recorded on Bornemann's website
`www-m8.ma.tum.de/m3/bornemann/challengebook/`
which accompanies *The Challenge*.

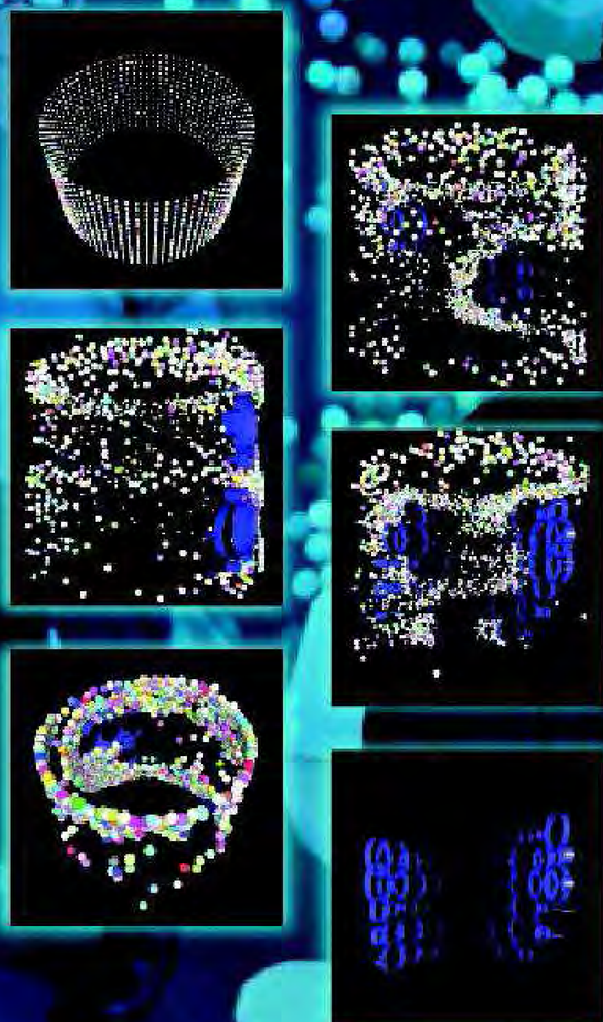
ISSN 0002-9920

Notices

of the American Mathematical Society

May 2005 Volume 52, Number 5





Simulating structure formation in the cosmos (see page 438)

Lists, Challenges, and Competitions

These have a long and primarily lustrous—social constructivist—history in mathematics.

- Consider the Hilbert Problems*, the Clay Institute's seven (million dollar) Millennium problems, or Dongarra and Sullivan's 'Top Ten Algorithms'.

In 2000, Sullivan and Dongarra wrote "Great algorithms are the poetry of computation," when they compiled a list of the 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century".†

- Newton's method was apparently ruled ineligible for consideration.

*See the late Ben Yandell's wonderful *The Honors Class: Hilbert's Problems and Their Solvers*, A K Peters, 2001.

†From "Random Samples", *Science* page 799, February 4, 2000. The full article appeared in the January/February 2000 issue of *Computing in Science & Engineering*. Dave Bailey wrote the description of 'PSLQ'.

I. The 20th century's Top Ten

#1. 1946: **The Metropolis Algorithm for Monte Carlo**. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.

#2. 1947: **Simplex Method for Linear Programming**. An elegant solution to a common problem in planning and decision-making.

#3. 1950: **Krylov Subspace Iteration Method**. A technique for rapidly solving the linear equations that abound in scientific computation.

#4. 1951: **The Decompositional Approach to Matrix Computations**. A suite of techniques for numerical linear algebra.

#5. 1957: **The Fortran Optimizing Compiler**. Turns high-level code into efficient computer-readable code.

#6. 1959: QR Algorithm for Computing Eigenvalues. Another crucial matrix operation made swift and practical.

#7. 1962: **Quicksort Algorithms for Sorting**. For the efficient handling of large databases.

#8. 1965: **Fast Fourier Transform**. Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

#9. 1977: **Integer Relation Detection**. A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

#10. 1987: **Fast Multipole Method**. A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

Eight of these appeared in the first two decades of serious computing. Most are multiply embedded in every major mathematical computing package.

- We turn to the story of a more recent highly successful challenge and associated book.

  The book under review also makes it clear that with the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" has little traction, especially for a newer generation of mathematicians who may readily take advantage of the maturation of computational packages such as Maple, Mathematica *and* MATLAB*.*
  (JMB, 2005)

- But we take a longer perspective.

Numerical Analysis Then and Now

Emphasizing quite how great an advance positional notation was, Ifrah writes:

> *A wealthy (15th Century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division – assuming that he has sufficient gifts – then you will have to send him to Italy.* (Georges Ifrah*)

*From page 577 of *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, translated from French, John Wiley, 2000.

Archimedes method

George Phillips has accurately called Archimedes the first numerical analyst. In the process of obtaining his famous estimate

$$3 + \frac{10}{71} < \pi < 3 + \frac{10}{70}$$

he had to master notions of recursion without computers, interval analysis without zero or positional arithmetic, and trigonometry without any of our modern analytic scaffolding ...

A modern computer algebra system can tell one that

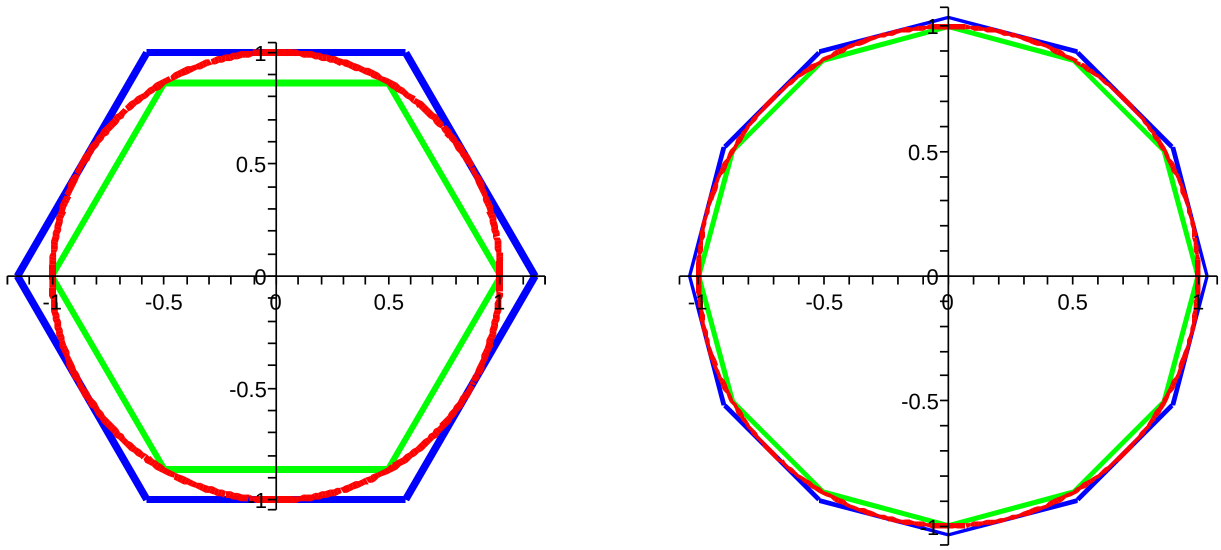
$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \tag{1}$$

since the integral may be interpreted as the area under a positive curve.

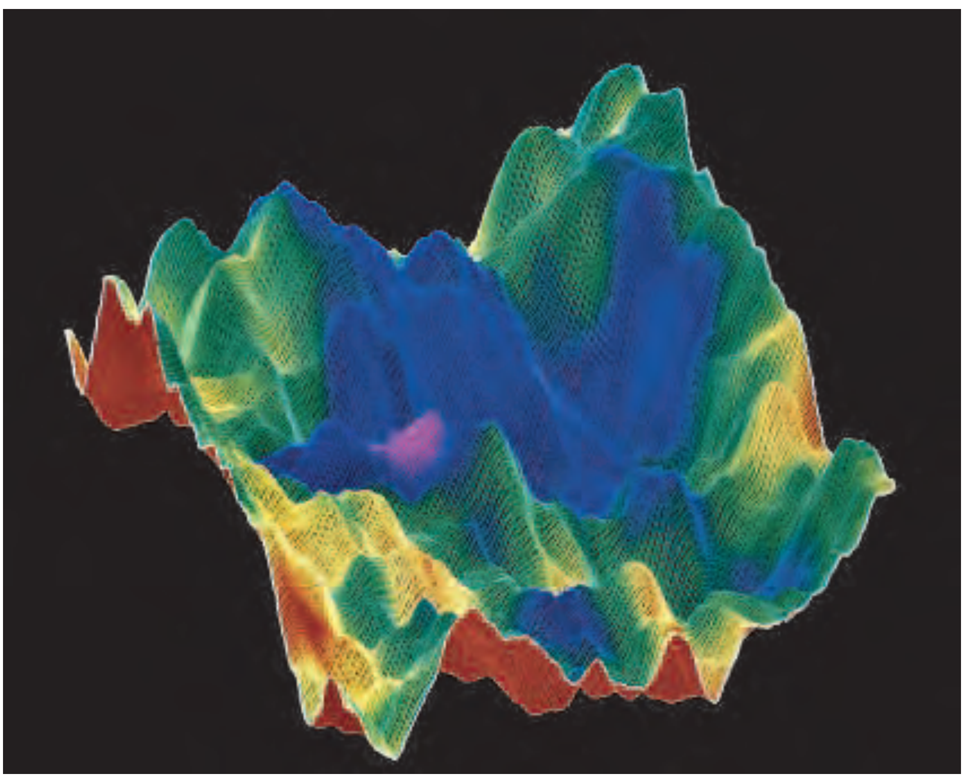
We are though no wiser as to why! If, however, we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4\,t - 4 \arctan(t)\,.$$

Now (1) is rigourously established by differentiation and an appeal to the Fundamental theorem of calculus. □



Archimedes' method for π with 6- and 12-gons



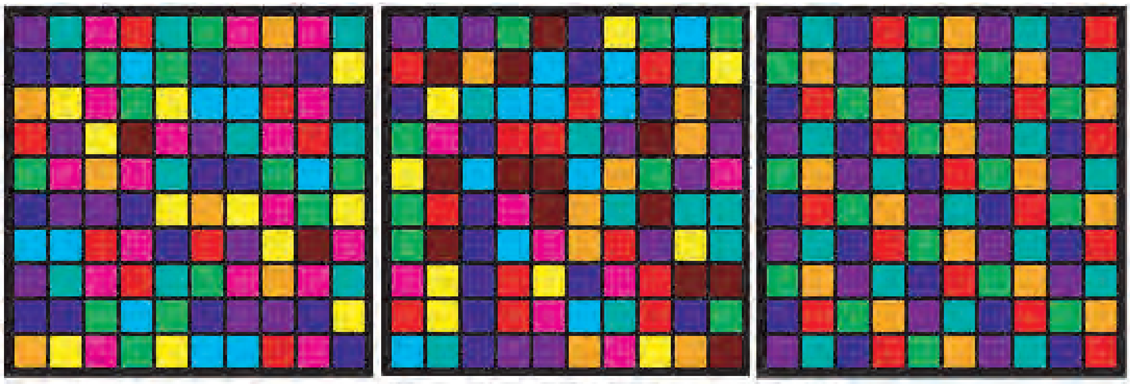
A random walk on one million digits of π

- While there were many fine arithmeticians over the next 1500 years, Ifrah's anecdote above shows how little had changed, other than to get worse, before the Renaissance.

- By the 19th Century, Archimedes had finally been outstripped both as a theorist, and as an (applied) numerical analyst:

  *In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labour. Thinking this is a good opportunity to illustrate the superiority of the method of W. G. Horner, not yet known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this point, as a Christmas exercise. I received several answers, agreeing with each other, to 50 places of decimals. In 1848, I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places.** (Augustus De Morgan)

*Quoted by Adrian Rice in "What Makes a Great Mathematics Teacher?" on page 542 of The *American Mathematical Monthly*, June-July 1999.



Archimedes: 223/71 < π < 22/7

A pictorial proof

- De Morgan seems to have been one of the first to mistrust William Shanks's epic computations of Pi—to 527, 607 and 727 places, noting there were too few sevens.

- But the error was only confirmed three quarters of a century later in 1944 by Ferguson with the help of a calculator in the last pre-computer calculations of π.*

◁ Until around 1950 a "computer" was still a person and ENIAC was an "Electronic Numerical Integrator and Calculator" on which Metropolis and Reitwiesner computed Pi to 2037 places in 1948 and confirmed that there were the expected number of sevens.

*A Guinness record for finding an error in math literature?

Reitwiesner, then working at the Ballistics Research Laboratory, Aberdeen Proving Ground in Maryland, starts his article with:

> *Early in June, 1949, Professor* John von Neumann *expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of* π *and* e *to many decimal places with a view to toward obtaining a statistical measure of the randomness of distribution of the digits.*

The paper notes that $e$appears to be *too random*—this is now proven—and ends by respecting an oft-neglected 'best-practice':

> *Values of the auxiliary numbers* $\operatorname{arccot} 5$ *and* $\operatorname{arccot} 239$ *to 2035D ... have been deposited in the library of Brown University and the UMT file of MTAC.*

- Just as layers of software, hardware & middleware have stabilized, so have their roles in scientific and especially mathematical computing.

- Thirty years ago, LP texts concentrated on 'Y2K'-like tricks for limiting storage demands.

  – Now serious users and researchers will often happily run large-scale problems in MATLAB and other broad spectrum packages, or rely on *NAG* library routines.

  – While such out-sourcing or commoditization of scientific computation and numerical analysis is not without its drawbacks, the analogy with automobile driving in 1905 and 2005 is apt.

- We are now in possession of mature—not to be confused with 'error-free'—technologies. We can be fairly comfortable that *Mathematica* is sensibly handling round-off or cancelation error, using reasonable termination criteria etc.

  – Below the hood, *Maple* is optimizing polynomial computations using tools like Horner's rule, running multiple algorithms when there is no clear best choice, and switching to reduced complexity (Karatsuba or FFT-based) multiplication when accuracy so demands.*

*Though, it would be nice if all vendors allowed as much peering under the bonnet as *Maple* does.

About the Contest

In a 1992 essay "*The Definition of Numerical Analysis*"*. Trefethen engagingly demolishes the conventional definition of Numerical Analysis as "*the science of rounding errors*". He explores how this hyperbolic view emerged and finishes by writing:

> I believe that the existence of finite algorithms for certain problems, together with other historical forces, has distracted us for decades from a balanced view of numerical analysis. ... *For guidance to the future we should study not Gaussian elimination and its beguiling stability properties, but the diabolically fast conjugate gradient iteration, or Greengard and Rokhlin's $O(N)$ multipole algorithm for particle simulations, or the exponential convergence of spectral methods for solving certain PDEs, or the convergence in $O(N)$ iteration achieved by multigrid methods for many kinds of problems, or even Borwein and Borwein's magical AGM iteration for determining 1,000,000 digits of π in the blink of an eye. That is the* heart *of numerical analysis.*

**SIAM News*, November 1992.

In *SIAM News* (Jan 2002), Trefethen liste ten diverse problems used in teaching *modern* graduate numerical analysis in Oxford. Readers were challenged to compute 10 digits of each, with a dollar per digit ($100) prize to the best entry. Trefethen wrote,

> "*If anyone gets 50 digits in total, I will be impressed.*"

- And he was, 94 teams from 25 nations submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem).
  - They included the late John Boersma working with Fred Simons and others, Gaston Gonnet (a *Maple* founder) and Robert Israel, a team containing Carl Devore, and the current authors variously working alone and with others.
  - An originally anonymous donor, William J. Browning, provided funds for a $100 award to each of the twenty perfect teams.
  - JMB, David Bailey* and Greg Fee entered, but failed to qualify for an award.†

*Bailey wrote the introduction to the book under review.

†We took Nick at his word and turned in 85 digits!

II. The Ten Digit Challenge Problems

*The purpose of computing is insight, not numbers.** (Richard Hamming)

#1. What is $\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}\cos(x^{-1}\log x)\,dx$?

#2. A photon moving at speed 1 in the x-y plane starts at $t=0$ at $(x,y)=(1/2,1/10)$ heading due east. Around every integer lattice point (i,j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t=10$?

#3. The infinite matrix A with entries $a_{11}=1$, $a_{12}=1/2$, $a_{21}=1/3$, $a_{13}=1/4$, $a_{22}=1/5$, $a_{31}=1/6$, etc., is a bounded operator on ℓ^2. What is $||A||$?

#4. What is the global minimum of the function

$$\exp(\sin(50x))+\sin(60e^y)+\sin(70\sin x)$$

$$+\sin(\sin(80y))-\sin(10(x+y))+(x^2+y^2)/4?$$

*In *Numerical Methods for Scientists and Engineers*, 1962.

#5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $||\cdot||_\infty$. What is $||f-p||_\infty$?

#6. A flea starts at $(0,0)$ on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4+\epsilon$, and west with probability $1/4-\epsilon$. The probability that the flea returns to $(0,0)$ sometime during its wanderings is $1/2$. What is ϵ?

#7. Let A be the 20000×20000 matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \cdots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i-j| = 1, 2, 4, 8, \cdots, 16384$. What is the $(1,1)$ entry of A^{-1}.

#8. A square plate $[-1,1] \times [-1,1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

#9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)] x^\alpha \sin(\alpha/(2 - x))\, dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

#10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Answers correct to 40 digits are at

`web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html`

About the Book and Its Authors

Success in solving these problems requires a broad knowledge of mathematics and numerical analysis, together with significant computational effort, to obtain solutions and ensure correctness of the results.

- The strengths and limitations of *Maple*, *Mathematica*, Matlab (The *3Ms*), and other software tools such as PARI or GAP, are strikingly revealed in these ventures.

- Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided.

In December 2002, Keller wrote:

> *To the Editor: ... found it surprising that no proof of the correctness of the answers was given. Omitting such proofs is the accepted procedure in scientific computing. However, in a contest for calculating precise digits, one might have hoped for more.*
>
> *Joseph B. Keller, Stanford University*

Keller's request for proofs as opposed to compelling evidence of correctness is, in this context, somewhat unreasonable and even in the long-term somewhat counter-productive.
Nonetheless, the *The Challenge* makes a complete and cogent response to Keller and much much more. The interest in the contest has extended to *The Challenge*, which has already received reviews in places such as *Science* where mathematics is not often seen.

- *Different readers, depending on temperament, tools and training will find the same problem more or less interesting* and more or less challenging.

- *Problems can be read independently*: multiple solution techniques are given, background, implementation details—variously in each of the *3Ms* or otherwise—and extensions are discussed.

- *Each problem has its own chapter and lead author*: Folkmar Bornemann, Dirk Laurie, Stan Wagon and Jörg Waldvogel come from 4 countries on 3 continents and did not know each other, though Dirk did visit Jörge and Stan visited Folkmar as they were finishing up.

Some High Spots

The book proves the growing power of collaboration, networking and the grid—both human and computational. A careful reading *yields proofs of correctness for all problems except* for #1, #3 and #5.

- For **#5** one difficulty is to develop a *robust interval implementation* for both complex computation and, more importantly, for the *Gamma function.* Error bounds for **#1** may be out of reach, but an analytic solution to **#3** seems tantalizingly close.

- The authors ultimately provided **10,000-digit solutions** to **nine** of the problems. They say that this improved their knowledge on several fronts as well as being 'cool'.
  - success with Integer Relation Methods often demands ultrahigh precision computation.

- **One** (and only one) problem remains totally intractable —by this rarefied measure. *As of today only 300 digits of #3 are known*.

Some Surprising Outcomes

The authors* were surprised by the following:

#1. The best algorithm for 10,000 digits was the trusty *trapezoidal rule*—a not uncommon personal experience of mine.

#2. Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can still find the position of the particle at time 2000 (not just time ten), which makes a fine exercise for very high-precision interval computation.

#4. Interval analysis algorithms can handle similar problems in higher dimensions. As a foretaste of future graphic tools, one can solve this problem using current *adaptive 3-D plotting* routines which can catch all the bumps.

As an optimizer by background this was the first problem my group solved using a damped Newton method.

*Stan Wagon and Folkmar Bornemann, private communications.

#5. While almost all canned optimization algorithms failed, *differential evolution*, a relatively new type of evolutionary algorithm worked quite well.

#6 This has an almost-closed form via *elliptic integrals* and leads to a study of random walks on hypercubic lattices, and *Watson integrals*

#9. The maximum parameter is expressible in terms of a *MeijerG function*. Unlike most contestants, *Mathematica* and *Maple* both figure this out.

- This is another measure of the changing environment.* It is a good idea—and not at all immoral—to data-mine and find out what your one of the 3Ms knows about your current object of interest. Thus, Maple says:

```
     The Meijer G function is defined by the inverse
                     Laplace transform

      MeijerG([as,bs],[cs,ds],z)
                   /
           1      |   GAMMA(1-as+y) GAMMA(cs-y)  y
      = ------   0   ------------------------- z dy
        2 Pi I    |   GAMMA(bs-y) GAMMA(1-ds+y)
                 /
                  L
    where ...
```

*As is *Lambert W*, see Brian Hayes' *Why W?*

Two Big Surprises

Two solutions really surprised the authors: #7 **Too Large to be Easy, Too Small to Be Hard**.

Not so long ago a 20,000 $\times$ 20,000 matrix was large enough to be hard. Using both *congruential* and *p-adic methods*, Dumas, Turner and Wan obtained a fully *symbolic* answer, a rational with a 97,000-digit numerator and like denominator. Wan has reduced the time needed to 15 minutes on one machine, from using many days on many machines.

- While p-adic analysis is parallelizable it is less easy than with congruential methods; *the need for better parallel algorithms lurks below the surface* of much modern computational math.

- The surprise here, though, is not that the solution is rational, but that it can be explicitly constructed.

  The chapter, like the others offers an interesting menu of numeric and exact solution strategies. Of course, in any numeric approach *ill-conditioning* rears its ugly head while the use of *sparsity* and other core topics come into play.

Problem #10: **Hitting the Ends**

(My personal favourite, for reasons that may be apparent.) Bornemann starts the chapter by exploring *Monte-Carlo methods*, which are shown to be impracticable.

- He then reformulates the problem *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate* harmonic measure *of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.

- This is then solved by a well chosen *sparse Cholesky* solver. At this point a reliable numerical value of

  $$\mathbf{3.837587979 \cdot 10^{-7}}$$

  is obtained.

  And the posed problem is solved numerically to the requisite 10 places.

But this is only the warm up ...

Analytic Solutions

We proceed to develop two analytic solutions, the first using *separation of variables** on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$p(a,b) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \operatorname{sech}\left((2k+1)\frac{\pi}{2}\rho\right) \qquad (2)$$

where $\rho := a/b$.

A second method using *conformal mappings*, yields

$$\operatorname{arccot} \rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \qquad (3)$$

where K is the *complete elliptic integral* of the first kind.

- It will not be apparent to one unfamiliar with inversion of elliptic integrals that (2) and (3) encode the same solution—though they must as the solution is unique in $(0,1)$—and each can now be used to solve for $\rho = 10$ to arbitrary precision.

*As with the trapezoidal rule, easy can be good.

Enter Srinivasa Ramanujan

Bornemann finally shows that, for far from simple reasons, the answer is

$$p = \frac{2}{\pi} \arcsin\left(k_{100}\right), \tag{4}$$

where

$$k_{100} :=$$

$$\left(\left(3-2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2$$

- No one anticipated a closed form like this—a simple composition of Pi, one arcsin and a few square roots.*

▷ Let me show how to finish up the feast.

*Actually fundamental units of real (quadratic/quartic) fields; solutions to Pell's equation.

An apt result in *Pi and the AGM* is that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) = \frac{1}{2}\arcsin k, \qquad (5)$$

exactly when k_{ρ^2} is parametrized by *theta functions* in terms of the *elliptic nome* as Jacobi discovered.

We have thus gotten

$$k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \qquad q := e^{-\pi\rho}. (6)$$

Comparing (5) and (2) we see that the solution is

$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7}$$

as asserted in (4).

- The explicit form follows from 19th century *modular function theory*. □

- If only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one.

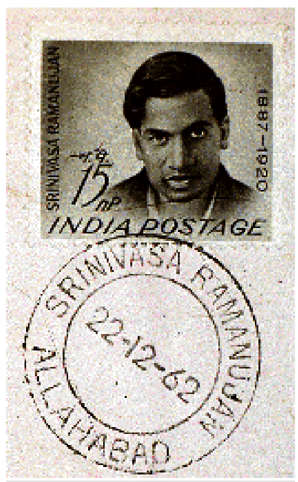
  – $k_{15/14}$ and k_{210} *share* their units (Pi & AGM).

A Singular Interlude

Indeed k_{210} is the *singular value* sent to Hardy in Ramanujan's famous 1913 letter of introduction—ignored by two other famous English mathematicians.

$$k_{210} := \left(\sqrt{2}-1\right)^2\left(\sqrt{3}-2\right)\left(\sqrt{7}-6\right)^2\left(8-3\sqrt{7}\right) \times \left(\sqrt{10}-3\right)^2\left(\sqrt{15}-\sqrt{14}\right)\left(4-\sqrt{15}\right)^2\left(6-\sqrt{35}\right)$$

matics. Four hours creative work a day is about the limit for a mathematician, he used to say. Lunch, a light meal, in hall. After lunch he loped off for a game of real tennis in the university court. (If it had been summer, he would have walked down to Fenner's to watch cricket.) In the late afternoon, a stroll back to his rooms. That particular day, though, while the timetable wasn't altered, internally things were not going according to plan. At the back of his mind, getting in the way of his complete pleasure in his game, the Indian manuscript nagged away. Wild theorems. Theorems such as he had never seen before, nor imagined. A fraud of genius? A question was forming itself in his mind. As it was Hardy's mind, the question was forming itself with epigrammatic clarity: is a fraud of genius more probable than an unknown mathematician of genius? Clearly the answer was no. Back in his rooms in Trinity, he had another look at the script. He sent word to Littlewood (probably by messenger, certainly not by telephone, for which, like all mechanical contrivances including fountain pens, he had a deep distrust) that they must have a discussion after hall.



That is an occasion at which one would have liked to be present. Hardy, with his combination of remorseless clarity and intellectual panache (he was very English, but in argument he showed the characteristics that Latin minds have often assumed to be their own): Littlewood, imaginative, powerful, humorous. Apparently it did not take them long. Before midnight they knew, and knew for certain. The writer of these manuscripts was a man of genius. That was as much as they could judge, that night. It was only later that Hardy decided that Ramanujan was, in terms of *natural* mathematical genius, in the class of Gauss and Euler: but that he could not expect, because of the defects of his education, and because he had come on the scene too late in the line of mathematical history, to make a contribution on the same scale.

GH Hardy (1877–1947)

CP Snow's description in
A Mathematician's Apology

A Modern Finale

Alternatively, armed only with the knowledge that the singular values are always algebraic we may finish with an *au courant* proof: numerically obtain the minimal polynomial from a high precision computation with (6) and recover the surds.

Maple allows the following

```
> Digits:=100:with(PolynomialTools):
> k:=s->evalf(EllipticModulus(exp(-Pi*sqrt(s)))):
> p:=latex(MinimalPolynomial(k(100),12)):
> 'Error',fsolve(p)[1]-evalf(k(100)); galois(p);
                                      -106
                 Error,  4 10

"8T9", {"D(4)[x]2", "E(8):2"}, "+", 16, {"(4 5)(6 7)",
  "(4 8)(15)(26)(3 7)", "(1 8)(2 3)(4 5)(6 7)",
 "(2 8)(1 3)(4 6)(5 7)"}
```

This finds the minimal polynomial for k_{100}, checks it to 100 places, tells us the *galois group*, and returns a latex expression 'p' which sets as:

$$\begin{aligned}1 \quad - \quad & 1658904\,\_X - 3317540\,\_X^2 + 1657944\,\_X^3 + 6637254\,\_X^4 \\ + \quad & 1657944\,\_X^5 - 3317540\,\_X^6 - 1658904\,\_X^7 + \_X^8,\end{aligned}$$

and is *self-reciprocal*:

It satisfies $p(x) = x^8 p(1/x)$.

This suggests taking a square root and we discover $y = \sqrt{k_{100}}$ satisfies

$$\begin{aligned} p(y) = 1 \;\; & - \;\; 1288\,y + 20\,y^2 - 1288\,y^3 - 26\,y^4 \\ & + \;\; 1288\,y^5 + 20\,y^6 + 1288\,y^7 + y^8. \end{aligned}$$

Now life is good. The prime factors of 100 are 2 and 5 prompting:

```
subs(_X=z,[op(((factor(p,{sqrt(2),sqrt(5)}))))]))
```

The code yields four quadratic terms, the desired one being

$$\begin{aligned} q = z^2 \;\; & + \;\; 322\,z - 228\,z\sqrt{2} + 144\,z\sqrt{5} - 102\,z\sqrt{2}\sqrt{5} \\ & + \;\; 323 - 228\,\sqrt{2} + 144\,\sqrt{5} - 102\,\sqrt{2}\sqrt{5}. \end{aligned}$$

For security,

```
w:=solve(q)[2]:  evalf[1000](k(100)-w^2);
```

gives a 1000-digit error check of $2.20226255 \cdot 10^{-998}$.

- We can work a little more to find, using one of the 3Ms, the beautiful form of k_{100} given in (4).

□

III. The Ten Symbolic Challenge Problems

Each of the following* requires numeric work—some times considerable—to facilitate whatever transpires thereafter.

#1. Compute the value of r for which the chaotic iteration $x_{n+1} = rx_n(1 - x_n)$, starting with some $x_0 \in (0, 1)$, exhibits a bifurcation between 4-way periodicity and 8-way periodicity.

Extra credit: This constant is an algebraic number of degree not exceeding 20. Find its minimal polynomial.

#2. Evaluate

$$\sum_{(m,n,p)\neq 0} \frac{(-1)^{m+n+p}}{\sqrt{m^2 + n^2 + p^2}}, \tag{7}$$

where convergence is over increasingly large cubes surrounding the origin.

Extra credit: Identify this constant.

*To appear in the *MAA Monthly*.

#3. Evaluate the sum

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3}.$$

Extra credit: Evaluate this constant as a multi-term expression involving well-known mathematical constants. This expression has seven terms, and involves $\pi, \log 2, \zeta(3)$, and $\mathrm{Li}_5(1/2)$.

Hint: The expression is "homogenous."

#4. Evaluate

$$\prod_{k=1}^{\infty} \left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{\left[\log_2\left(1+\frac{1}{k(k+2)}\right)\right]}$$

Extra credit: Evaluate this constant in terms of a less-well-known mathematical constant.

#5. Given $a, b, \eta > 0$, define

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}.$$

Calculate $R_1(2,2)$.

Extra credit: Evaluate this constant as a two-term expression involving a well-known mathematical constant.

#6. Calculate the expected distance between two random points on different sides of the unit square.

Hint: This can be expressed in terms of integrals as

$$\frac{2}{3}\int_0^1\int_0^1 \sqrt{x^2+y^2}\,dx\,dy + \frac{1}{3}\int_0^1\int_0^1 \sqrt{1+(y-u)^2}\,du\,dy.$$

Extra credit: Express this constant as a three-term expression involving algebraic constants and the natural logarithm with an algebraic argument.

Similarly:

#7. Calculate the expected distance between two random points on different faces of the unit cube.

Hint: This can be expressed in terms of integrals as

$$\frac{4}{5}\int_0^1\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+(z-w)^2}\,dw\,dx\,dy\,dz+$$
$$\frac{1}{5}\int_0^1\int_0^1\int_0^1\int_0^1\sqrt{1+(y-u)^2+(z-w)^2}\,du\,dw\,dy\,dz.$$

Extra credit: Express this constant as a six-term expression involving algebraic constants and two natural logarithms.

Answers to all ten are detailed in our paper [Bailey, Borwein, Kapoor and Weisstein].

- The final three we finish by further discussing...

#8. Calculate

$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)\, dx. \tag{8}$$

Extra credit: Express this constant as an analytic expression.

Hint: It is *not* what it first appears to be.

#9. Calculate

$$\sum_{i>j>k>l>0} \frac{1}{i^3 j k^3 l}.$$

Extra credit: Express this constant as a single well-known mathematical constant.

Solution. In the notation of Lecture II:

$$\zeta(3,1,3,1) = \frac{2\,\pi^8}{10!},$$

and is the second case of *Zagier's conjecture*, now proven (see APPENDIX I, D).

#10. Evaluate

$$W_1 \;=\; \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{3-\cos(x)-\cos(y)-\cos(z)}\, dx\, dy\, dz.$$

Extra credit: Express this constant in terms of the Gamma function.

History and Context

The challenge of showing that the value of $\pi_2 < \pi/8$ was posed by Bernard Mares, Jr., along with the problem of showing

$$\pi_1 := \int_0^\infty \prod_{n=1}^\infty \cos\left(\frac{x}{n}\right)\, dx < \frac{\pi}{4}. \tag{9}$$

This is indeed true, although the error is remarkably small, as we shall see.

Solution The computation of a high-precision numerical value for this integral is rather challenging, due in part to the oscillatory behavior of $\prod_{n\geq 1}\cos(x/n)$ but mostly due to the difficulty of computing high-precision evaluations of the integrand function.

Let $f(x)$ be the integrand function. We can write

$$f(x) \;=\; \cos(2x)\left[\prod_1^m \cos(x/k)\right]\exp(f_m(x)), \tag{10}$$

where we choose $m > x$, and where

$$f_m(x) \;=\; \sum_{k=m+1}^\infty \log\cos\left(\frac{x}{k}\right). \tag{11}$$

The log cos evaluation can be expanded as follows:

$$\log\cos\left(\frac{x}{k}\right) = \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!}\left(\frac{x}{k}\right)^{2j},$$

where B_{2j} are *Bernoulli numbers*. Note that since $k > m > x$ in (11), this series converges. We can now write

$$f_m(x) = \sum_{k=m+1}^{\infty}\sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!}\left(\frac{x}{k}\right)^{2j},$$

which after interchanging the sums gives

$$f_m(x) = -\sum_{j=1}^{\infty} \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}\left[\sum_{k=m+1}^{\infty}\frac{1}{k^{2j}}\right]x^{2j}.$$

or as follows:

$$f_m(x) = -\sum_{j=1}^{\infty} \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}\left[\zeta(2j) - \sum_{k=1}^{m}\frac{1}{k^{2j}}\right]x^{2j}.$$

We have more compactly

$$f_m(x) = -\sum_{j=1}^{\infty} a_j b_{j,m} x^{2j},$$

where

$$a_j = \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}} \qquad b_{j,m} = \zeta(2j) - \sum_{k=1}^{m} 1/k^{2j}. \quad (12)$$

With this evaluation scheme for $f(x)$ in hand, the integral (8) can be computed using, for instance, the *tanh-sinh quadrature* algorithm, which can be implemented fairly easily on a personal computer or workstation, and which is also well-suited for highly parallel processing .

- This algorithm approximates an integral $f(x)$ on $[-1, 1]$ by transforming it to an integral on $(-\infty, \infty)$, using the change of variable $x = g(t)$, where $g(t) = \tanh(\pi/2 \cdot \sinh t)$:

$$\begin{aligned} \int_{-1}^{1} f(x)\,dx &= \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt \\ &= h \sum_{j=-\infty}^{\infty} w_j f(x_j) + E(h). \end{aligned} \quad (13)$$

  Here $x_j = g(hj)$ and $w_j = g'(hj)$ are abscissas and weights for the tanh-sinh quadrature scheme (which can be pre-computed), and $E(h)$ is the error in this approximation.

- The tanh-sinh quadrature algorithm is designed for a finite integration interval. The simple substitution $s = 1/(x+1)$ reduces again to an integral from 0 to 1.

In spite of the substantial precomputation required, the calculation requires only about one minute, using Bailey's *ARPREC* software package The first 100 digits of the result are:

<u>0.392699081698724154807830422909937860524646</u>543418723
159592681228516209324713993854617901651274745536677 7

The *Inverse Symbolic Calculator*, e.g., suggests this is likely $\pi/8$. But a careful comparison with $\pi/8$:

<u>0.39269908169872415480783042290993786052464</u>6174921888
227621868074038477050785776124828504353167764633497...,

reveals they *differ* by approximately $\mathbf{7.407 \times 10^{-43}}$.

- These two values are provably distinct. The reason is governed by the fact that

$$\sum_{n=1}^{55} \frac{1}{2n+1} > 2 > \sum_{n=1}^{54} \frac{1}{2n+1}.$$

⋗ We do not know a concise closed-form evaluation of this constant.

Further History and Context

Recall the *sinc* function

$$\text{sinc}(x) \;:=\; \frac{\sin(x)}{x}.$$

Consider, the seven highly oscillatory integrals below.

$$I_1 \;:=\; \int_0^\infty \text{sinc}(x)\,dx \;=\; \frac{\pi}{2},$$

$$I_2 \;:=\; \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right)\,dx \;=\; \frac{\pi}{2},$$

$$I_3 \;:=\; \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right)\text{sinc}\left(\frac{x}{5}\right)\,dx \;=\; \frac{\pi}{2},$$

$$\ddots$$

$$I_6 \;:=\; \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right)\cdots\text{sinc}\left(\frac{x}{11}\right)\,dx \;=\; \frac{\pi}{2},$$

$$I_7 \;:=\; \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right)\cdots\text{sinc}\left(\frac{x}{13}\right)\,dx \;=\; \frac{\pi}{2}.$$

However,

$$\begin{aligned} I_8 &:= \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right)\cdots\text{sinc}\left(\frac{x}{15}\right)\,dx \\ &= \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi \\ &\approx 0.499999999992646\pi. \end{aligned}$$

- When shown this, a friend using a well-known computer algebra package, and the software vendor concluded there was a "bug" in the software.

- Not so! It is easy to see that the limit of these integrals is $2\pi_1$.

Fourier analysis, via *Parseval's theorem*, links

$$I_N \;:=\; \int_0^\infty \mathrm{sinc}(a_1 x)\mathrm{sinc}\,(a_2 x)\cdots\mathrm{sinc}\,(a_N x)\;dx$$

with the volume of the polyhedron P_N given by

$$P_N \;:=\; \{x : |\sum_{k=2}^{N} a_k x_k| \le a_1, |x_k| \le 1, 2 \le k \le N\},$$

where $x := (x_2, x_3, \cdots, x_N)$.

If we let

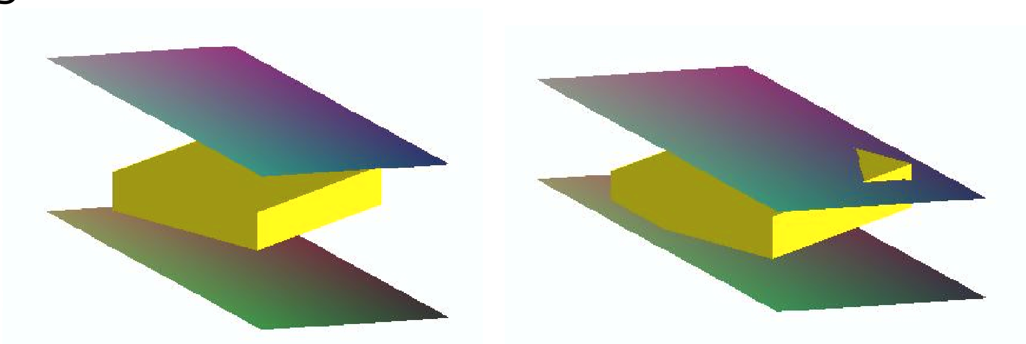
$$C_N \;:=\; \{(x_2, x_3, \cdots, x_N) : -1 \le x_k \le 1, 2 \le k \le N\},$$

then

$$I_N \;=\; \frac{\pi}{2a_1}\frac{\mathrm{Vol}(P_N)}{\mathrm{Vol}(C_N)}.$$

- Thus, the value drops precisely when the constraint
$$\sum_{k=2}^{N} a_k x_k \le a_1$$
becomes *active* and *bites* into the hypercube C_N; this occurs exactly when $\sum_{k=2}^{N} a_k > a_1$.

- $\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$, but on addition of $\frac{1}{15}$, the sum exceeds 1, the volume drops, and $I_N = \frac{\pi}{2}$ no longer holds.



Before and after the bite

- A similar analysis applies to π_2. Moreover, it is fortunate that we began with π_1 or the falsehood of the identity analogous to that displayed above would have been much harder to see.

#10. History and Context

The integral arises in Gaussian and spherical models of ferromagnetism and in the theory of *random walks* (as in extensions of Trefethen #6). It leads to one of the most impressive closed-form evaluations of an equivalent integral due to G.N. Watson:

$$\begin{aligned}\widehat{W} &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{3-\cos(x)-\cos(y)-\cos(z)}\,dx\,dy\,dz \\ &= \frac{1}{96}(\sqrt{3}-1)\,\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right) \qquad (14) \\ &= 4\pi\left(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6}\right)\mathrm{K}^2(k_6),\end{aligned}$$

where $k_6 = \left(2-\sqrt{3}\right)\left(\sqrt{3}-\sqrt{2}\right)$ is the *sixth singular value*.

The most self contained derivation of this very subtle result is due to Joyce and Zucker.

Solution. We apply the formula

$$\frac{1}{\lambda} = \int_0^\infty e^{-\lambda t}\,dt, \qquad \mathrm{Re}(\lambda) > 0 \qquad (15)$$

to W_3. The 3-dimension integral is reducible to a single integral by using

$$\frac{1}{\pi}\int_0^\infty \exp(t\cos\theta)d\theta = I_0(t) \qquad (16)$$

is the *modified Bessel function of the first kind*.

It follows from this that

$$W_3 = \int_0^\infty \exp(-3t) I_0^3(t)\,dt.$$

which evaluates to arbitrary precision giving:

$$W_3 = 0.5054620197173260060520040532271 40\ldots.$$

Finally an integer relation hunt to express $\log W$ in terms of $\log\pi, \log 2, \log\Gamma(k/24)$ and $\log(\sqrt{3}-1)$ will produce (14).

- We may also write W_3 only as a product of $\Gamma-$values.

This is what our *Mathematician's ToolKit* returned:

```
0= -1.* log[w3] + -1.* log[gamma[1/24]] + 4.*log[gamma[3/24]] +
                  -8.*log[gamma[5/24]] + 1.* log[gamma[7/24]] +
                   14.*log[gamma[9/24]]+-6.*log[gamma[11/24]] +
                 -9.*log[gamma[13/24]] +18.*log[gamma[15/24]] +
                 -2.*log[gamma[17/24]] +-7.*log[gamma[19/24]]
```

- which is proven by comparing the result with (14) and establishing the implicit Γ - representation of $(\sqrt{3}-1)^2/96$.

- Similar searches suggest there is no similar four dimensional closed form.

- We found that W_4 is not expressible as a product of powers of $\Gamma(k/120)$ (for $0 < k < 120$) with coefficients of less than 12 digits.

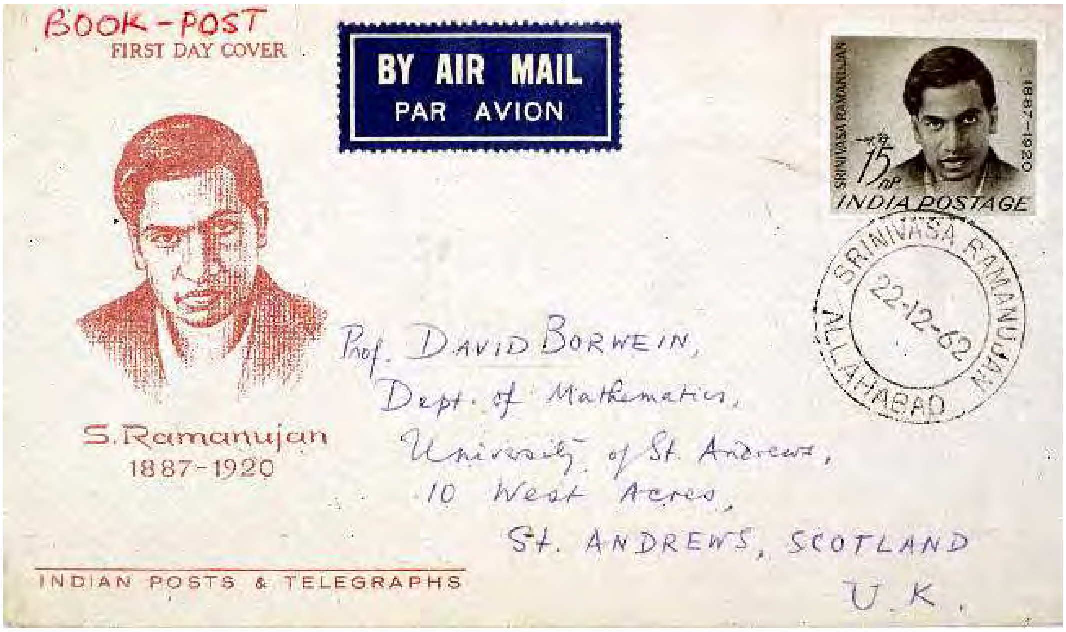
  – This does not, of course, rule out the possibility of a larger relation, but it does cast doubt, experimentally, that such a relation exists.

  – enough to stop looking!

Advanced Collaborative Environment

CONCLUSION

The many techniques and types of mathematics used are a wonderful advert for multi-field, multi-person, multi-computer, multi-package collaboration.



- Edwards comments in his recent *Essays on Constructive Mathematics* that his own preference for constructivism was forged by experience of computing in the fifties, when computing power was as he notes "trivial by today's standards".

My similar attitudes were cemented primarily by the ability in the early days of personal computers to decode—with the help of *APL*—exactly the sort of work by Ramanujan which finished #10.

CARATHÉODORY and CHRÉTIEN

I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science. (Constantin Carathéodory, 1936)

- Addressing the MAA (retro-digital data-mining?)

A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven. (Jean Chrétien)

The then Prime Minister, explaining in 2002 how Canada would determine if Iraq had WMDs, sounds a lot like Bertrand Russell!

REFERENCES

1. D. Bailey, J. Borwein, V. Kapoor and E. Weisstein, "Ten Problems in Experimental Mathematics," *MAA Monthly,* accepted April 2005. [CoLab Preprint 270].*

2. Jonathan M. Borwein and Robert Corless, "Emerging Tools for Experimental Mathematics," *MAA Monthly*, **106** (1999), 889–909. [CECM 98:110]

3. D.H. Bailey and J.M. Borwein, "Experimental Mathematics: Recent Developments and Future Outlook," pp, 51-66 in Vol. I of *Mathematics Unlimited — 2001 and Beyond,* B. Engquist & W. Schmid (Eds.), Springer-Verlag, 2000. [CECM 99:143]

*All references are at D-drive and `www.cecm.sfu.ca/preprints`.

4. J. Dongarra, F. Sullivan, "The top 10 algorithms," *Computing in Science & Engineering*, **2** (2000), 22–23.
(See `personal/jborwein/algorithms.html`.)

5. J.M. Borwein and P.B. Borwein, "Challenges for Mathematical Computing," *Computing in Science & Engineering*, **3** (2001), 48–53. [CECM 00:160].

6. J.M. Borwein and D.H. Bailey), **Mathematics by Experiment: Plausible Reasoning in the 21st Century**, and **Experimentation in Mathematics: Computational Paths to Discovery**, (with R. Girgensohn,) AK Peters Ltd, 2003-04.

7. J.M. Borwein and T.S Stanway, "Knowledge and Community in Mathematics," *The Mathematical Intelligencer*, in Press, 2005.

▶ The web site is at **www.expmathbook.info**

APPENDIX I: INTEGER RELATIONS

The USES of LLL and PSLQ

▶ A vector $(x_1, x_2, \cdots, x_n)$ of reals *possesses an integer relation* if there are integers a_i not all zero with

$$0 = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.$$

PROBLEM: Find a_i if such exist. If not, obtain lower bounds on the size of possible a_i.

- $(n = 2)$ *Euclid's algorithm* gives solution.

- $(n \geq 3)$ Euler, Jacobi, Poincare, Minkowski, Perron, others sought method.

- *First general algorithm* in 1977 by **Ferguson** & Forcade. Since '77: **LLL** (in Maple), HJLS, PSOS, **PSLQ** ('91, *parallel* '99).

▶ Integer Relation Detection was recently ranked among "the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century." J. Dongarra, F. Sullivan, *Computing in Science & Engineering* **2** (2000), 22–23.

Also: Monte Carlo, Simplex, Krylov Subspace, QR Decomposition, Quicksort, ..., FFT, Fast Multipole Method.

A. ALGEBRAIC NUMBERS

Compute α to sufficiently high precision (O(n^2)) and apply LLL to the vector

$$(1, \alpha, \alpha^2, \cdots, \alpha^{n-1}).$$

- Solution integers a_i are coefficients of a polynomial likely satisfied by α.

- If no relation is found, exclusion bounds are obtained.

B. FINALIZING FORMULAE

▶ If we suspect an identity PSLQ is powerful.

- (*Machin's Formula*) We try `lin_dep` on

$$[\arctan(1), \arctan(\frac{1}{5}), \arctan(\frac{1}{239})]$$

and recover `[1, -4, 1]`. That is,

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}).$$

[Used on all serious computations of π from 1706 (100 digits) to 1973 (1 million).]

- (*Dase's 'mental' Formula*) We try `lin_dep` on

$$[\arctan(1), \arctan(\frac{1}{2}), \arctan(\frac{1}{5}), \arctan(\frac{1}{8})]$$

and recover `[-1, 1, 1, 1]`. That is,

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8}).$$

[Used by Dase for 200 digits in 1844.]

C. ZETA FUNCTIONS

▶ The *zeta function* is defined, for $s > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

• Thanks to *Apéry* (1976) it is well known that

$$S_2 := \zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}$$

$$A_3 := \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

$$S_4 := \zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}$$

▶ These results *strongly* suggest that

$$\aleph_5 := \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5 \binom{2k}{k}}$$

is a simple rational or algebraic number. Yet, **PSLQ shows**: if $\aleph_5$ satisfies a polynomial of degree ≤ 25 the Euclidean norm of coefficients *exceeds* 2×10^{37}.

D. ZAGIER'S CONJECTURE

For $r \geq 1$ and $n_1, \ldots, n_r \geq 1$, consider:

$$L(n_1, \ldots, n_r; x) := \sum_{0 < m_r < \ldots < m_1} \frac{x^{m_1}}{m_1^{n_1} \ldots m_r^{n_r}} .$$

Thus

$$L(n; x) = \frac{x}{1^n} + \frac{x^2}{2^n} + \frac{x^3}{3^n} + \cdots$$

is the classical *polylogarithm*, while

$$\begin{aligned} L(n, m; x) &= \frac{1}{1^m} \frac{x^2}{2^n} + (\frac{1}{1^m} + \frac{1}{2^m}) \frac{x^3}{3^n} + (\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m}) \frac{x^4}{4^n} \\ &+ \cdots, \\ L(n, m, l; x) &= \frac{1}{1^l} \frac{1}{2^m} \frac{x^3}{3^n} + (\frac{1}{1^l} \frac{1}{2^m} + \frac{1}{1^l} \frac{1}{3^m} + \frac{1}{2^l} \frac{1}{3^m}) \frac{x^4}{4^n} + \cdots . \end{aligned}$$

- The series converge absolutely for $|x| < 1$ and conditionally on $|x| = 1$ unless $n_1 = x = 1$.

These polylogarithms

$$L(n_r, \ldots, n_1; x) = \sum_{0<m_1<\ldots<m_r} \frac{x^{m_r}}{m_r^{n_r} \ldots m_1^{n_1}},$$

are *determined uniquely* by the **differential equations**

$$\frac{d}{dx} L(\mathbf{n_r}, \ldots, n_1; x) = \frac{1}{x} L(\mathbf{n_r - 1}, \ldots, n_2, n_1; x)$$

if $n_r \geq 2$ and

$$\frac{d}{dx} L(\mathbf{n_r}, \ldots, n_2, n_1; x) = \frac{1}{1-x} L(\mathbf{n_{r-1}}, \ldots, n_1; x)$$

if $n_r = 1$ with the *initial conditions*

$$L(n_r, \ldots, n_1; 0) = 0$$

for $r \geq 1$ and

$$L(\emptyset; x) \equiv 1.$$

Set $\overline{s} := (s_1, s_2, \ldots, s_N)$. Let $\{\overline{s}\}_n$ denotes concatenation, and $w := \sum s_i$.

Then every *periodic* polylogarithm leads to a function

$$L_{\overline{s}}(x,t) := \sum_n L(\{\overline{s}\}_n; x) t^{wn}$$

which solves an algebraic ordinary differential equation in x, and leads to nice *recurrences*.

A. In the simplest case, with $N = 1$, the ODE is $\mathbf{D_s F = t^s F}$ where

$$D_s := \left((1-x)\frac{d}{dx}\right)^1 \left(x\frac{d}{dx}\right)^{s-1}$$

and the solution (by series) is a generalized hypergeometric function:

$$L_{\overline{s}}(x,t) = 1 + \sum_{n\geq 1} x^n \frac{t^s}{n^s} \prod_{k=1}^{n-1} \left(1 + \frac{t^s}{k^s}\right),$$

as follows from considering $D_s(x^n)$.

B. Similarly, for $N = 1$ and negative integers

$$L_{\overline{-s}}(x,t) := 1 + \sum_{n\geq 1} (-x)^n \frac{t^s}{n^s} \prod_{k=1}^{n-1} \left(1 + (-1)^k \frac{t^s}{k^s}\right),$$

and $L_{\overline{-1}}(2x-1,t)$ solves a hypergeometric ODE.

▶ Indeed

$$L_{\overline{-1}}(1,t) = \frac{1}{\beta(1+\frac{t}{2}, \frac{1}{2} - \frac{t}{2})}.$$

C. We may obtain ODEs for eventually periodic Euler sums. Thus, $L_{-2,1}(x,t)$ is a solution of

$$\begin{aligned}
t^6 F &= x^2(x-1)^2(x+1)^2\, D^6 F \\
&+ x(x-1)(x+1)(15x^2-6x-7)\, D^5 F \\
&+ (x-1)(65x^3+14x^2-41x-8)\, D^4 F \\
&+ (x-1)(90x^2-11x-27)\, D^3 F \\
&+ (x-1)(31x-10)\, D^2 F + (x-1)\, DF.
\end{aligned}$$

- This leads to a four-term recursion for $F = \sum c_n(t)x^n$ with initial values $c_0 = 1, c_1 = 0, c_2 = t^3/4, c_3 = -t^3/6$, and the ODE can be simplified.

We are now ready to prove Zagier's conjecture. Let $F(a, b; c; x)$ denote the *hypergeometric function*. Then:

Theorem 1 (BBGL) *For* $|x|, |t| < 1$ *and integer* $n \geq 1$

$$\begin{aligned}\sum_{n=0}^{\infty} \quad & L(\underbrace{3, 1, 3, 1, \ldots, 3, 1}_{n-fold}; x)\, t^{4n} \\ = \quad & F\left(\frac{t(1+i)}{2}, \frac{-t(1+i)}{2}; 1; x\right) \\ \times \quad & F\left(\frac{t(1-i)}{2}, \frac{-t(1-i)}{2}; 1; x\right).\end{aligned} \tag{17}$$

Proof. Both sides of the putative identity start

$$1 + \frac{t^4}{8}x^2 + \frac{t^4}{18}x^3 + \frac{t^8 + 44t^4}{1536}x^4 + \cdots$$

and are *annihilated* by the differential operator

$$D_{31} := \left((1-x)\frac{d}{dx}\right)^2 \left(x\frac{d}{dx}\right)^2 - t^4 \,.$$

QED

- Once discovered — and it was discovered after much computational evidence — this can be checked variously in Mathematica or Maple (e.g., in the package *gfun*)!

Corollary 2 *(***Zagier Conjecture***)*

$$\zeta(\underbrace{3,1,3,1,\ldots,3,1}_{n-fold}) = \frac{2\,\pi^{4n}}{(4n+2)!} \qquad (18)$$

Proof. We have

$$F(a,-a;1;1) = \frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin \pi a}{\pi a}$$

where the first equality comes from Gauss's evaluation of $F(a,b;c;1)$.

Hence, setting $x = 1$, in (17) produces

$$F\left(\frac{t(1+i)}{2}, \frac{-t(1+i)}{2}; 1; 1\right) F\left(\frac{t(1-i)}{2}, \frac{-t(1-i)}{2}; 1; 1\right)$$

$$= \frac{2}{\pi^2 t^2} \sin\left(\frac{1+i}{2}\pi t\right) \sin\left(\frac{1-i}{2}\pi t\right)$$

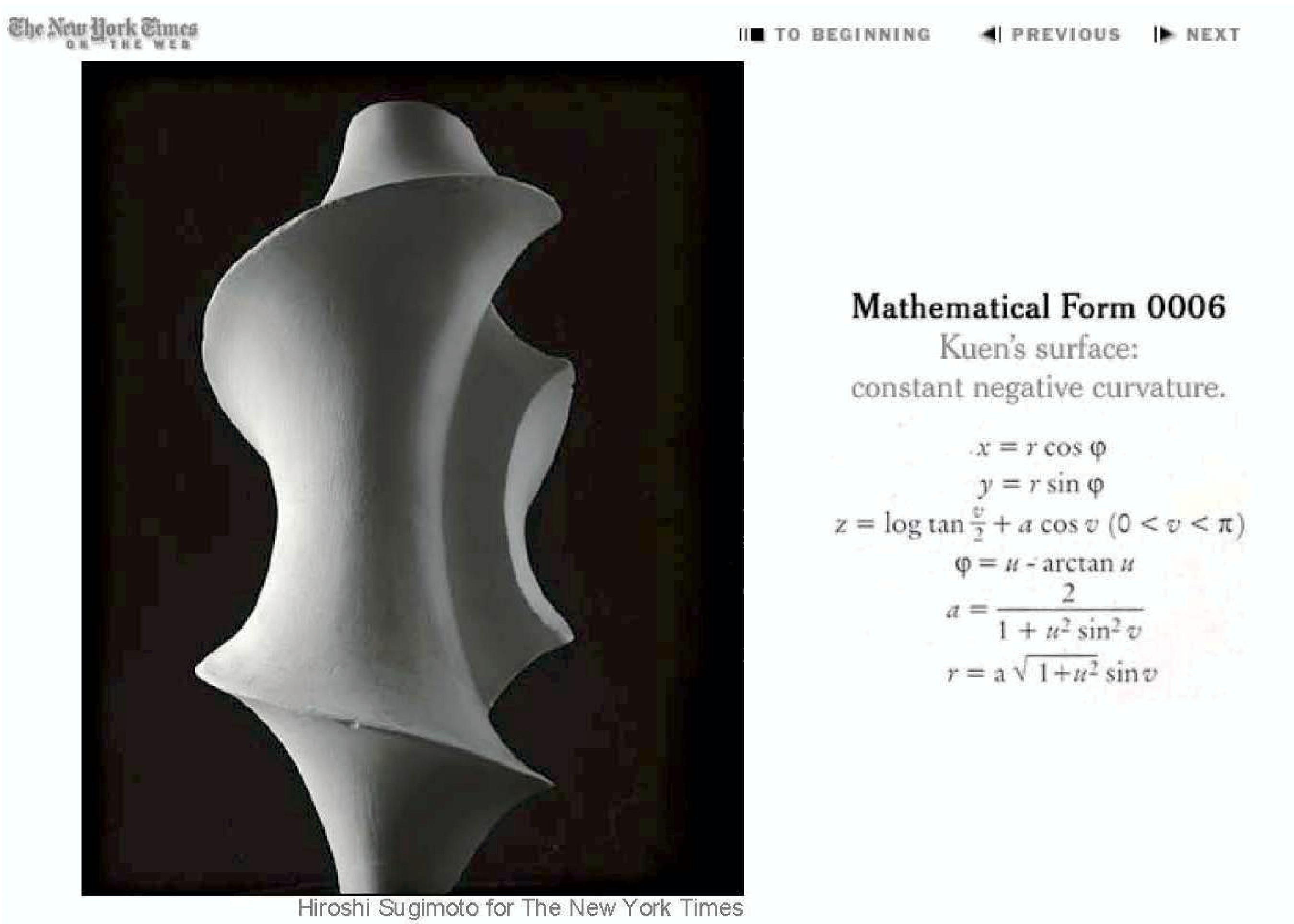
$$= \frac{\cosh \pi t - \cos \pi t}{\pi^2 t^2} = \sum_{n=0}^{\infty} \frac{2\pi^{4n} t^{4n}}{(4n+2)!}$$

on using the Taylor series of cos and cosh. Comparing coefficients in (17) ends the proof. **QED**

► What other deep Clausen-like hypergeometric factorizations lurk within?

• If one suspects that (2) holds, once one can compute these sums well, it is easy to verify many cases numerically and be entirely convinced.

♠ This is the *unique* non-commutative analogue of Euler's evaluation of $\zeta(2n)$.



Felix Klein's heritage

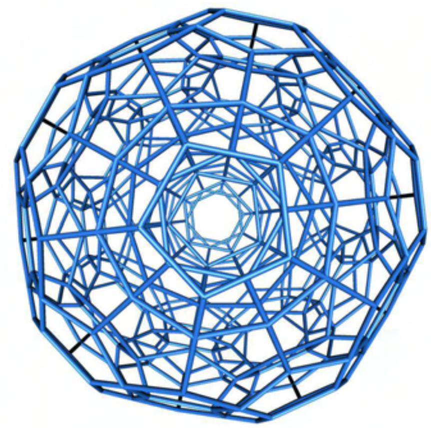
Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ...

I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane. Augustus de Morgan (1806–71).

- First President of the LMS, he was equally influential as an educator and a researcher

- There is evidence young children see more naturally in three than two dimensions

Donald Coxeter's (1907–2003) **octahedral kaleidoscope** built in Liverpool (circa 1925)



4D Coxeter polytope with 120 do-decahedral faces

- In a **1997** paper, Coxeter showed his friend M.C. Escher, knowing no math, had achieved "mathematical perfection" in etching *Circle Limit III*. "Escher did it by instinct," Coxeter wrote, "I did it by trigonometry."

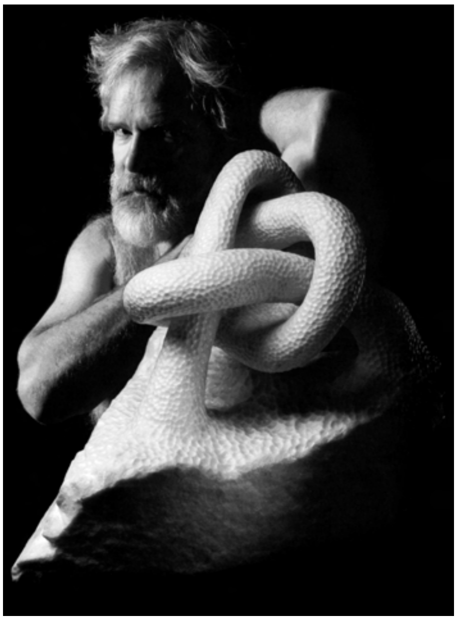
David Mumford recently noted that Donald Coxeter placed great value on working out details of complicated explicit examples:

In my book, Coxeter has been one of the most important 20th century mathematicians —not because he started a new perspective, but because he deepened and extended so beautifully an older esthetic. The classical goal of geometry is the exploration and enumeration of geometric configurations of all kinds, their symmetries and the constructions relating them to each other. The goal is not especially to prove theorems but to discover these perfect objects and, in doing this,theorems are only a tool that imperfect humans need to reassure themselves that they have seen them correctly. (David Mumford, 2003)

20th C. MATHEMATICAL MODELS

Ferguson's **"Eight-Fold Way"** sculpture

The Fergusons won the 2002 Communications Award, of the Joint Policy Board of Mathematics. The citation runs:



> They *have dazzled the mathematical community and a far wider public with exquisite sculptures embodying mathematical ideas, along with artful and accessible essays and lectures elucidating the mathematical concepts.*

It has been known for some time that the *hyperbolic volume* V of the **figure-eight knot complement** is

$$V = 2\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}} \sum_{k=n}^{2n-1} \frac{1}{k}$$

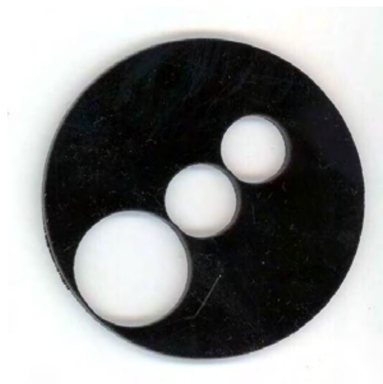
$$= 2.029883212819307250042405108549\ldots$$

Ferguson's **“Figure-Eight Knot Complement”**
sculpture

In 1998, British physicist David Broadhurst conjectured $V/\sqrt{3}$ is a *rational linear combination* of

$$C_j = \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n(6n+j)^2}. \tag{19}$$

Ferguson's subtractive image of the **BBP Pi formula**



Indeed, as Broadhurst found, using PSLQ (*Ferguson's Integer Relation Algorithm*):

$$V = \frac{\sqrt{3}}{9} \sum_{n=0}^{\infty} \frac{(-1)^n}{27^n} \times \left\{ \frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2} \right\}.$$

- Entering the following code in the *Mathematician's Toolkit*, at **www.expmath.info**:

```
v = 2 * sqrt[3] * sum[1/(n*binomial[2*n,n])
  * sum[1/k,{k, n,2*n-1}], {n, 1, infinity}]

  pslq[v/sqrt[3],
  table[sum[(-1)^n/(27^n*(6*n+j)^2),
  {n, 0, infinity}], {j, 1, 6}]]
```

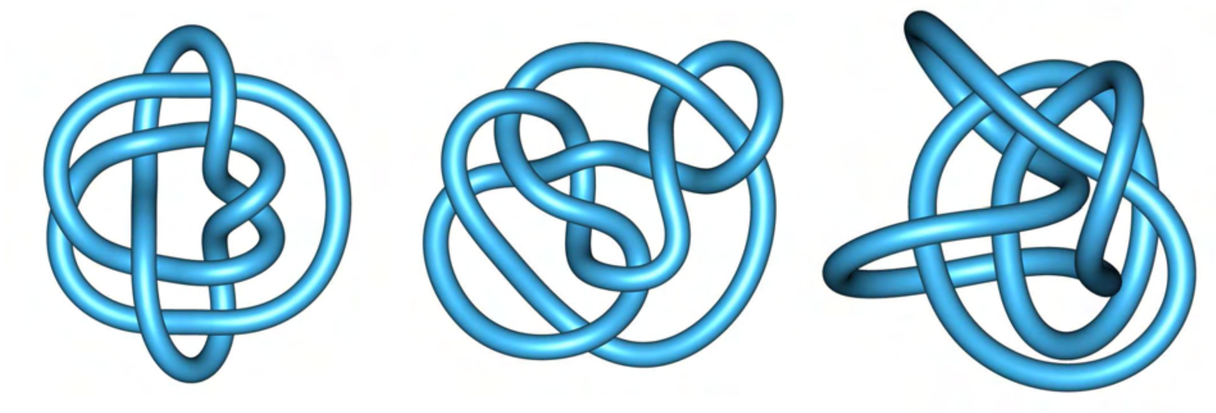
  recovers the solution vector

  (9, -18, 18, 24, 6, -2, 0)

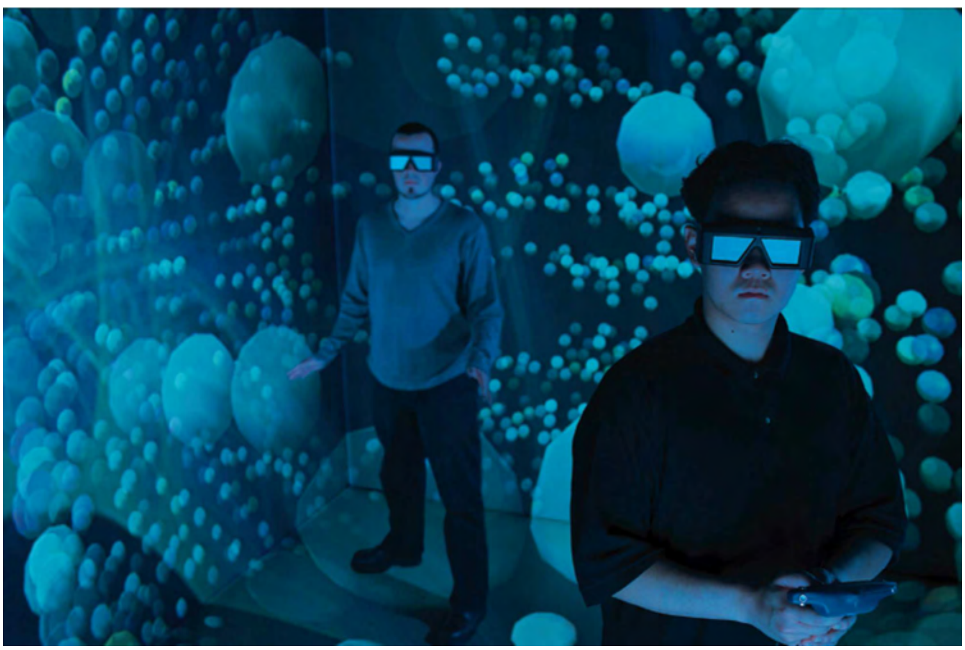
- The *first proof* that this formula holds is given in our recent book

- The formula is inscribed on each cast of the sculpture-marrying both sides of Helaman!

21st C. MATHEMATICAL MODELS



Knots 10_{161} (L) and 10_{162} (C) agree (R)*



In a NewMedia Cave or Plato's?

*KnotPlot: from Little (1899) to Perko (1974) and on