Maximality of the sum of a maximally monotone linear relation and a maximally monotone operator

Dedicated to Petar Kenderov on the occasion of his seventieth birthday

Jonathan M. Borwein* and Liangjin Yao†

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Abstract

The most famous open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximally monotone operators provided that Rockafellar's constraint qualification holds.

In this paper, we prove the maximal monotonicity of $A+B$ provided that A, B are maximally monotone and A is a linear relation, as soon as Rockafellar's constraint qualification holds: $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. Moreover, $A+B$ is of type (FPV).

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1 Introduction

Throughout this paper, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from X to X^*, i.e., for

*CARMA, University of Newcastle, Newcastle, New South Wales 2308, Australia. E-mail: `jonathan.borwein@newcastle.edu.au`. Laureate Professor at the University of Newcastle and Distinguished Professor at King Abdul-Aziz University, Jeddah.

†CARMA, University of Newcastle, Newcastle, New South Wales 2308, Australia. E-mail: `liangjin.yao@newcastle.edu.au`.

every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of A. Recall that A is *monotone* if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A \; \forall (y,y^*) \in \operatorname{gra} A, \tag{1}$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say (x,x^*) is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is *of type (FPV)* if for every open convex set $U \subseteq X$ such that $U \cap \operatorname{dom} A \neq \varnothing$, the implication

$$x \in U \text{ and } (x,x^*) \text{ is monotonically related to } \operatorname{gra} A \cap (U \times X^*) \Rightarrow (x,x^*) \in \operatorname{gra} A$$

holds. We say A is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. Monotone operators have proven to be important objects in modern Optimization and Analysis; see, e.g., the books [2, 8, 11, 12, 15, 21, 22, 19, 32, 33, 34] and the references therein. We adopt standard notation used in these books: $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$ is the *domain* of A. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C, $\operatorname{bdry} C$ is the *boundary* of C, $\operatorname{aff} C$ is the *affine hull* of C and $\overline{C}$ is the norm *closure* of C. We set $C^\perp := \{x^* \in X^* \mid (\forall c \in C)\, \langle x^*, c\rangle = 0\}$ and $S^\perp := \{x^{**} \in X^{**} \mid (\forall s \in S)\, \langle x^{**}, s\rangle = 0\}$ for a set $S \subseteq X^*$. We define ${}^{ic}C$ by

$${}^{ic}C := \begin{cases} {}^{i}C, & \text{if } \operatorname{aff} C \text{ is closed;} \\ \varnothing, & \text{otherwise,} \end{cases}$$

where ${}^{i}C$ [32] is the *intrinsic core* or *relative algebraic interior* of C, defined by ${}^{i}C := \{a \in C \mid \forall x \in \operatorname{aff}(C-C), \exists \delta > 0, \forall \lambda \in [0,\delta] : a + \lambda x \in C\}$.

The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{otherwise.} \end{cases} \tag{2}$$

If $D \subseteq X$, we set $C - D = \{x - y \mid x \in C, y \in D\}$. For every $x \in X$, the normal cone operator of C at x is defined by $N_C(x) := \big\{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^*\rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) = \varnothing$, if $x \notin C$. For $x, y \in X$, we set $[x,y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. Given $f\colon X \to\,]-\infty, +\infty]$, we set $\operatorname{dom} f := f^{-1}(\mathbb{R})$. We say f is *proper* if $\operatorname{dom} f \neq \varnothing$. Let f be proper. Then $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\, \langle y - x, x^*\rangle + f(x) \leq f(y)\big\}$ is the *subdifferential operator* of f. We also set $P_X : X \times X^* \to X\colon (x,x^*) \mapsto x$. Finally, the *open unit ball* in X is denoted by $U_X := \big\{x \in X \mid \|x\| < 1\big\}$, the *closed unit ball* in X is denoted by $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, and $\mathbb{N} := \{1, 2, 3, \ldots\}$. We denote by $\longrightarrow$ and $\rightharpoonup_{\mathrm{w}^*}$ the norm convergence and weak* convergence of nets, respectively.

Let A and B be maximally monotone operators from X to X^*. Clearly, the *sum operator* $A + B\colon X \rightrightarrows X^*\colon x \mapsto Ax + Bx := \big\{a^* + b^* \mid a^* \in Ax \text{ and } b^* \in Bx\big\}$ is monotone. Rockafellar established the following very important result in 1970.

Theorem 1.1 (Rockafellar's sum theorem) (See [18, Theorem 1] or [8].) *Suppose that X is reflexive. Let $A, B : X \rightrightarrows X^*$ be maximally monotone. Assume that A and B satisfy the classical* constraint qualification

$$\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing.$$

Then $A + B$ is maximally monotone.

The most significant open problem in the theory concerns the maximal monotonicity of the sum of two maximally monotone operators in general Banach spaces, which is called the "sum problem". Some recent developments on the sum problem can be found in Simons' monograph [22] and [5, 6, 7, 8, 10, 28, 14, 25, 29, 30, 31]. It is known, among other things, that the sum theorem holds under Rockafellar's constraint qualification when both operators are of dense type or when each operator has nonempty domain interior [8, Ch. 8] and [27].

Here we focus on the case when A is a maximally monotone linear relation, and B is maximally monotone such that $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. In Theorem 3.1 we shall show that $A+B$ is maximally monotone.

The remainder of this paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. The proof of our main result (Theorem 3.1) forms the bulk of Section 3. In Section 4 various other consequences and related results are presented.

2 Auxiliary Results

We start with a well known result from Rockafellar.

Fact 2.1 (Rockafellar) (See [17, Theorem 1] or [22, Theorem 27.1 and Theorem 27.3].) *Let $A : X \rightrightarrows X^*$ be maximal monotone with* $\operatorname{int} \operatorname{dom} A \neq \varnothing$. *Then* $\operatorname{int} \operatorname{dom} A = \operatorname{int} \overline{\operatorname{dom} A}$ *and* $\overline{\operatorname{dom} A}$ *is convex.*

The Fitzpatrick function below is a very useful tool in Monotone Operator Theory, which by now has been applied comprehensively.

Fact 2.2 (Fitzpatrick) (See [13, Corollary 3.9].) *Let $A\colon X \rightrightarrows X^*$ be maximally monotone, and set*

(3) $$F_A\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big),$$

the Fitzpatrick function *associated with A. Then for every $(x, x^*) \in X \times X^*$, the inequality $\langle x, x^*\rangle \leq F_A(x, x^*)$ is true, and equality holds if and only if $(x, x^*) \in \operatorname{gra} A$.*

Fact 2.3 (See [26, Theorem 3.4 and Corollary 5.6], or [22, Theorem 24.1(b)].) *Let* $A, B : X \rightrightarrows X^*$ *be maximally monotone operators. Assume* $\bigcup_{\lambda>0} \lambda [P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)]$ *is a closed subspace. If*

$$F_{A+B} \geq \langle \cdot, \cdot \rangle \text{ on } \quad X \times X^*, \tag{4}$$

then $A+B$ *is maximally monotone.*

We next introduce some properties of type (FPV) operators.

Fact 2.4 (Simons) (See [22, Theorem 46.1].) *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation. Then* A *is of type (FPV).*

Fact 2.5 (Simons and Verona-Verona) (See [22, Theorem 44.1] or [23].) *Let* $A : X \rightrightarrows X^*$ *be maximally monotone. Suppose that for every closed convex subset* C *of* X *with* $\operatorname{dom} A \cap \operatorname{int} C \neq \varnothing$, *the operator* $A + N_C$ *is maximally monotone. Then* A *is of type (FPV).*

Next we present a useful sufficient condition for the sum problem to have a positive resolution (see also [6]).

Fact 2.6 (Voisei and Zălinescu) (See [28, Corollary 4].) *Let* $A, B : X \rightrightarrows X^*$ *be maximally monotone. Assume that* ${}^{ic}(\operatorname{dom} A) \neq \varnothing, {}^{ic}(\operatorname{dom} B) \neq \varnothing$ *and* $0 \in {}^{ic}[\operatorname{dom} A - \operatorname{dom} B]$. *Then* $A+B$ *is maximally monotone.*

Fact 2.7 (See [4, Lemma 2.9].) *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $z \in X \cap (A0)^{\perp}$. *Then* $z \in \overline{\operatorname{dom} A}$.

Fact 2.8 (See [3, Lemma 2.5].) *Let* C *be a nonempty closed convex subset of* X *such that* $\operatorname{int} C \neq \varnothing$. *Let* $c_0 \in \operatorname{int} C$ *and suppose that* $z \in X \smallsetminus C$. *Then there exists* $\lambda \in \left]0,1\right[$ *such that* $\lambda c_0 + (1-\lambda)z \in \operatorname{bdry} C$.

Fact 2.9 (Boundedness below) (See [9, Fact 4.1].) *Let* $A : X \rightrightarrows X^*$ *be monotone and* $x \in \operatorname{int} \operatorname{dom} A$. *Then there exist* $\delta > 0$ *and* $M > 0$ *such that* $x + \delta B_X \subseteq \operatorname{dom} A$ *and* $\sup_{a \in x + \delta B_X} \|Aa\| \leq M$. *Assume that* (z, z^*) *is monotonically related to* $\operatorname{gra} A$. *Then*

$$\langle z - x, z^* \rangle \geq \delta \|z^*\| - (\|z - x\| + \delta) M. \tag{5}$$

Before we turn to our main result, we need the following technical lemma.

Lemma 2.10 *Let* $A : X \rightrightarrows X^*$ *be a monotone linear relation, and let* $B : X \rightrightarrows X^*$ *be a maximally monotone operator. Suppose that* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. *Suppose also that* $(z, z^*) \in X \times X^*$ *is monotonically related to* $\operatorname{gra}(A+B)$, *and that* $z \in \operatorname{dom} A$. *Then* $z \in \operatorname{dom} B$.

Proof. We can and do suppose that $(0,0) \in \operatorname{gra} A \cap \operatorname{gra} B$ and $0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B$. Suppose to the contrary that $z \notin \operatorname{dom} B$. Then we have $z \neq 0$. We claim that

$$N_{[0,z]} + B \quad \text{is maximally monotone.} \tag{6}$$

Since $z \neq 0$, then we have $\frac{1}{2}z \in {}^{ic}(\operatorname{dom} N_{[0,z]})$. Clearly, ${}^{ic}(\operatorname{dom} B) \neq \varnothing$ and $0 \in {}^{ic}[\operatorname{dom} A - \operatorname{dom} B]$. By Fact 2.6, $N_{[0,z]} + B$ is maximally monotone and hence (6) holds. Since $(z, z^*) \notin \operatorname{gra}(N_{[0,z]} + B)$, there exist $\lambda \in [0,1]$ and $x^*, y^* \in X^*$ such that $(\lambda z, x^*) \in \operatorname{gra} N_{[0,z]}$, $(\lambda z, y^*) \in \operatorname{gra} B$ and

$$\langle z - \lambda z, z^* - x^* - y^* \rangle < 0. \tag{7}$$

Since $(\lambda z, x^*) \in \operatorname{gra} B$ and $z \notin \operatorname{dom} B$, $\lambda < 1$. Then by (7),

$$\langle z, -x^* \rangle + \langle z, z^* - y^* \rangle = \langle z, z^* - x^* - y^* \rangle < 0. \tag{8}$$

Since $(\lambda z, x^*) \in \operatorname{gra} N_{[0,z]}$, we have $\langle z - \lambda z, x^* \rangle \leq 0$. Then $\langle z, -x^* \rangle \geq 0$. Thus (8) implies that

$$\langle z, z^* - y^* \rangle < 0. \tag{9}$$

Let $a^* \in A(\lambda z)$. By the assumption, we have

$$\langle z - \lambda z, z^* - a^* - y^* \rangle \geq 0.$$

Then we have $\langle z, z^* - a^* - y^* \rangle \geq 0$ and hence

$$\langle z, z^* - y^* \rangle \geq \langle z, a^* \rangle. \tag{10}$$

Now we show that

$$\langle z, a^* \rangle \geq 0. \tag{11}$$

We consider two cases.

Case 1: $\lambda = 0$. Then $a^* \in A0$. Since $z \in \operatorname{dom} A$ and A is monotone, [1, Proposition 5.1(i)] implies that $\langle z, a^* \rangle = 0$. Hence (10) holds.

Case 2: $\lambda \neq 0$. Since $(\lambda z, a^*) \in \operatorname{gra} A$, $\langle \lambda z, a^* \rangle \geq 0$ and hence $\langle z, a^* \rangle \geq 0$. Hence (11) holds.

Combining (10) and (11),

$$\langle z, z^* - y^* \rangle \geq 0, \quad \text{which contradicts (9).}$$

Hence $z \in \operatorname{dom} B$. ■

Remark 2.11 Lemma 2.10 generalizes [4, Lemma 2.10] in which B is assumed to be a convex subdifferential.

We now come to our central result.

3 Main Result

The proof of Theorem 3.1 in part follows that of [30, Theorem 3.1].

Theorem 3.1 (Linear sum theorem) *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $B : X \rightrightarrows X^*$ *be maximally monotone. Suppose that* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$*. Then* $A + B$ *is maximally monotone.*

Proof. After translating the graphs if necessary, we can and do assume that $0 \in \operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B$ and that $(0, 0) \in \operatorname{gra} A \cap \operatorname{gra} B$. By Fact 2.2, $\operatorname{dom} A \subseteq P_X(\operatorname{dom} F_A)$ and $\operatorname{dom} B \subseteq P_X(\operatorname{dom} F_B)$. Hence,

$$\bigcup_{\lambda>0} \lambda\big(P_X(\operatorname{dom} F_A) - P_X(\operatorname{dom} F_B)\big) = X. \tag{12}$$

Thus, by Fact 2.3, it suffices to show that

$$F_{A+B}(z, z^*) \geq \langle z, z^*\rangle, \quad \forall (z, z^*) \in X \times X^*. \tag{13}$$

Take $(z, z^*) \in X \times X^*$. Then

$$\begin{aligned} &F_{A+B}(z, z^*) \\ &= \sup_{\{x,x^*,y^*\}} \left[\langle x, z^*\rangle + \langle z, x^*\rangle - \langle x, x^*\rangle + \langle z - x, y^*\rangle - \iota_{\operatorname{gra} A}(x, x^*) - \iota_{\operatorname{gra} B}(x, y^*)\right]. \end{aligned} \tag{14}$$

Assume to the contrary that

$$F_{A+B}(z, z^*) + \lambda < \langle z, z^*\rangle, \tag{15}$$

where $\lambda > 0$.

Now by (15),

$$(z, z^*) \text{ is monotonically related to } \operatorname{gra}(A + B). \tag{16}$$

We claim that

$$z \notin \operatorname{dom} A. \tag{17}$$

Indeed, if $z \in \operatorname{dom} A$, apply (16) and Lemma 2.10 to get $z \in \operatorname{dom} B$. Thus $z \in \operatorname{dom} A \cap \operatorname{dom} B$ and hence $F_{A+B}(z, z^*) \geq \langle z, z^*\rangle$ which contradicts (15). This establishes (17).

By (15) and the assumption that $(0, 0) \in \operatorname{gra} A \cap \operatorname{gra} B$, we have

$$\sup\left[\langle 0, z^*\rangle + \langle z, A0\rangle - \langle 0, A0\rangle + \langle z, B0\rangle\right] = \sup_{a^* \in A0, b^* \in B0} \left[\langle z, a^*\rangle + \langle z, b^*\rangle\right] < \langle z, z^*\rangle.$$

Thus, since $A0$ is a linear subspace,

$$z \in X \cap (A0)^{\perp}. \tag{18}$$

Then, by Fact 2.7, we have

$$z \in \overline{\operatorname{dom} A}. \tag{19}$$

Combining (17) and (19), we have

$$z \in \overline{\operatorname{dom} A} \backslash \operatorname{dom} A. \tag{20}$$

Set

$$U_n := z + \tfrac{1}{n} U_X, \quad \forall n \in \mathbb{N}. \tag{21}$$

By (20), $(z, z^*) \notin \operatorname{gra} A$ and $U_n \cap \operatorname{dom} A \neq \varnothing$. Since $z \in U_n$ and A is type of (FPV) by Fact 2.4, there exists $(a_n, a_n^*)_{n\in\mathbb{N}}$ in $\operatorname{gra} A$ with $a_n \in U_n, n \in \mathbb{N}$ such that

$$\langle z, a_n^*\rangle + \langle a_n, z^*\rangle - \langle a_n, a_n^*\rangle > \langle z, z^*\rangle, \quad \forall n \in \mathbb{N}. \tag{22}$$

Then by (15) and (18) we have

$$a_n \neq 0, \quad \forall n \in \mathbb{N} \quad \text{and} \quad a_n \longrightarrow z. \tag{23}$$

Now we claim that

$$z \in \overline{\operatorname{dom} B}. \tag{24}$$

Suppose to the contrary that $z \notin \overline{\operatorname{dom} B}$.

As (23), there exists $K_1 \in \mathbb{N}$ such that $a_n \notin \operatorname{dom} B, \forall n \geq K_1$. For convenience, we can and do suppose that

$$a_n \notin \operatorname{dom} B, \quad \forall n \in \mathbb{N}. \tag{25}$$

Since $0 \in \operatorname{int} \operatorname{dom} B$, by Fact 2.8 and Fact 2.1, there exists $\delta \in]0, 1[$ such that

$$\delta z \in \operatorname{bdry} \overline{\operatorname{dom} B}. \tag{26}$$

Now consider the operator: $B + N_{[0,a_n]}$. Following the corresponding lines of the proof of Lemma 2.10 and by (23), $B + N_{[0,a_n]}$ is maximally monotone for every $n \in \mathbb{N}$.

Because $a_n \notin \operatorname{dom} B$ by (25), $a_n \notin \operatorname{dom} B \cap [0, a_n] = \operatorname{dom}(B + N_{[0,a_n]})$ for every $n \in \mathbb{N}$. Thus, $(a_n, z^*) \notin \operatorname{gra}(B + N_{[0,a_n]})$. Thence there exist $\beta_n \in [0, 1]$, $w_n^* \in B(\beta_n a_n)$ and $v_n^* \in N_{[0,a_n]}(\beta_n a_n)$ such that

$$\langle a_n - \beta_n a_n, z^* - w_n^*\rangle < \langle a_n - \beta_n a_n, v_n^*\rangle \leq 0, \quad \forall n \in \mathbb{N}. \tag{27}$$

Since $\beta_n \in [0,1]$, there is a convergent subsequence of $(\beta_n)_{n\in\mathbb{N}}$, which, for convenience, we still denote by $(\beta_n)_{n\in\mathbb{N}}$. Now $\beta_n \longrightarrow \beta$, where $\beta \in [0,1]$. Then by (23),

$$\beta_n a_n \longrightarrow \beta z. \tag{28}$$

We claim that

$$\beta \leq \delta < 1. \tag{29}$$

Indeed, suppose to the contrary that $\beta > \delta$. By (28), $\beta z \in \overline{\operatorname{dom} B}$. Then by $0 \in \operatorname{int}\operatorname{dom} B$ and [32, Theorem 1.1.2(ii)], $\delta z = \frac{\delta}{\beta}\beta z \in \operatorname{int}\overline{\operatorname{dom} B}$, which contradicts (26). Hence (29) holds.

We can and do suppose that $\beta_n < 1$ for every $n \in \mathbb{N}$. By (27),

$$\langle a_n, z^* - w_n^* \rangle < 0, \quad \forall n \in \mathbb{N}. \tag{30}$$

Since $(0,0) \in \operatorname{gra} A$, $\langle a_n, a_n^* \rangle \geq 0, \forall n \in \mathbb{N}$. Then by (22), we have

$$\langle z, \beta_n a_n^* \rangle + \langle \beta_n a_n, z^* \rangle - \beta_n^2 \langle a_n, a_n^* \rangle \geq \langle \beta_n z, a_n^* \rangle + \langle \beta_n a_n, z^* \rangle - \beta_n \langle a_n, a_n^* \rangle \geq \beta_n \langle z, z^* \rangle. \tag{31}$$

Hence, by (31),

$$\langle z - \beta_n a_n, \beta_n a_n^* \rangle \geq \langle \beta_n z - \beta_n a_n, z^* \rangle. \tag{32}$$

Since $\operatorname{gra} A$ is a linear subspace and $(a_n, a_n^*) \in \operatorname{gra} A$, $(\beta_n a_n, \beta_n a_n^*) \in \operatorname{gra} A$. By (15), we have

$$\begin{aligned} \lambda <& \langle z - \beta_n a_n, z^* - w_n^* - \beta_n a_n^* \rangle = \langle z - \beta_n a_n, z^* - w_n^* \rangle + \langle z - \beta_n a_n, -\beta_n a_n^* \rangle \\ &\leq \langle z - \beta_n a_n, z^* - w_n^* \rangle - \langle \beta_n z - \beta_n a_n, z^* \rangle \quad \text{(by (32))}. \end{aligned}$$

Then

$$\lambda \leq \langle z - \beta_n a_n, z^* - w_n^* \rangle - \langle \beta_n z - \beta_n a_n, z^* \rangle. \tag{33}$$

We again consider two cases:

Case 1: $(w_n^*)_{n\in\mathbb{N}}$ is bounded. By the Banach-Alaoglu Theorem (see [20, Theorem 3.15]), there exist a weak* convergent *subnet* $(w_\gamma^*)_{\gamma\in\Gamma}$ of $(w_n^*)_{n\in\mathbb{N}}$ such that

$$w_\gamma^* \rightharpoonup_{\mathrm{w}^*} w_\infty^* \in X^*. \tag{34}$$

Combine (23), (28) and (34), we pass to the limit along the given subnet of (33) to deduce that

$$\lambda \leq \langle z - \beta z, z^* - w_\infty^* \rangle. \tag{35}$$

By (29), on dividing by $(1-\beta)$ on both sides of (35) we get

$$\langle z, z^* - w_\infty^* \rangle \geq \frac{\lambda}{1-\beta} > 0. \tag{36}$$

On the other hand, by (23) and (34), on passing to the limit along the same subnet in (30) we see that

$$\langle z, z^* - w_\infty^* \rangle \leq 0, \tag{37}$$

which contradicts (36).

Case 2: $(w_n^*)_{n\in\mathbb{N}}$ is unbounded. After passing to a subsequence if necessary, we assume that $\|w_n^*\| \neq 0, \forall n \in \mathbb{N}$ and that $\|w_n^*\| \longrightarrow +\infty$. By the Banach-Alaoglu Theorem again, there exist a weak* convergent *subnet* $(w_\nu^*)_{\nu\in I}$ of $(w_n^*)_{n\in\mathbb{N}}$ such that

$$\frac{w_\nu^*}{\|w_\nu^*\|} \rightharpoondown_{\mathrm{w}^*} y_\infty^* \in X^*. \tag{38}$$

As $0 \in \operatorname{int}\operatorname{dom} B$ and using Fact 2.9, there exist $\rho > 0$ and $M > 0$ such that

$$\langle \beta_n a_n, \tfrac{w_n^*}{\|w_n^*\|} \rangle \geq \rho - \frac{(\|\beta_n a_n\| + \rho)M}{\|w_n^*\|}, \quad \forall n \in \mathbb{N}. \tag{39}$$

Combining (28) and (38),and taking the limit in (39) along the subnet, we obtain

$$\langle \beta z, y_\infty^* \rangle \geq \rho. \tag{40}$$

Then we have $\beta \neq 0$ and thus $\beta > 0$. By (40),

$$\langle z, y_\infty^* \rangle \geq \frac{\rho}{\beta} > 0. \tag{41}$$

Dividing by $\|w_n^*\|$ in (33) and taking the weak* limit in (33) along the subnet, it follows from (28) and (38) that

$$\langle z - \beta z, -y_\infty^* \rangle \geq 0. \tag{42}$$

By (29),

$$\langle z, y_\infty^* \rangle \leq 0, \quad \text{which contradicts (41).}$$

Combining all the cases above, we obtain $z \in \overline{\operatorname{dom} B}$.

Next, we show that

$$F_{A+B}(tz, tz^*) \geq t^2\langle z, z^* \rangle, \quad \forall t \in \left]0, 1\right[. \tag{43}$$

Let $t \in \left]0,1\right[$. By $0 \in \operatorname{int}\operatorname{dom} B$, Fact 2.1 and [32, Theorem 1.1.2(ii)], we have

$$tz \in \operatorname{int}\overline{\operatorname{dom} B}. \tag{44}$$

Fact 2.1 implies that

$$tz \in \operatorname{int}\operatorname{dom} B. \tag{45}$$

Set

$$H_n := tz + \tfrac{1}{n}U_X, \quad \forall n \in \mathbb{N}.$$

Since $\operatorname{dom} A$ is a linear subspace, $tz \in \overline{\operatorname{dom} A} \backslash \operatorname{dom} A$ by (20). Then $H_n \cap \operatorname{dom} A \neq \varnothing$. Since $(tz, tz^*) \notin \operatorname{gra} A$ and $tz \in H_n$, and A is of type (FPV) by Fact 2.4, there exists $(b_n, b_n^*)_{n\in\mathbb{N}}$ in $\operatorname{gra} A$ such that $b_n \in H_n$ and

$$\langle tz, b_n^*\rangle + \langle b_n, tz^*\rangle - \langle b_n, b_n^*\rangle > t^2\langle z, z^*\rangle, \quad \forall n \in \mathbb{N}. \tag{46}$$

As $tz \in \operatorname{int}\operatorname{dom} B$ and $b_n \longrightarrow tz$, by Fact 2.9, there exist $N \in \mathbb{N}$ and $K > 0$ such that

$$b_n \in \operatorname{int}\operatorname{dom} B \quad \text{and} \quad \sup_{v^*\in B(b_n)} \|v^*\| \leq K, \quad \forall n \geq N. \tag{47}$$

Hence

$$\begin{aligned}
F_{A+B}(tz, tz^*) &\geq \sup_{\{c^*\in B(b_n)\}} \left[\langle b_n, tz^*\rangle + \langle tz, b_n^*\rangle - \langle b_n, b_n^*\rangle + \langle tz - b_n, c^*\rangle\right], \quad \forall n \geq N\\
&\geq \sup_{\{c^*\in B(b_n)\}} \left[t^2\langle z, z^*\rangle + \langle tz - b_n, c^*\rangle\right], \quad \forall n \geq N \quad (\text{by } (46))\\
&\geq \sup\left[t^2\langle z, z^*\rangle - K\|tz - b_n\|\right], \quad \forall n \geq N \quad (\text{by } (47))\\
&\geq t^2\langle z, z^*\rangle \quad (\text{by } b_n \longrightarrow tz). \qquad (48)
\end{aligned}$$

Hence $F_{A+B}(tz, tz^*) \geq t^2\langle z, z^*\rangle$.

We have proved that (43) holds. Since $(0,0) \in \operatorname{gra}(A+B)$ and $A+B$ is monotone, we have $F_{A+B}(0,0) = \langle 0, 0\rangle = 0$. Since F_{A+B} is convex, (43) implies that

$$tF_{A+B}(z, z^*) = tF_{A+B}(z, z^*) + (1-t)F_{A+B}(0,0) \geq F_{A+B}(tz, tz^*) \geq t^2\langle z, z^*\rangle, \quad \forall t \in \left]0, 1\right[.$$

Letting $t \longrightarrow 1^-$ in the above inequality, we obtain

$$F_{A+B}(z, z^*) \geq \langle z, z^*\rangle. \tag{49}$$

Therefore, (13) holds, and $A+B$ is maximally monotone. ■

Remark 3.2 Theorem 3.1 generalizes the main results in [3, 4, 30].

We now establish the promised corollary:

Corollary 3.3 (FPV property of the sum) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Let $B : X \rightrightarrows X^*$ be maximally monotone. If $\operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \neq \varnothing$, then $A+B$ is of type (FPV).*

Proof. By Theorem 3.1, $A + B$ is maximally monotone. Let C be a nonempty closed convex subset of X, and suppose that $\operatorname{dom}(A + B) \cap \operatorname{int} C \neq \varnothing$. Let $x_1 \in \operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B$ and $x_2 \in \operatorname{dom}(A + B) \cap \operatorname{int} C$. Then $x_1, x_2 \in \operatorname{dom} A$, $x_1 \in \operatorname{int}\operatorname{dom} B$ and $x_2 \in \operatorname{dom} B \cap \operatorname{int} C$. Hence $\lambda x_1 + (1-\lambda)x_2 \in \operatorname{int}\operatorname{dom} B$ for every $\lambda \in]0,1]$ by Fact 2.1 and [32, Theorem 1.1.2(ii)] and so there exists $\delta \in]0,1]$ such that $\lambda x_1 + (1-\lambda)x_2 \in \operatorname{int} C$ for every $\lambda \in [0,\delta]$.

Thus, $\delta x_1 + (1-\delta)x_2 \in \operatorname{dom} A \cap \operatorname{int}\operatorname{dom} B \cap \operatorname{int} C$. By Fact 2.6 or [6, Theorem 9(i)], $B + N_C$ is maximally monotone. Then, by Theorem 3.1 (applied A and $B + N_C$ to A and B), $A + B + N_C = A + (B + N_C)$ is maximally monotone. By Fact 2.5, $A + B$ is of type (FPV). ■

Note that with $A = 0$ we recover the fact that a maximally monotone mapping is type (FPV) when its domain has nonempty interior.

Remark 3.4 The proof of Corollary 3.3 was adapted from that of [30, Corollary 3.3]. Moreover, Corollary 3.3 generalizes [30, Corollary 3.3].

4 Further Consequences

The next result reduces all sum theorems to linear ones.

Proposition 4.1 *Let* $A, B : X \rightrightarrows X^*$ *be monotone such that* $\operatorname{dom} A \cap \operatorname{dom} B \neq \varnothing$, *let*

$$C := \{(x,x) \in X \times X \mid x \in X\}.$$

Let $T : X \times X \rightrightarrows X^* \times X^*$ *be defined by*

$$T(x,y) := \big(Ax, By\big).$$

Then $A + B$ *is maximally monotone if and only if* $T + N_C$ *is maximally monotone.*

Proof. We have

$$N_C(x,x) = \big\{(x^*, -x^*) \mid x^* \in X^*\big\}, \quad \forall x \in X. \tag{50}$$

"⇒": Clearly, $T+N_C$ is monotone. Let $\big((x_0,y_0),(x_0^*,y_0^*)\big)\in(X\times X)\times(X^*\times X^*)$ be monotonically related to $\operatorname{gra}(T+N_C)$. Now we show that $\big((x_0,y_0),(x_0^*,y_0^*)\big)\in\operatorname{gra}(T+N_C)$. Then by (50),

$$\begin{aligned}
&\Big\langle(x_0,y_0)-(a,a),(x_0^*,y_0^*)-(a^*,b^*)-(x^*,-x^*)\Big\rangle\geq 0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\\
&\quad\forall x^*\in X^*\\
&\Rightarrow\Big\langle(x_0-a,y_0-a),(x_0^*-a^*-x^*,y_0^*-b^*+x^*)\Big\rangle\geq 0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\\
&\quad\forall x^*\in X^*\\
&\Rightarrow\langle x_0-a,x_0^*-a^*\rangle+\langle y_0-a,y_0^*-b^*\rangle\geq 0,\quad\langle x_0-a,-x^*\rangle+\langle y_0-a,x^*\rangle=0,\\
&\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\forall x^*\in X^*\\
&\Rightarrow\langle x_0-a,x_0^*-a^*\rangle+\langle y_0-a,y_0^*-b^*\rangle\geq 0,\ x_0=y_0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B\\
&\Rightarrow\langle x_0-a,x_0^*+y_0^*-a^*-b^*\rangle\geq 0,\ x_0=y_0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B\\
&\Rightarrow x_0^*+y_0^*\in(A+B)x_0,\ x_0=y_0\quad\text{(since } A+B \text{ is maximal monotone)}\\
&\Rightarrow\exists v^*\in X^*,\quad x_0^*+v^*\in Ax_0,y_0^*-v^*\in Bx_0,\ x_0=y_0\\
&\Rightarrow\big((x_0,y_0),(x_0^*,y_0^*)\big)\in\operatorname{gra}(T+N_C)\quad\text{(by (50))}.
\end{aligned}$$

.

Hence $T+N_C$ is maximally monotone.

"⇐": Let $(z,z^*)\in X\times X^*$ be monotonically related to $\operatorname{gra}(A+B)$.

$$\begin{aligned}
&\langle z-a,z^*-a^*-b^*\rangle\geq 0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B\\
&\Rightarrow\langle z-a,\frac{z^*}{2}-a^*\rangle+\langle z-a,\frac{z^*}{2}-b^*\rangle+\langle z-a,-x^*\rangle+\langle z-a,x^*\rangle\geq 0,\\
&\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\forall x^*\in X^*\\
&\Rightarrow\langle z-a,\frac{z^*}{2}-a^*-x^*\rangle+\langle z-a,\frac{z^*}{2}-b^*+x^*\rangle\geq 0,\\
&\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\forall x^*\in X^*\\
&\Rightarrow\Big\langle(z,z)-(a,a),(\frac{z^*}{2},\frac{z^*}{2})-(a^*,b^*)-(x^*,-x^*)\Big\rangle\geq 0,\quad\forall(a,a^*)\in\operatorname{gra}A,(a,b^*)\in\operatorname{gra}B,\\
&\quad\forall x^*\in X^*\\
&\Rightarrow\Big\langle(z,z)-\mathbf{w},(\frac{z^*}{2},\frac{z^*}{2})-\mathbf{w}^*\Big\rangle\geq 0,\quad\forall(\mathbf{w},\mathbf{w}^*)\in\operatorname{gra}(T+N_C)\quad\text{(by (50))}\\
&\Rightarrow(\frac{z^*}{2},\frac{z^*}{2})\in(T+N_C)(z,z)\quad\text{(since } T+N_C \text{ is maximal monotone)}\\
&\Rightarrow\exists v^*,\quad(\frac{z^*}{2},\frac{z^*}{2})\in(Az,Bz)+(v^*,-v^*)\quad\text{(by (50))}.\\
&\Rightarrow z^*=\frac{z^*}{2}+\frac{z^*}{2}\in(A+B)z.
\end{aligned}$$

Hence $A+B$ is maximally monotone. ■

It is important to note that exchanging the roles of A and B in Theorem 3.1 leads to a much tougher linear sum problem.

Remark 4.2 Let A, B be maximally monotone such that $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$ (i.e., they satisfy Rockafellar's constraint qualification), let T, C be defined as in Proposition 4.1. Then we have

$$\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} T - \operatorname{dom} N_C\right] = X \times X.$$

Note that T is maximally monotone (see the corresponding lines of proof of [10, Proposition 3.13]) and N_C is a maximally monotone linear relation. If the following conjecture is true then by Proposition 4.1, $A + B$ is maximally monotone and hence the general sum theorem holds.

> ***Conjecture*** *Let* $S : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $T : X \rightrightarrows X^*$ *be maximally monotone such that* $\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} S - \operatorname{dom} T\right] = X$. *Then* $S + T$ *is maximally monotone.*

In a related manner we have:

Corollary 4.3 *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $B : X \rightrightarrows X^*$ *be maximally monotone such that* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. *Let* $C := \{(x, x) \in X \times X \mid x \in X\}$ *and let* $T : X \times X \rightrightarrows X^* \times X^*$ *be defined by* $T(x, y) := \big(Ax, By\big)$. *Then* $T + N_C$ *is maximally monotone.*

Proof. Apply Theorem 3.1 and Proposition 4.1 directly. ■

In consequence, we are left with the following unresolved and interesting questions.

Open problem 4.4 *Let* $A : X \to X^*$ *be a continuous monotone linear operator, and let* $B : X \rightrightarrows X^*$ *be maximally monotone. Is* $A + B$ *necessarily maximally monotone* ?

Open problem 4.5 *Let* $f : X \to \left]-\infty, +\infty\right]$ *be a proper lower semicontinuous convex function, and let* $B : X \rightrightarrows X^*$ *be maximally monotone with* $\operatorname{dom} \partial f \cap \operatorname{int} \operatorname{dom} B \neq \varnothing$. *Is* $\partial f + B$ *necessarily maximally monotone* ?

Finally we recapitulate the conjecture after Remark 4.2.

Open problem 4.6 *Let* $A : X \rightrightarrows X^*$ *be a maximally monotone linear relation, and let* $B : X \rightrightarrows X^*$ *be maximally monotone such that* $\bigcup_{\lambda>0} \lambda \left[\operatorname{dom} A - \operatorname{dom} B\right] = X$. *Is* $A+B$ *necessarily maximally monotone* ?

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