MAA Online
The Mathematical Association of America
Carleton College

Honours Seminar based on
MAA Summer Seminar
Experimental Math in Action
(Carleton College July 15-20, 2007)

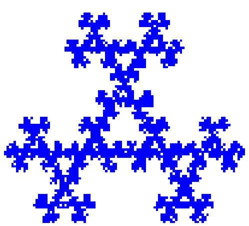
Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

``intuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.

Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning."

George Polya



DALHOUSIE UNIVERSITY
Inspiring Minds

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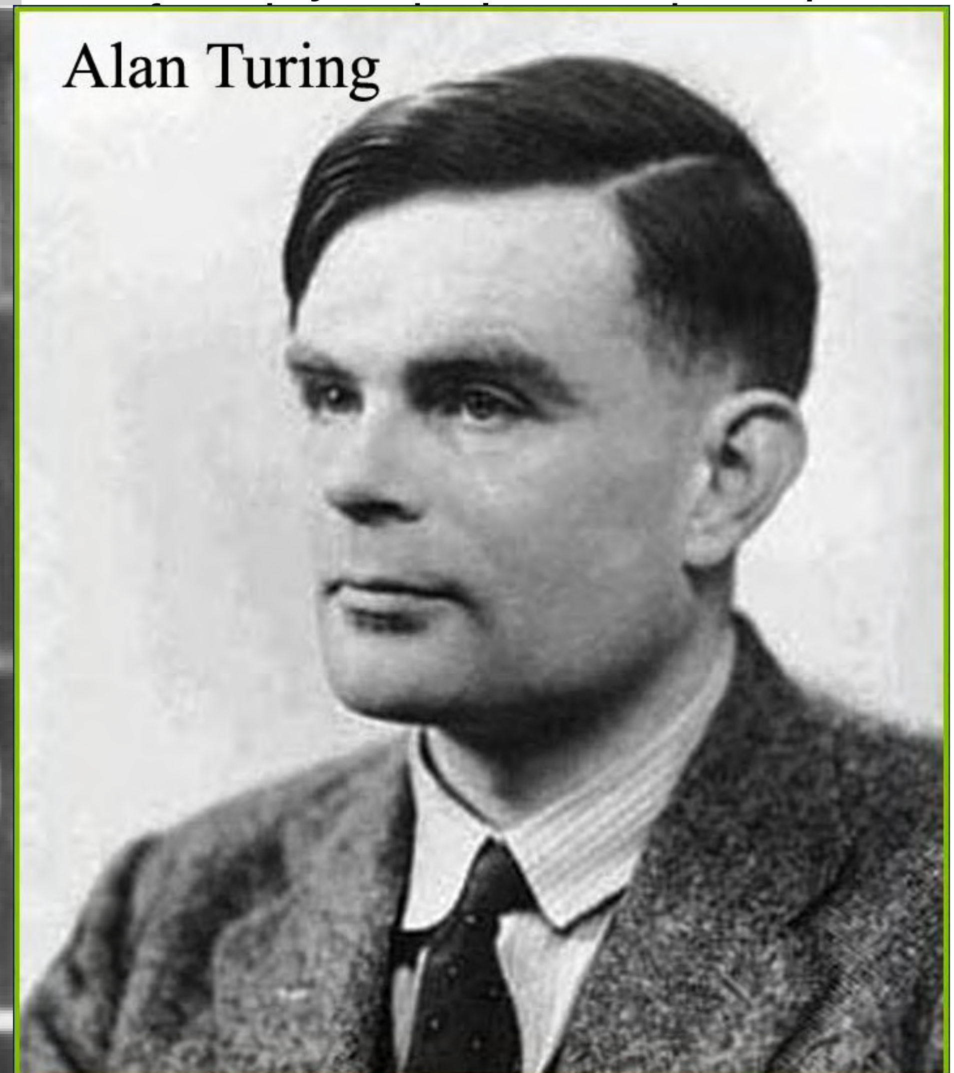
Experimental Mathematics in Action: Insight from Computation

Abstract: The influence of the computer on mathematics might be compared to the influence the discovery of the microscope had on biology, or the telescope on astronomy. Like those sciences we now have a tool that allows us to see previously unimaginable phenomena. We are still in the very early days of beginning to understand the effect and usefulness of this new tool.

"If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics." (**Kurt Godel,**1951)

Seventy-five years ago Godel (and Turing) overturned the mathematical apple cart entirely deductively; but he held quite



Jon Borwein's Math Resource Portal

The following is a list of useful math tools.

Utilities

1. ISC2.0: The Inverse Symbolic Calculator
2. EZ Face : An interface for evaluation of Euler sums and Multiple Zeta Values
3. 3D Function Grapher
4. GraPHedron: Automated and computer assisted conjectures in graph theory
5. Julia and Mandelbrot Set Explorer
6. Embree-Trefethen-Wright pseudospectra and eigenproblem

Reference

7. The On-Line Encyclopedia of Integer Sequences
8. Finch's Mathematical Constants
9. The Digital Library of Mathematical Functions
10. The Prime Pages

Content

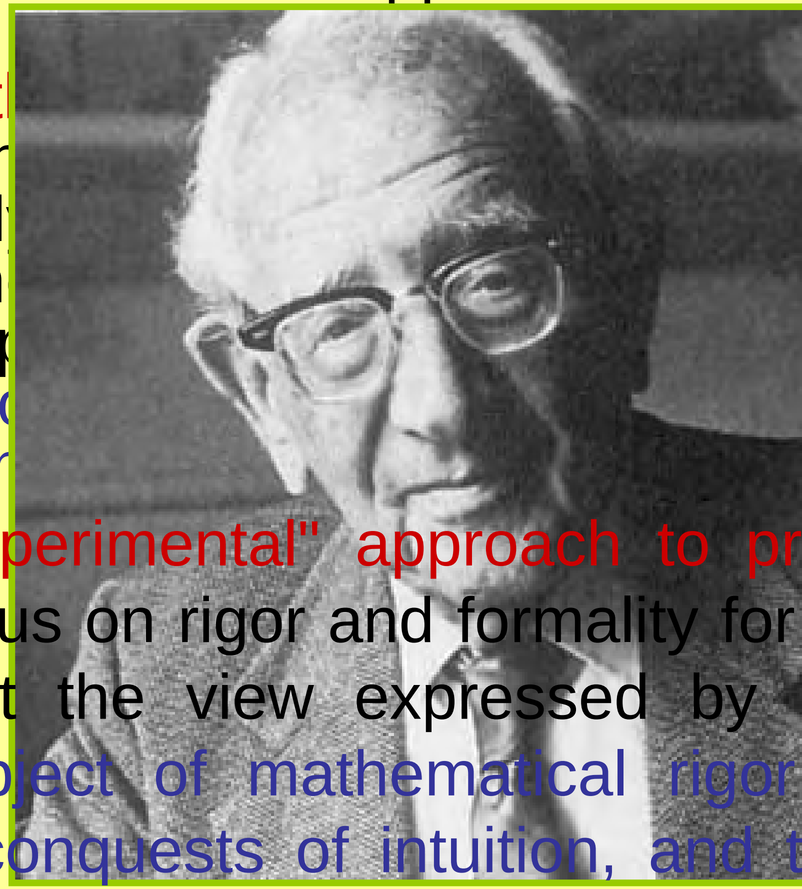
11. Experimental Mathematics Website
12. Wolfram Mathworld
13. Planet Math
14. Numbers, Constants, and Computation
15. Wikipedia: Mathematics

ICCOPT 2007 Short Course

16. Jon's Lectures

Polya Made Plausible by Computers

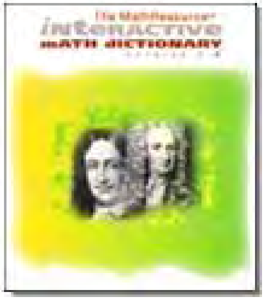
"A mathematical deduction appears to Descartes as a chain of
conclusions, … e steps. What is
needed for t… intuitive insight at
each step wh… on attained by that
step evidentl… lows from formerly
acquired kn… y by intuition or
indirectly by p… hat in teaching high
school age y… size intuitive insight
more than, ar… soning."



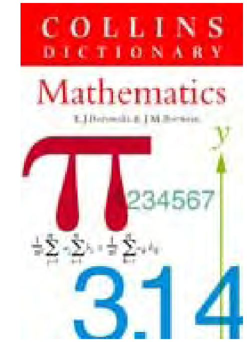
George Polya (1887-1985)

"This "quasi-experimental" approach to proof can help to de-emphasis a focus on rigor and formality for its own sake, and to instead support the view expressed by Hadamard when he stated "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.""

George Polya in ***Mathematical discovery***: *On understanding, learning, and teaching problem solving* (Combined Ed.), New York, Wiley, 1981.



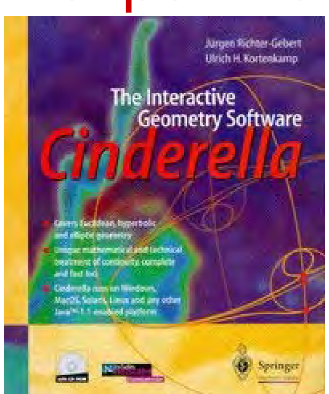
FURTHER ABSTRACT

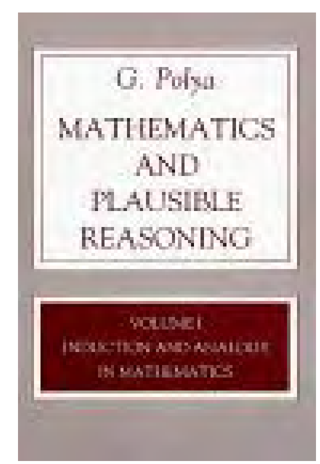


"RESOURCES not COURSES"

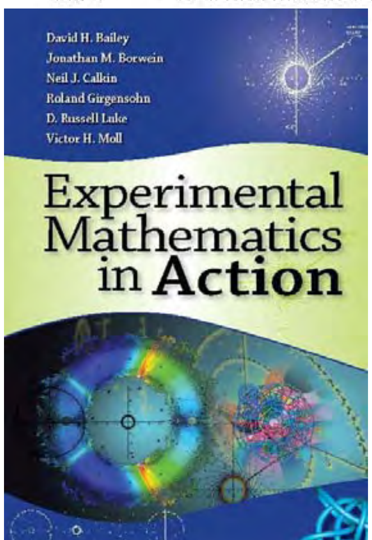
This introductory lecture is based on my new book **Experiment-al Math in Action** (also http://ddrive.cs.dal.ca/~isc/portal and www.experimentalmath.info) and my principal aim will be to expose you to the incredible mathematical insight that can be gained through computation and experimentation.

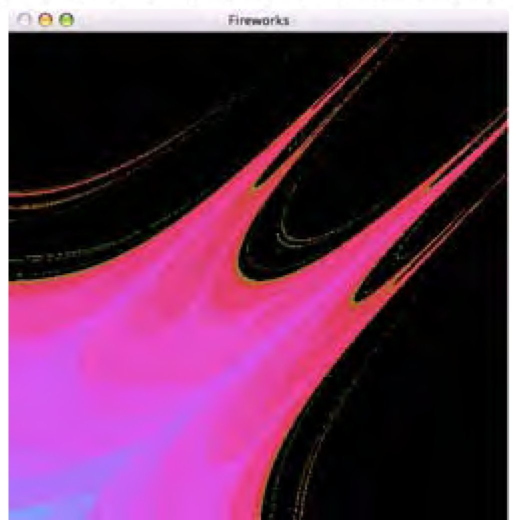
My goal here (and the book) is to "present a coherent variety of accessible examples of modern mathematics where intelligent computing plays a significant role and in so doing to highlight some of the key algorithms and to teach some of the key experimental approaches."

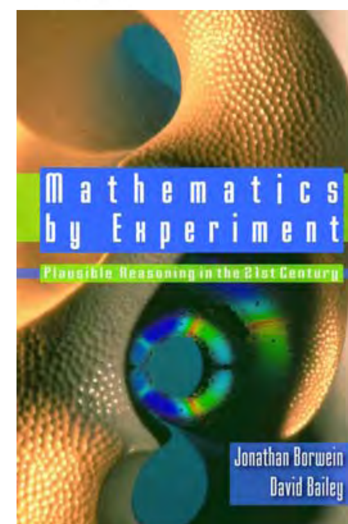




I. Experimental Mathematics a Philosophical Introduction







Experimental Mathematics in Action

David H. Bailey
Jonathan M. Borwein
Neil J. Calkin
Roland Girgensohn
D. Russell Luke
Victor H. Moll

The last twenty years have been witness to a fundamental shift in the way mathematics is practiced. With the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" no longer has any traction for a newer generation of mathematicians that can really take advantage of computer-aided research, especially given the scope and availability of modern computational packages such as Maple, Mathematica, and MATLAB. The authors provide a coherent variety of accessible examples of modern mathematics subjects in which intelligent computing plays a significant role.

Advance Praise for *Experimental Mathematics in Action*

"Experimental mathematics has not only come of age but is quickly maturing, as this book shows so clearly. The authors display a vast range of mathematical understanding and connection while at the same time delineating various ways in which experimental mathematics is and can be undertaken, with startling effect."

—**Prof. John Mason**, Open University and University of Oxford

"Computing is to mathematics as telescope is to astronomy: it might not explain things, but it certainly shows 'what's out there.' The authors are expert in the discovery of new mathematical 'planets,' and this book is a beautifully written exposé of their values, their methods, their subject, and their enthusiasm about it. A must read."

—**Prof. Herbert S. Wilf**, author of *generatingfunctionology*

"From within the ideological blizzard of the young field of Experimental Mathematics comes this tremendous, clarifying book. The authors—all experts—convey this complex new subject in the best way possible; namely, by fine example. Let me put it this way: Discovering this book is akin to finding an emerald in a snowdrift."

BAILEY
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LUKE
MOLL

Experimental Mathematics in Action

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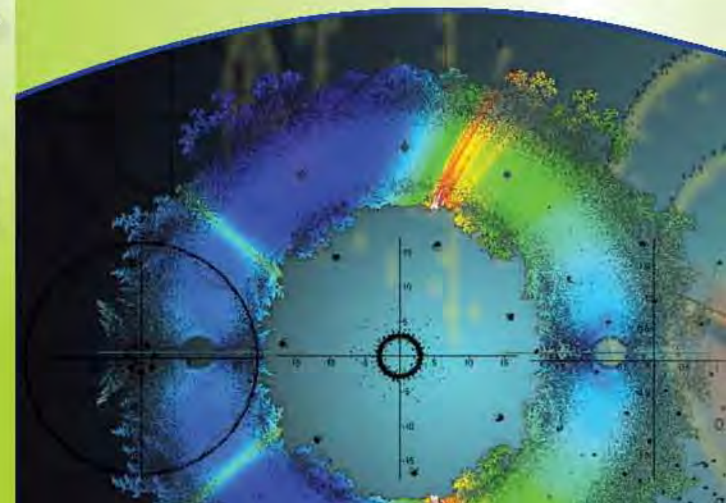


FOUR FORMS of EXPERIMENTS

Kantian examples: generating "*the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms.*"

The Baconian experiment is a contrived as opposed to a natural happening, it *"is the consequence of `trying things out' or even of merely messing about."*

Aristotelian demonstrations: *"apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."*

The most important is **Galilean**: "a *critical experiment -- one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."*

the only form which will make Experimental Mathematics a serious enterprise.

From Peter Medawar (1915-87) *Advice to a Young Scientist* (1979)

A Paraphrase of Hersh's Humanism

1. **Mathematics is human**. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.
2. **Mathematical knowledge is Fallible.** As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations.*
3. **There are different versions of proof or rigo**r. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977 (1997), is just one example of an emerging nontraditional standard of rigor.
4. **Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics**. Aristotelian logic isn't necessarily always the best way of deciding.
5. **Mathematical objects are a special variety of a social-cultural-historical object**. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

``Fresh Breezes in the Philosophy of Mathematics'', *MAA Monthly,* Aug 1995, 589-594.

A Paraphrase of

Ernest's Social Constructivism

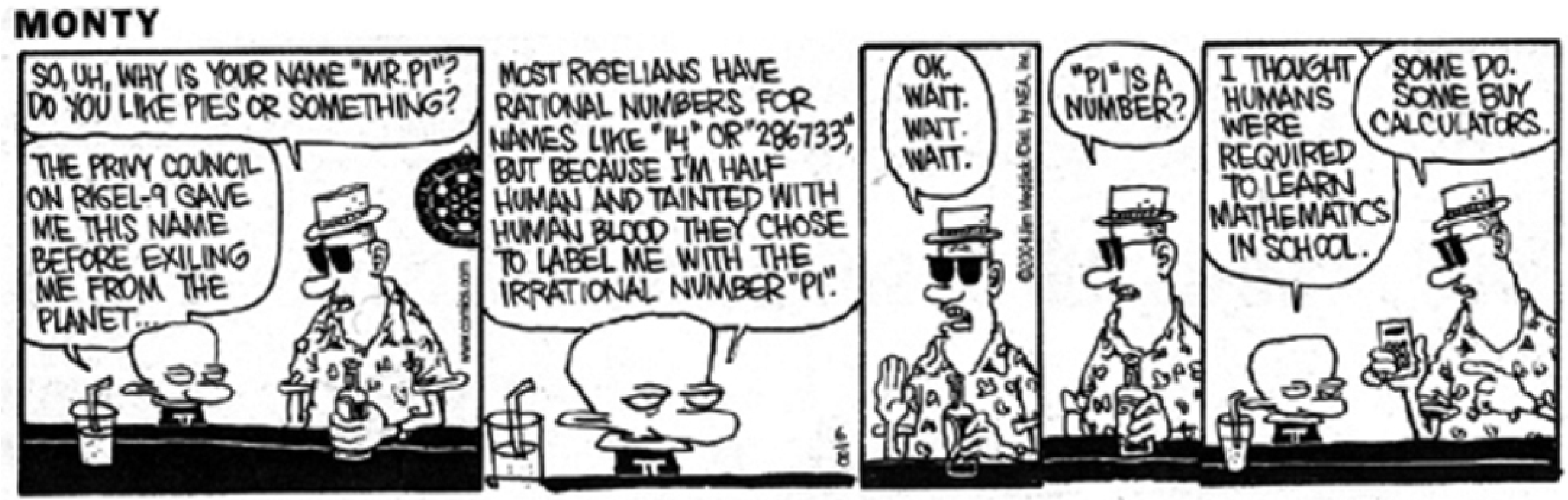
The idea that what is accepted as mathematical knowledge is, *to some degree*, dependent upon a community's methods of knowledge acceptance is central to the social constructivist school of mathematical philosophy.

> *The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge* (Paul Ernest)

Associated most notably with his *Social Constructivism as a Philosophy of Mathematics,* Ernest, an English Mathematician and Professor in Philosophy of Mathematics Education, carefully traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others, social constructivism seeks to define mathematical knowledge and epistemology through social structure and interaction of the mathematical community and society as a whole.

DISCLAIMER: Social Constructivism is not Cultural Relativism

Mr Pi


MONTY
SO, UH, WHY IS YOUR NAME "MR. PI"? DO YOU LIKE PIES OR SOMETHING?
THE PRIVY COUNCIL ON RIGEL-9 GAVE ME THIS NAME BEFORE EXILING ME FROM THE PLANET...
MOST RIGELIANS HAVE RATIONAL NUMBERS FOR NAMES LIKE "14" OR "286733", BUT BECAUSE I'M HALF HUMAN AND TAINTED WITH HUMAN BLOOD THEY CHOSE TO LABEL ME WITH THE IRRATIONAL NUMBER "PI".
OK. WAIT. WAIT. WAIT.
"PI" IS A NUMBER?
I THOUGHT HUMANS WERE REQUIRED TO LEARN MATHEMATICS IN SCHOOL.
SOME DO. SOME BUY CALCULATORS.

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures

   Detailed examples are given later

5. Exploring a possible result to see if it merits formal proof

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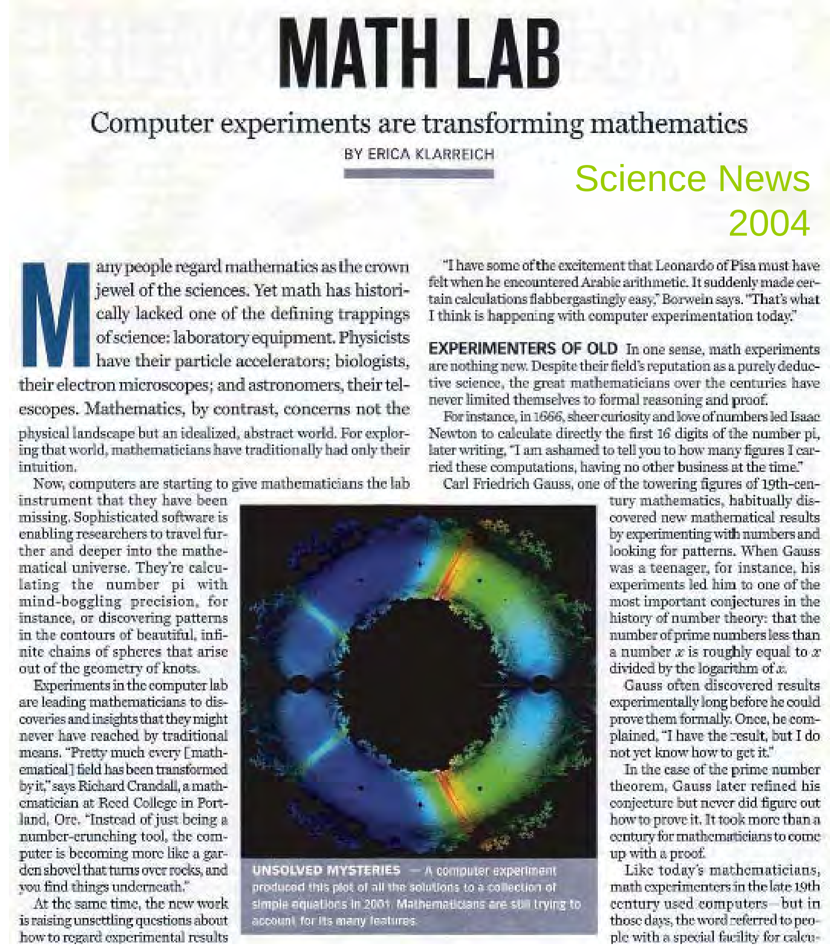
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results

Science News 2004



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

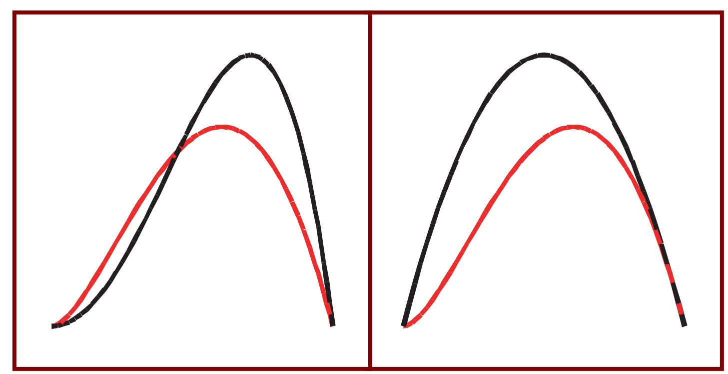
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

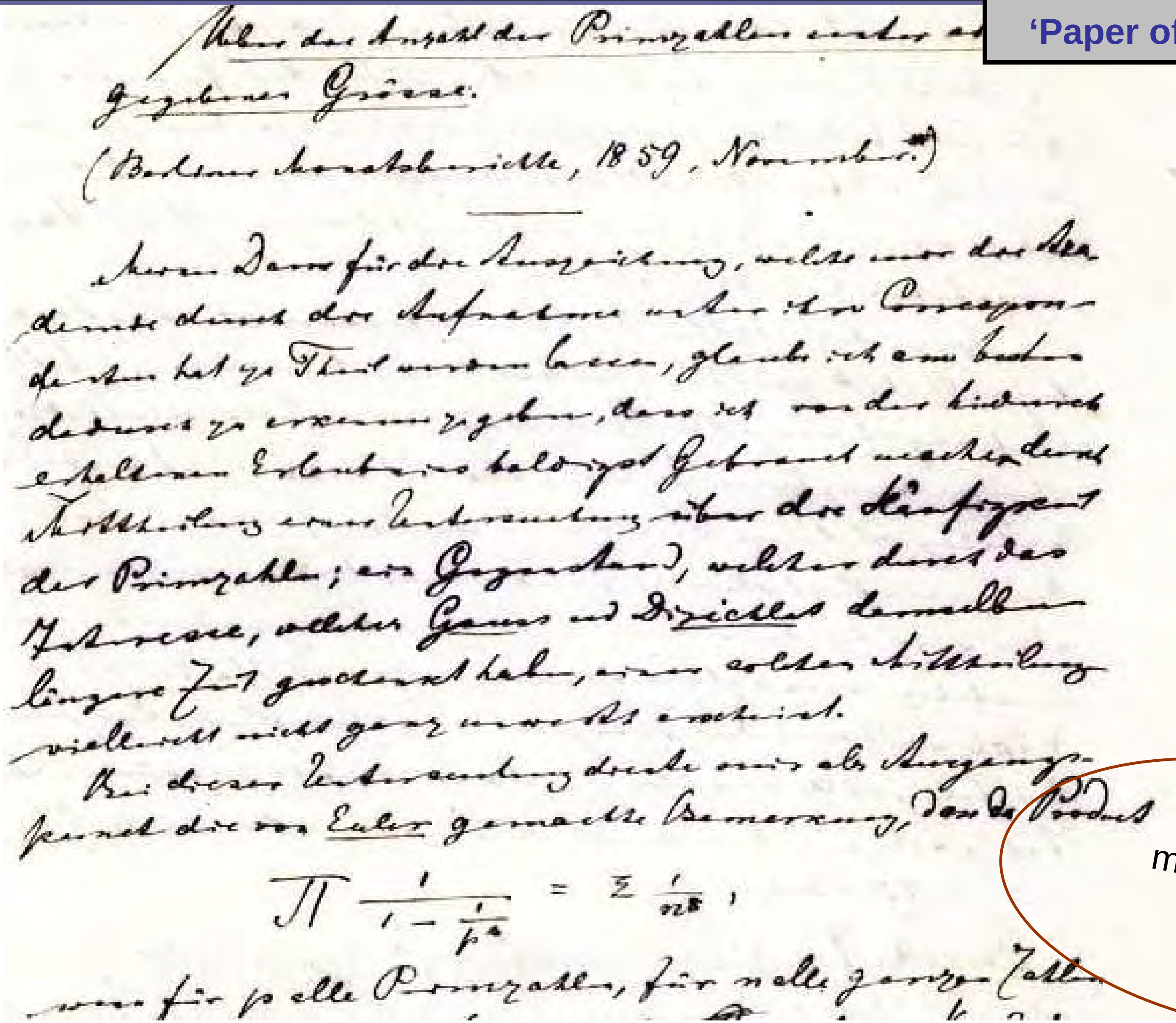


Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

Uber die Anzahl der Primzahlen unter einer Gegebenen Grosse

On the number of primes less than a given quantity

Riemann's six page 1859
'Paper of the Millennium'?


Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.*)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

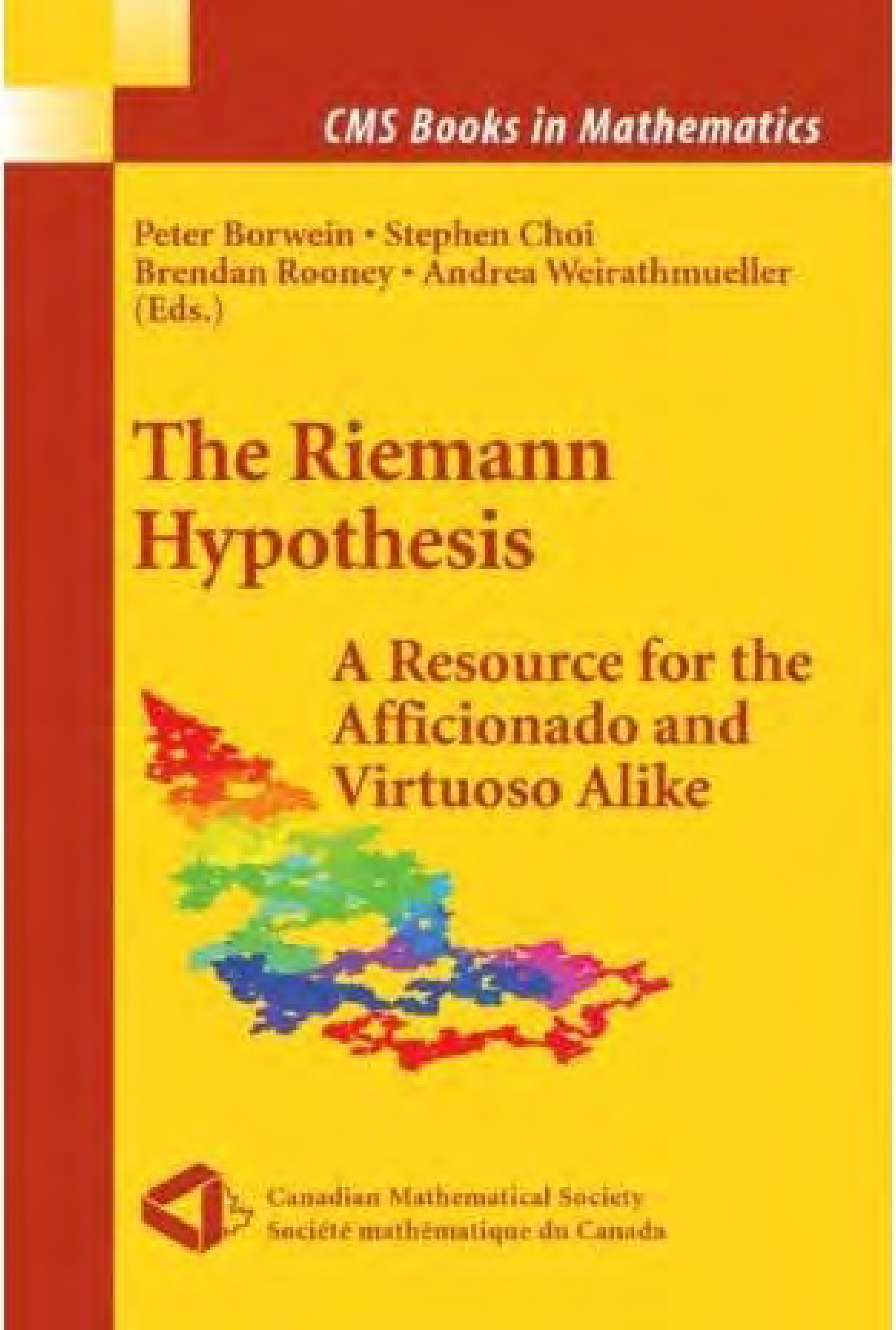
Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen

RH is so important because it yields precise results on distribution and behaviour of primes

Euler's product makes the key link between **primes and ζ**



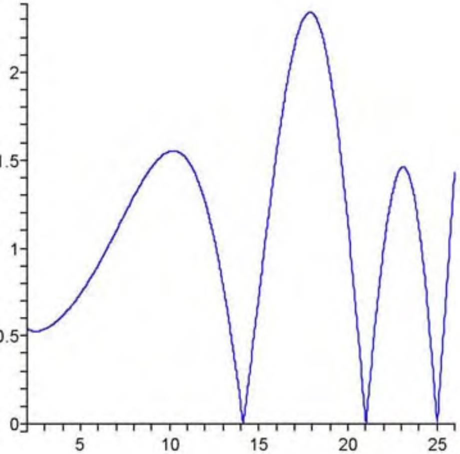
Buy this book

- Formulae
- History
- Equivalences

And almost all the key papers on the #1 problem in Mathematics

Dilcher/Taylor
series editors

The Modulus of Zeta and the **Riemann Hypothesis** (A Millennium Problem) Made Concrete



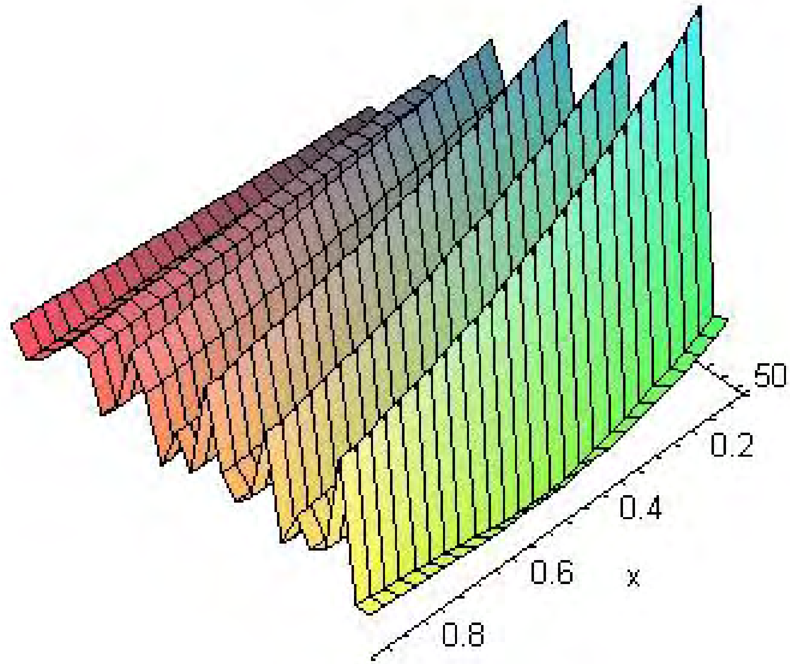
The imaginary parts of first 4 zeroes are:

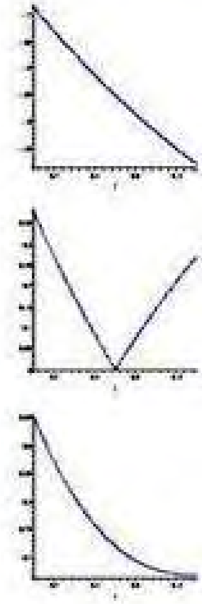
14.134725142
21.022039639

25.010857580
30.424876126

The first 1.5 billion are on the *critical line*

Yet at 10^{22} the ***"Law of small numbers"*** still rules (Odlyzko)





Curves at and around the 1st zero

'All non-real zeros have real part one-half'

(The Riemann Hypothesis)

Note the **monotonicity** of **$x \mapsto |\zeta(x+iy)|$** is **equivalent to RH** discovered in a Calgary class in 2002 by Zvengrowski and Saidak

Things Computer are Good For

- **High Precision Arithmetic:** the microscope
- **Formal Power-Series manipulations:** $\theta_2 \; \theta_3^2$
- **Continued Fractions:** changing representations
- **Partial Fractions :** changing representations
- **Pade' Approximations:** changing representations
- **Recursion Solving:** 'rsolve' and 'gfun'
- **Integer Relation Algorithms:** 'identify'
- **Creative Telescoping:** Wilf-Zeilberger
- **Pictures, Pictures, Pictures:** simulation

Generating Functions in Maple

Sums of 2, 3, 4 squares: what we can tell the easy way

```
> r:=sum(q^(n^2),n=0..20);
```

$$r := 1+q+q^4+q^9+q^{16}+q^{25}+q^{36}+q^{49}+q^{64}+q^{81}+q^{100}+q^{121}+q^{144}+q^{169}+q^{196}+q^{225}+q^{256}+q^{289}+q^{324}+q^{361}+q^{400}$$

$p \equiv 1 \bmod 4$ iff $p = n^2 + m^2$ (Fermat)

```
> series(r^2,q,50);
```

$$1+2q+q^2+2q^4+2q^5+q^8+2q^9+2q^{10}+2q^{13}+2q^{16}+2q^{17}+q^{18}+2q^{20}+4q^{25}+2q^{26}+2q^{29}+q^{32}+2q^{34}+2q^{36}+2q^{37}+2q^{40}+2q^{41}+2q^{45}+2q^{49}+O(q^{50})$$

8N+7 is not a sum of 3 squares

```
> series(r^3,q,50);
```

$$1+3q+3q^2+q^3+3q^4+6q^5+3q^6+3q^8+6q^9+6q^{10}+3q^{11}+q^{12}+6q^{13}+6q^{14}+3q^{16}+9q^{17}+6q^{18}+3q^{19}+6q^{20}+6q^{21}+3q^{22}+3q^{24}+9q^{25}+12q^{26}+4q^{27}+12q^{29}+6q^{30}+3q^{32}+6q^{33}+9q^{34}+6q^{35}+6q^{36}+6q^{37}+9q^{38}+6q^{40}+15q^{41}+6q^{42}+3q^{43}+3q^{44}+12q^{45}+6q^{46}+q^{48}+9q^{49}+O(q^{50})$$

```
> series(r^4,q,50);
```

$$1+4q+6q^2+4q^3+5q^4+12q^5+12q^6+4q^7+6q^8+16q^9+18q^{10}+12q^{11}+8q^{12}+16q^{13}+24q^{14}+12q^{15}+5q^{16}+24q^{17}+30q^{18}+16q^{19}+18q^{20}+28q^{21}+24q^{22}+12q^{23}+12q^{24}+28q^{25}+42q^{26}+28q^{27}+12q^{28}+36q^{29}+48q^{30}+16q^{31}+6q^{32}+36q^{33}+42q^{34}+36q^{35}+29q^{36}+28q^{37}+48q^{38}+28q^{39}+18q^{40}+48q^{41}+60q^{42}+28q^{43}+24q^{44}+60q^{45}+48q^{46}+24q^{47}+8q^{48}+44q^{49}+O(q^{50})$$

All numbers are sums of four squares (Lagrange)

And what Sloane tells us ...

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,1,4,9,25,64,169,441

Search

Hints

Search: **1, 1, 4, 9, 25, 64, 169, 441**

Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A007598 F(n)^2, where F() = Fibonacci numbers A000045. +20
(Formerly M3364) 36

0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, 7921, 20736, 54289, 142129, 372100, 974169, 2550409, 6677056, 17480761, 45765225, 119814916, 313679521, 821223649, 2149991424, 5628750625, 14736260449, 38580030724 (list; graph; listen)

OFFSET	0,4
COMMENT	a(n)*(-1)^(n+1) = (2*(1-T(n,-3/2))/5), n>=0, with Chebyshev's polynomials T(n,x) of the first kind, is the r=-1 member of the r-family of sequences S_r(n) defined in A092184 where more information can be found. W. Lang (wolfdieter.lang_AT_physik_DOT_uni-karlsruhe_DOT_de), Oct 18 2004
REFERENCES	A. T. Benjamin and J. J. Quinn, Proofs that really count: the art of combinatorial proof, M.A.A. 2003, id. 8. R. Honsberger, Mathematical Gems III, M.A.A., 1985, p. 130. R. P. Stanley, Enumerative Combinatorics I, Example 4.7.14, p. 251.
LINKS	D. Foata and G.-N. Han, Nombres de Fibonacci et polynomes orthogonaux. T. Mansour, A note on sum of k-th power of Horadam's sequence T. Mansour, Squaring the terms of an ell-th order linear recurrence P. Stanica, Generating functions, weighted and non-weighted sums of powers...

And what Sloane tells us …

FORMULA	a(0) = 0, a(1) = 1; a(n) = a(n-1) + Sum(a(n-i)) + k, 0 <= i < n where k = 1 when n is odd, or k = -1 when n is even. E.g. a(2) = 1 = 1 + (1 + 1 + 0) - 1, a(3) = 4 = 1 + (1 + 1 + 0) + 1, a(4) = 9 = 4 + (4 + 1 + 1 + 0) - 1, a(5) = 25 = 9 + (9 + 4 + 1 + 1 + 0) + 1. - Sadrul Habib Chowdhury (adil040(AT)yahoo.com), Mar 02 2004 G.f.: x(1-x)/((1+x)(1-3x+x^2)). a(n)=2a(n-1)+2a(n-2)-a(n-3), n>2. a(0)=0, a(1)=1, a(2)=1. a(-n)=a(n). (1/5)[2*Fibonacci(2n+1) - Fibonacci(2n) - 2(-1)^n]. - R. Stephan, May 14 2004 a(n) = F(n-1)F(n+1) - (-1)^n = A059929(n-1) - A033999(n). a(n) = right term of M^n * [1 0 0] where M = the 3X3 matrix [1 2 1 / 1 1 0 / 1 0 0]. M^n * [1 0 0] = [a(n+1) A001654(n) a(n)]. E.g. a(4) = 9 since M^4 * [1 0 0] = [25 15 9] = [a(5) A001654(4) a(4)]. - Gary W. Adamson (qntmpkt(AT)yahoo.com), Dec 19 2004 Sum_(j=0..2n) binomial(2n,j) a(j)= 5^(n-1) A005248(n+1) for n>=1 [P. Stanica]. sum_(j=0..2n+1) binomial(2n+1,j) a(j)=5^n A001519(n+1) [P. Stanica]. - Richard J. Mathar (mathar(AT)strw.leidenuniv.nl), Oct 16 2006

- What a wonderful resource!
- The more technical the result, the less we will learn from Sloane and the more from Salvy-Zimmerman

Eight Roles for Computation

4.1 Eight Roles for Computation

I next recapitulate eight roles for computation that Bailey and I discuss in our two recent books [9, 10]:

#1. **Gaining insight and intuition or just knowledge.** Working algorithmically with mathematical objects almost inevitably adds insight to the processes one is studying. At some point even just the careful aggregation of data leads to better understanding.

#2. **Discovering new facts, patterns and relationships.** The number of *additive partitions* of a positive integer n, $p(n)$, is *generated* by

$$P(q) := 1 + \sum_{n\geq 1} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}. \quad (2)$$

Thus, $p(5) = 7$ since

Revisit later today

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Developing (2) is a fine introduction to enumeration via *generating functions*. Additive partitions are harder to handle than multiplicative factorizations, but they are very interesting, [10, Chapter 4]. Ramanujan used Major MacMahon's table of $p(n)$ to intuit remarkable deep congruences such as

$$p(\mathbf{5n+4}) \equiv 0 \mod \mathbf{5}, \quad p(\mathbf{7n+5}) \equiv 0 \mod \mathbf{7}, \quad p(\mathbf{11n+6}) \equiv 0 \mod \mathbf{11},$$

E versus Pi: Continued Fractions

```
> evalf(exp(1));convert(%,confrac);
2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427427466391932003059921817413596629 \
043572900334295260595630738132328627943490763233829880753195251019011573834187930702154089149934884167509244761460668082264800168477411853742345442437107539077744992069551702761838606261331384583000752044933826560297606737113200709328709127443747047230696977209310141692836819025515108657463772111252389784425056953696770785449969967946864454905987931636889230098793 \
1

[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 1, 1,
    22, 1, 1, 24, 1, 1, 26, 1, 1, 28, 1, 1, 30, 1, 1, 32, 1, 1, 34, 1, 1, 36, 1, 1, 38, 1, 1, 40, 1,
    1, 42, 1, 1, 44, 1, 1, 46, 1, 1, 48, 1, 1, 50, 1, 1, 52, 1, 1, 54, 1, 1, 56, 1, 1, 58, 1, 1, 60,
    1, 1, 62, 1, 1, 64, 1, 1, 66, 1, 1, 68, 1, 1, 70, 1, 1, 72, 1, 1, 74, 1, 1, 76, 1, 1, 78, 1, 1,
    80, 1, 1, 82, 1, 1, 84, 1, 1, 86, 1, 1, 88, 1, 1, 90, 1, 1, 92, 1, 1, 94, 1, 1, 96, 1, 1, 98, 1,
    1, 100, 1, 1, 102, 1, 1, 104, 1, 1, 106, 1, 1, 108, 1, 1, 110, 1, 1, 112, 1, 1, 114, 1, 1,
    116, 1, 1, 118, 1, 1, 120, 1, 1, 122, 1, 1, 124, 1, 1, 126, 1, 1, 128, 1, 1, 130, 1, 1, 132,
    1, 1, 134, 1, 1, 136, 1, 1, 138, 1, 1, 140, 1, 1, 142, 1, 1, 144, 1, 1, 146, 1, 1, 148, 1, 1,
    150, 1, 1, 152, 1, 1, 154, 1, 1, 156, 1, 1, 158, 1, 1, 160, 1, 1, 162, 1, 1, 164, 1, 1, 166,
    1, 1, 168, 1, 1, 170, 1, 1, 172, 1, 1, 174, 1, 1, 176, 1, 1, 178, 1, 1, 180, 1, 1, 182, 1, 1,
    184, 1, 1, 186, 1, 1, 188, 1, 1, 190, 1, 1, 192, 1, 1, 194, 1, 1, 196, 1, 1, 198, 1, 1, 200,
    1, 1, 202, 1, 1, 204, 1, 1, 206, 1, 1, 208, 1, 1, 210, 1, 1, 212, 1, 1, 214, 1, 1, 216, 1, 1,
    218, 1, 1, 220, 1, 1, 222, 1, 1, 275 ]
```

```
> evalf(Pi);convert(%,confrac);
```

3.141592653589793238462643383279502884197169399375105820974944592307 8 \
16406286208998628034825342117067982148086513282306647093844609550 \
58223172535940812848111745028410270193852110555964462294895493038 \
19644288109756659334461284756482337867831652712019091456485669234 \
60348610454326648213393607260249141273724587006606315588174881520 \
92096282925409171536436789259036001133053054882046652138414695194 \
15116094330572703657595919530921861173819326117931051185480744623 \
799627495673518857527248912279381830119491

41% one's? Gauss-Kuzmin

[3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6,
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1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, 24, 1, 2, 1, 3, 1,
2, 1, 1, 10, 2, 5, 4, 1, 2, 2, 8, 1, 5, 2, 2, 26, 1, 4, 1, 1, 8, 2, 42, 2, 1, 7, 3, 3, 1, 1, 7, 2, 4,
9, 7, 2, 3, 1, 57, 1, 18, 1, 9, 19, 1, 2, 18, 1, 3, 7, 30, 1, 1, 1, 3, 3, 3, 1, 2, 8, 1, 1, 2, 1,
15, 1, 2, 13, 1, 2, 1, 4, 1, 12, 1, 1, 3, 3, 28, 1, 10, 3, 2, 20, 1, 1, 1, 1, 4, 1, 1, 1, 5, 3, 2,
1, 6, 1, 4, 1, 120, 2, 1, 1, 3, 1, 23, 1, 15, 1, 3, 7, 1, 16, 1, 2, 1, 21, 2, 1, 1, 2, 9, 1, 6, 4,
127, 14, 5, 1, 3, 13, 7, 9, 1, 1, 1, 1, 1, 5, 4, 1, 1, 3, 1, 1, 29, 3, 1, 1, 2, 2, 1, 3, 1, 1, 1, 3,
1, 1, 10, 3, 1, 3, 1, 2, 1, 12, 1, 4, 1, 1, 1, 1, 7, 1, 1, 2, 1, 11, 3, 1, 7, 1, 4, 1, 48, 16, 1, 4,
5, 2, 1, 1, 4, 3, 1, 2, 3, 1, 2, 2, 1, 2, 5, 20, 1, 1, 5, 4, 1, 436, 8, 1, 2, 2, 1, 1, 1, 1, 1, 5, 1,
2, 1, 3, 6, 11, 4, 3, 1, 1, 1, 2, 5, 4, 6, 9, 1, 5, 1, 5, 15, 1, 11, 24, 4, 4, 5, 2, 1, 4, 1, 6, 1,
1, 1, 4, 3, 2, 2, 1, 1, 2, 1, 58, 5, 1, 2, 1, 2, 1, 1, 2, 2, 7, 1, 15, 1, 4, 8, 1, 1, 4, 2, 1, 1, 1,
3, 1, 1, 1, 2, 1, 1, 1, 1, 1, 9, 1, 4, 3, 15, 1, 2, 1, 13, 1, 1, 1, 3, 24, 1, 2, 4, 10, 5, 12, 3, 3,
21, 1, 2, 1, 34, 1, 1, 1, 4, 15, 1, 4, 44, 1, 4, 20776, 1, 1, 1, 1, 1, 1, 1, 23, 1, 7, 2, 1, 94,
55, 1, 1, 2, 1, 1, 3, 1, 1, 32, 5, 1, 14, 1, 1, 1, 1, 1, 3, 50, 2, 16, 5, 1, 2, 1, 4, 6, 3, 1, 3, 3,
1, 2, 2, 2, 5, 2, 2, 2, 28, 1, 1, 13, 1, 4, 1]

from relatively limited data like

$$
\begin{aligned}
P(q) \quad = \quad & 1 + q + 2\,q^2 + 3\,q^3 + \underline{5}\,q^{\mathbf{4}} + \overline{7}\,q^{\mathbf{5}} + 11\,q^6 + 15\,q^7 \\
+ \quad & 22\,q^8 + \underline{30}\,q^{\mathbf{9}} + 42\,q^{10} + 56\,q^{11} + \overline{77}\,q^{\mathbf{12}} + 101\,q^{13} + \underline{135}\,q^{\mathbf{14}} \\
+ \quad & 176\,q^{15} + 231\,q^{16} + 297\,q^{17} + 385\,q^{18} + \overline{\underline{490}}\,q^{\mathbf{19}} \\
+ \quad & 627\,q^{20}b + 792\,q^{21} + 1002\,q^{22} + \cdots + p(200)q^{200} + \cdots
\end{aligned} \tag{3}
$$

Cases $5n+4$ and $7n+5$ are flagged in (3). Of course, it is markedly easier to (heuristically) confirm than find these fine examples of *Mathematics: the science of patterns.*[24] The study of such congruences—much assisted by symbolic computation—is very active today.

#3. **Graphing to expose mathematical facts, structures or principles**. Consider Nick Trefethen's fourth challenge problem as described in [5, 8]. It requires one to find ten good digits of:

4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4?$$

As a foretaste of future graphic tools, one can solve this problem graphically and interactively using current *adaptive 3-D plotting* routines which can catch all the bumps. This does admittedly rely on trusting a good deal of software.

#4. Rigourously testing and especially falsifying conjectures. I hew to the Popperian scientific view that we primarily falsify; but that as we perform more and more testing experiments without such falsification we draw closer to firm belief in the truth of a conjecture such as: *the polynomial* $P(n) = n^2 - n + p$ *has prime values for all* $n = 0, 1, \ldots, p-2$, *exactly for* Euler's lucky prime numbers, *that is,* p*= 2, 3, 5, 11, 17, and 41.*[25]

#5. Exploring a possible result to see if it *merits* formal proof. A conventional deductive approach to a hard multi-step problem really requires establishing all the subordinate lemmas and propositions needed along the way—especially if they are highly technical and un-intuitive. Now some may be independently interesting or useful, but many are only worth proving if the entire expedition pans out. Computational experimental mathematics provides tools to survey the landscape with little risk of error: only if the view from the summit is worthwhile, does one lay out the route carefully. I discuss this further at the end of the next Section.

#6. Suggesting approaches for formal proof. The proof of the *cubic theta function identity* discussed on [10, pp. 210] shows how a fully intelligible human proof can be obtained entirely by careful symbolic computation.

[24] The title of Keith Devlin's 1996 book, [21].

[25] See [55] for the answer.

#6: example on next page

Let $a(q) := \sum_{m,n} q^{m^2+nm+n^2}$, $b(q) := \sum_{m,n} (\omega)^{n-m} q^{m^2+nm+n^2}$,

$c(q) := \sum_{m,n} q^{(m+1/3)^2+(n+1/3)(m+1/3)+(n+1/3)^2}$, where $\omega = e^{2i\pi/3}$

Then a, b, c solve Fermat's eq'n: $a^3 = b^3 + c^3$.

#7. Computing replacing lengthy hand derivations. Who would wish to verify the following prime factorization by hand?

$$\begin{aligned} & 6422607578676942838792549775208734746307 \\ = \quad & (2140992015395526641)(1963506722254397)(1527791). \end{aligned}$$

Surely, what we value is understanding the underlying algorithm, not the human work?

#8. Confirming analytically derived results. This is a wonderful and frequently accessible way of confirming results. Even if the result itself is not computationally checkable, there is often an accessible corollary. An assertion about bounded operators on Hilbert space may have a useful consequence for three-by-three matrices. It is also an excellent way to error correct, or to check calculus examples before giving a class.

$\int_0^\infty \frac{\mathrm{sech}(t)\tanh(t)}{t}\, dt = 4\frac{G}{\pi}$ using residues.

The Key Cubic Discovery: series $\mapsto$ product

```
> b:=N->sum(sum(cos(2*Pi*(n-m)/3)*q^(n^2+n*m+m^2),n=-N..N),m=-N..N);
```

$$b := N \to \sum_{m=-N}^{N} \left(\sum_{n=-N}^{N} \cos\left(\frac{2\pi(n-m)}{3}\right) q^{(n^2+nm+m^2)} \right)$$

```
> convert(series(b(10),q,50),polynom);;
```

$$1 - 3q + 6q^3 - 3q^4 - 6q^7 + 6q^9 + 6q^{12} - 6q^{13} - 3q^{16} - 6q^{19} + 12q^{21} - 3q^{25} + 6q^{27} - 6q^{28} - 6q^{31} + 6q^{36} - 6q^{37} + 12q^{39} - 6q^{43} + 6q^{48} - 9q^{49}$$

```
> s2p(%,q);
```

$$(1-q)^3(1-q^2)^3(1-q^3)^2(1-q^4)^3(1-q^5)^3(1-q^6)^2(1-q^7)^3(1-q^8)^3(1-q^9)^2$$
$$(1-q^{10})^3(1-q^{11})^3(1-q^{12})^2(1-q^{13})^3(1-q^{14})^3(1-q^{15})^2(1-q^{16})^3(1-q^{17})^3$$
$$(1-q^{18})^2(1-q^{19})^3(1-q^{20})^3(1-q^{21})^2(1-q^{22})^3(1-q^{23})^3(1-q^{24})^2(1-q^{25})^3$$
$$(1-q^{26})^3(1-q^{27})^2(1-q^{28})^3(1-q^{29})^3(1-q^{30})^2(1-q^{31})^3(1-q^{32})^3(1-q^{33})^2$$
$$(1-q^{34})^3(1-q^{35})^3(1-q^{36})^2(1-q^{37})^3(1-q^{38})^3(1-q^{39})^2(1-q^{40})^3(1-q^{41})^3$$
$$(1-q^{42})^2(1-q^{43})^3(1-q^{44})^3(1-q^{45})^2(1-q^{46})^3(1-q^{47})^3(1-q^{48})^2(1-q^{49})^3$$

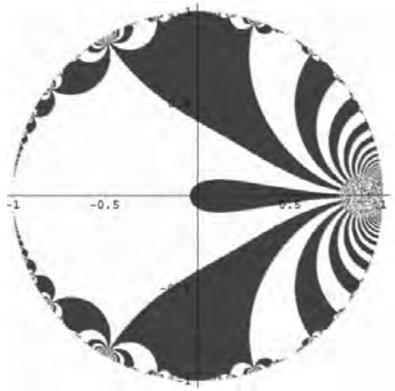
Ten Things to Try Them On, I

1. **Identify**
   1.4331274267223117583171834557759918204315127679060

2. **Compute** the following to 50 digits for N=1,2,3,4,5 and **explain** the answer

$$4\sum_{n=0}^{5\cdot 10^N} \frac{(-1)^n}{2n+1}$$

3. **Find** the first three numbers expressible as the sum of two cubes in exactly two ways. The first is **$1729=12^3+1=10^3+9^3$**.



4. **Evaluate**

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

5. **Evaluate** for *sinc*(x) =sin(x)/x

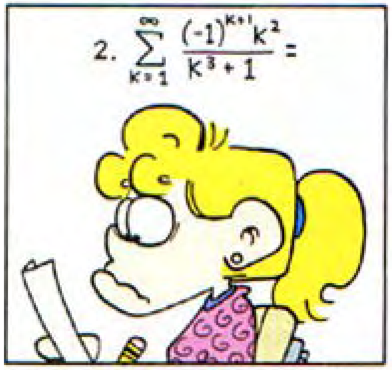
$$\frac{1}{2} + \sum_{n=1}^{\infty} sinc(n)\, sinc(n/3)\, sinc(n/5) \cdots sinc(n/23)\, sinc(n/29)$$

$$= \int_0^{\infty} sinc(x)\, sinc(x/3)\, sinc(x/5) \cdots sinc(x/23)\, sinc(x/29)\, dx$$

Ten Things to Try Them On, II

6. Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3+1}$$



7. and 8. Determine

$$\sum_{n=1}^{\infty} \frac{o\left(2^n\right)}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{e\left(2^n\right)}{2^n}$$

where $o(n))$ $((e(n))$ count the number of **odd (even) digits** in n. Thus $o(901) = 2, e(901) = 1, o(811) = 2$.

9. Determine the behaviour of the dynamical system

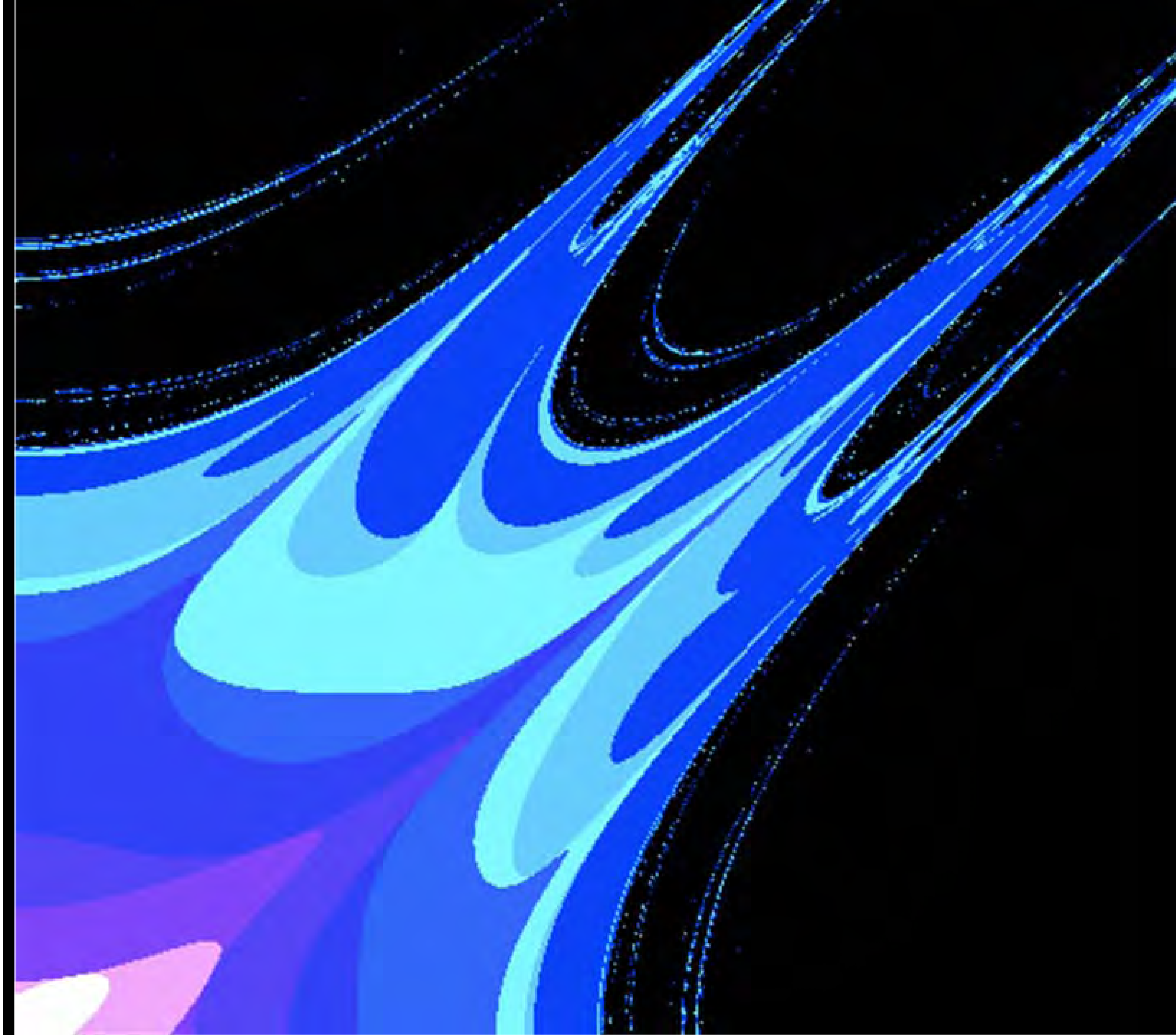
$$(x, y) \mapsto (y, x^2 - y^2) \text{ as } (x_0, y_0) \text{ ranges over } R^2.$$

10. Minimize

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4$$

#9. Plotting the Region of Convergence

The truth?



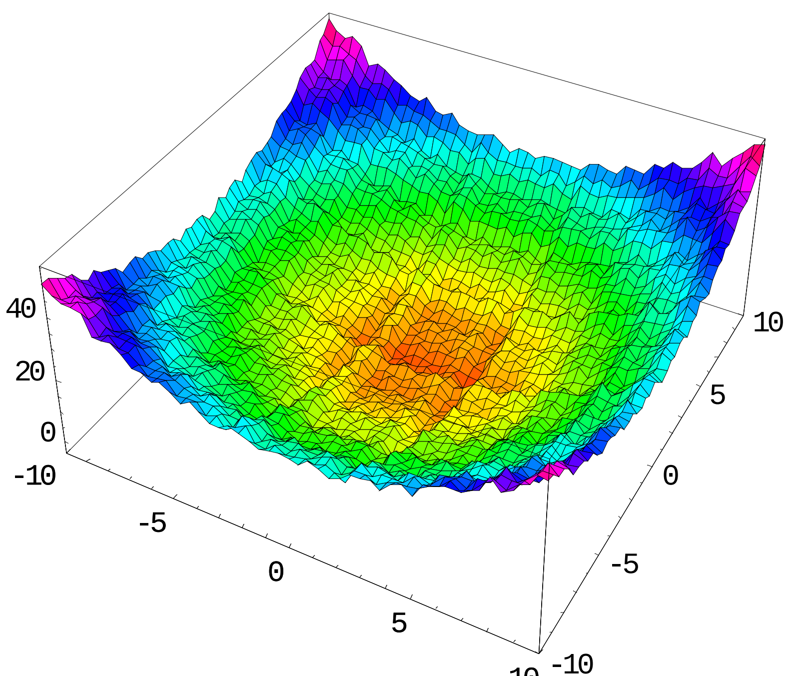
#10. Nick Trefethen's 100 Digit/100 Dollar Challenge, Problem 4 (SIAM News, 2002)

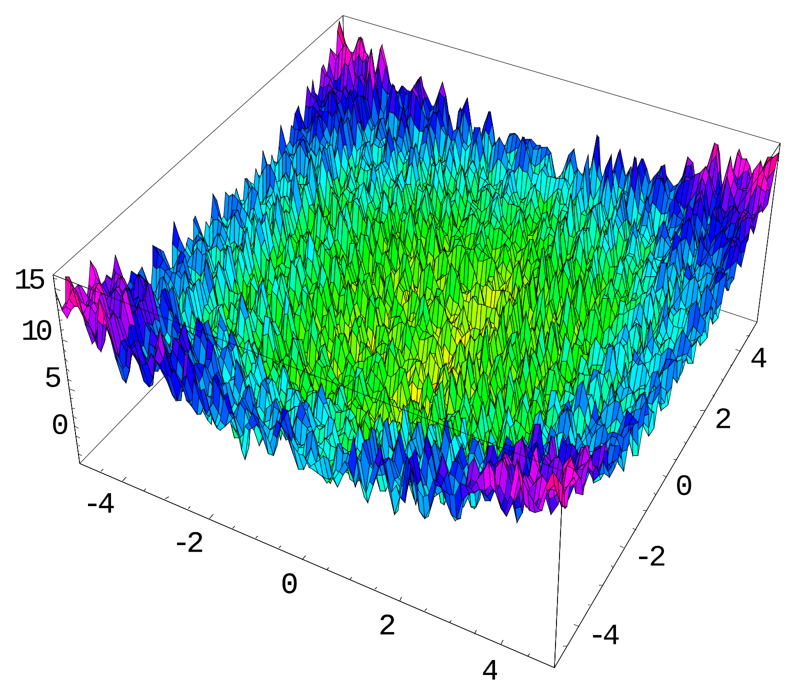
What is the global minimum of the function:

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x)$$

$$+ \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4?$$

- no bounds are given.





#2 SEEING PATTERNS in PARTITIONS

The number of **additive partitions** of n, p(n), is *generated* by

$$1+\sum_{n=1}^{\infty} p(n)q^n = \frac{1}{\prod_{n\geq 1}(1-q^n)}. \qquad (1)$$

Thus, $p(5)=7$ since

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

- Developing (1) is good introduction to *enumeration via generating functions*.

- Additive partitions are harder to handle than multiplicative factorizations, but they may be introduced in the elementary school curriculum with questions like: "How many `trains` of a given length can be built with Cuisenaire rods?"

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable deep congruences like

$$p(5n+4) \equiv 0 \bmod 5, \quad p(7n+5) \equiv 0 \bmod 7, \quad p(11n+6) \equiv 0 \bmod 11$$

from relatively limited data like

$$\begin{aligned} P(q) = 1 \quad & + \quad q + 2\,q^2 + 3\,q^3 + \underline{5}\,q^4 + \overline{7}\,q^{\mathbf{5}} + 11\,q^6 + 15\,q^7 \\ & + \quad 22\,q^8 + \underline{30}\,q^9 + 42\,q^{10} + 56\,q^{11} + \overline{77}\,q^{\mathbf{12}} \\ & + \quad 101\,q^{13} + \underline{135}\,q^{14} + 176\,q^{15} + 231\,q^{16} \\ & + \quad 297\,q^{17} + 385\,q^{18} + \overline{\underline{490}}\,q^{\mathbf{19}} \\ & + \quad 627\,q^{20} b + 792\,q^{21} + 1002\,q^{22} \\ & + \quad \cdots + p(200) q^{200} + \cdots \end{aligned}$$

- Cases $5n+4$ and $7n+5$ are flagged above.

Current research abounds!

Of course, it is easier to (heuristically) confirm than to find these fine examples of **Mathematics: the Science of Patterns,** Keith Devlin's 1997 book.

IS HARD or EASY BETTER?

A modern computationally driven question is:

How hard is $p(n)$ to compute?

- In **1900**, it took the father of combinatorics, Major Percy MacMahon (1854–1929), months to compute $p(200)$ using recursions developed from (1).

- By **2000**, *Maple* would produce $p(200)$ in seconds if one simply demands the 200'th term of the Taylor series. A few years earlier it required being careful to compute the series for $\prod_{n\geq 1}(1-q^n)$ *first* and *then* the series for the *reciprocal* of that series!

PENTAGONAL NUMBER THEOREM

- This baroque event is occasioned by *Euler's pentagonal number theorem*

$$\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)n/2}.$$

Try the cube of both sides

- The reason is that, if one takes the series for (1), the software has to deal with **200** terms on the bottom. But the series for $\prod_{n\geq 1}(1-q^n)$, has only to handle the **23** non-zero terms in series in the pentagonal number theorem.

If introspection fails, we can find the pentagonal numbers occurring above in Sloane's on-line `**Encyclopedia of Integer Sequences**':
www.research.att.com/personal/njas/sequences/eisonline.html

A CAVEAT

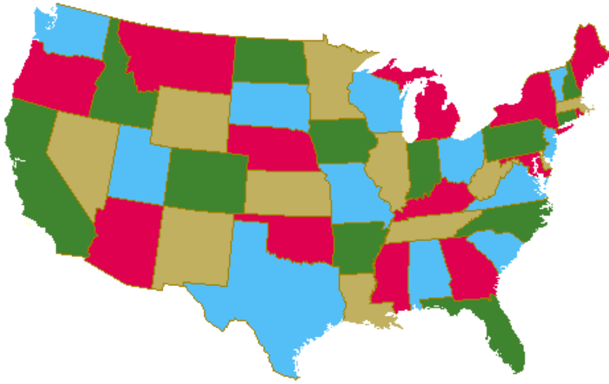
The difficulty of estimating the size of p(n) analytically---so as to avoid enormous or unattainable computational effort---led to some marvelous mathematical advances By researchers including Hardy and Ramanujan, and Rademacher.

- The corresponding ease of computation may now act as a retardant to insight.
- New mathematics is often discovered only when prevailing tools run totally out of steam.

This raises a caveat against mindless computing: *Will a student or researcher discover structure when it is easy to compute without needing to think about it? Today, she may thoughtlessly compute p(500) which a generation ago took much, much pain and insight?*

Grand Challenges in Mathematics (CISE 2000)

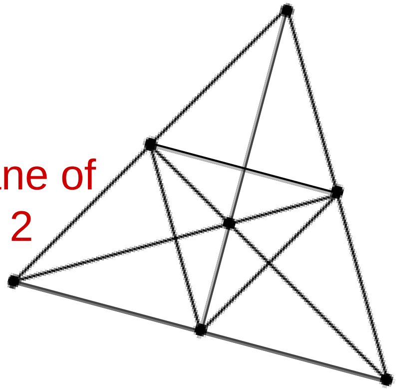
Are few and far between



- **Four Colour Theorem** (1976,1997)
- **Kepler's problem** (Hales, 2004-11)

On next slide

- Nonexistence of **Projective Plane of Order 10**
  - **10^2+10+1** lines and points on each other (n+1 fold)
    - 2000 Cray hrs in 1990
    - next similar case:18 needs10^{12} hours?
    - Or a **Quantum Computer**



Fano plane of order 2

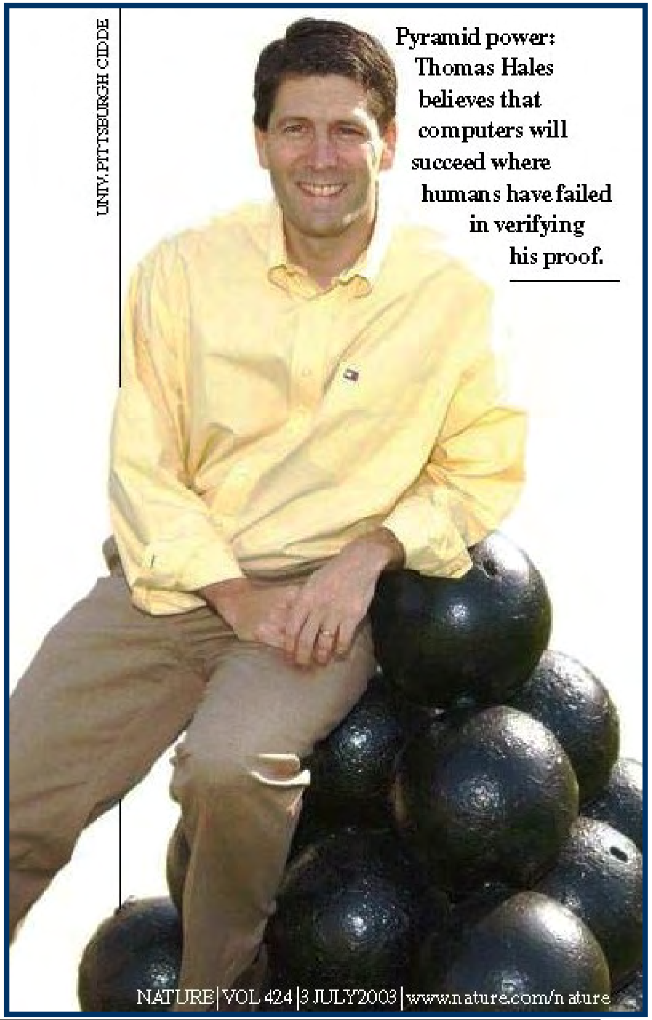
- **Fermat's Last Theorem** (Wiles 1993, 1994)
  - By contrast, any counterexample was too big to find (1985)

$$x^N + y^N = z^N, N > 2$$

has only trivial integer solutions

- **Kepler's** conjecture the densest way to stack spheres is in a pyramid
  - oldest problem in discrete geometry?
  - most interesting recent example of computer assisted proof relies on **CPLEX**!
  - published in *Annals of Mathematics* with an "only 99% checked" disclaimer
  - Many varied reactions. *In Math, Computers Don't Lie. Or Do They?* (NYT, 6/4/04)

- **Famous earlier examples**: Four Color Theorem and Non-existence of a Projective Plane of Order 10.
  - the three raise quite distinct questions - both real and specious
  - as does status of classification of Finite Simple Groups and Poincare' conjecture



Formal Proof theory (code validation) has received an unexpected boost: automated proofs *may* now exist of the Four Color Theorem, Godel and Prime Number Theorem

- COQ: *When is a proof a proof* ? Economist, April 2005

Cultural Mathematics

A French meteo in poor taste

REFERENCES

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J.M. Borwein, ``The Experimental Mathematician: The Pleasure of Discovery and the Role of Proof,'' *International Journal of Computers for Mathematical Learning*, **10** (2005), 75-108.

D.H. Bailey and J.M Borwein, "Experimental Mathematics: Examples, Methods and Implications," *Notices Amer. Math. Soc.*, **52** No. 5 (2005), 502-514.

Enigma

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

• J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.