October 14-15, 2011 13th

Midwest Optimization Meeting & Workshop on Large Scale Optimization and Applications

Hosted by the Fields Institute

http://www.fields.utoronto.ca/programs/scientific/11-12/optimization_mtg/

Friday October 14, 2011, 9:00 a.m. EDT (12.00pm AEST)

Nov Dec Dec Dec Dec Dec

Lonely Planet's top 10 cities

10:30 AEST Mon Nov 1 2010
Adam Bub

10 images in this story

Travel experts Lonely Planet have named the top 10 cities for 2011 in their annual travel bible, *Best in Travel 2011*. The top-listed cities win points for their local cultures, value for money, and overall va-va-voom. So which cities make the cut? Find out here, from 10 to 1...

What do you think of the list? **Tell us here!**

Related links: Lonely Planet destination videos

A weekend in Newcastle

Images: ThinkStock/Getaway

9. Newcastle, Australia 2 of 10

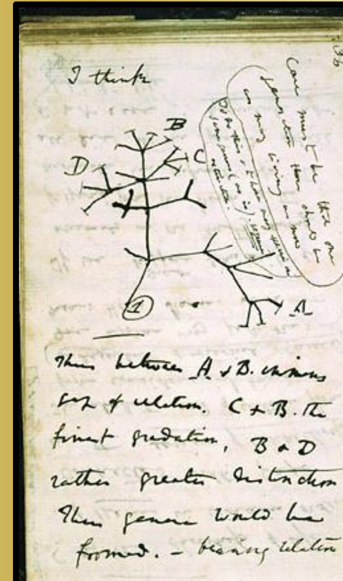
Midwest Optimization Meeting
& Workshop on
Large Scale Optimization and Applications

October 14th 2011

Douglas-Ratchford iterations in the absence of convexity

Alan Turing's Enigma

Charles Darwin's notes



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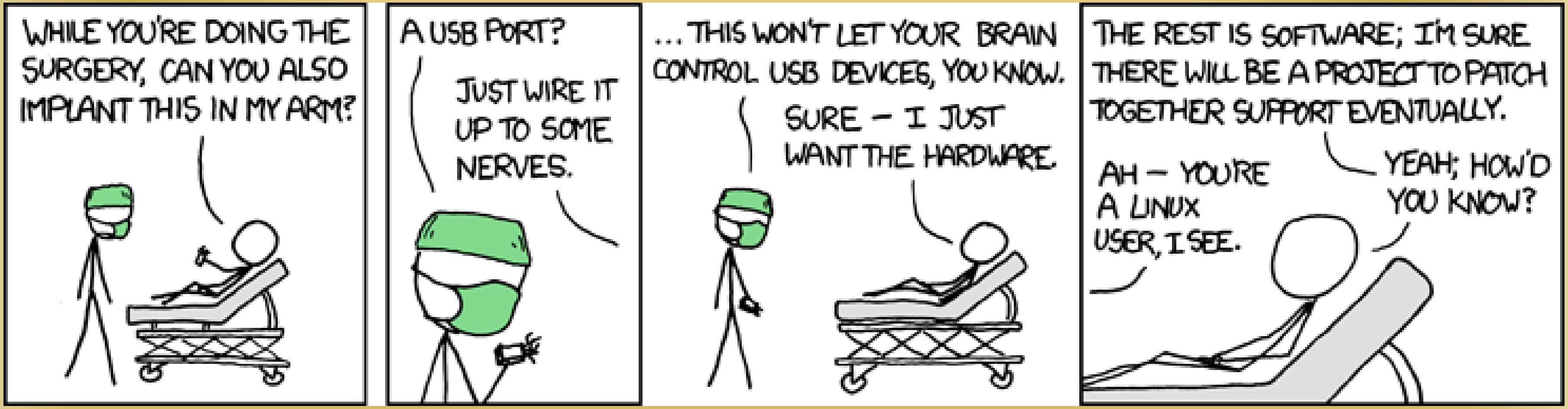
Revised 16-10-2011: joint work with B. Sims.
Thanks also to Ulli Kortenkamp, Fran Aragon, Matt Skerritt and Chris Maitland

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

THE REST IS SOFTWARE

"It was my luck (perhaps my bad luck) to be the world chess champion during the critical years in which computers challenged, then surpassed, human chess players. Before 1994 and after 2004 these duels held little interest." - Garry Kasparov, 2010



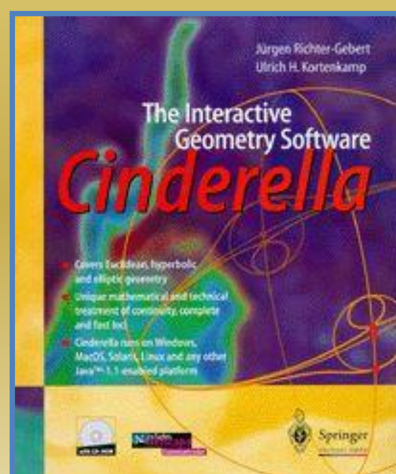
- Likewise much of current Optimization Theory

ABSTRACT

- The **Douglas-Rachford iteration** scheme, introduced half a century ago in connection with nonlinear heat flow problems, aims to find a point common to two or more closed constraint sets.
  - **Convergence is ensured when the sets are convex** subsets of a Hilbert space, however, despite the absence of satisfactory theoretical justification, the scheme has been routinely used to successfully solve a diversity of practical optimization or feasibility problems in which one or more of the constraints involved is non-convex.
- As a first step toward addressing this deficiency, **we provide convergence results for a proto-typical non-convex (phase-recovery) scenario**: Finding a point in the intersection of the Euclidean sphere and an affine subspace.

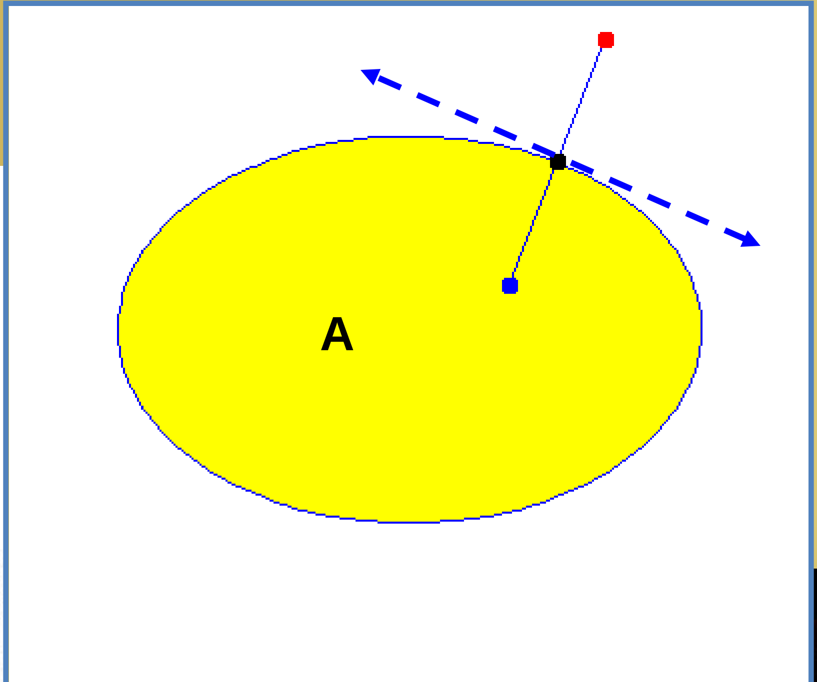
AN INTERACTIVE PRESENTATION

- Much of my lecture will be interactive using the interactive geometry package *Cinderella* and the html applets
  - www.carma.newcastle.edu.au/~jb616/reflection.html
  - www.carma.newcastle.edu.au/~jb616/expansion.html
  - www.carma.newcastle.edu.au/~jb616/lm-june.html

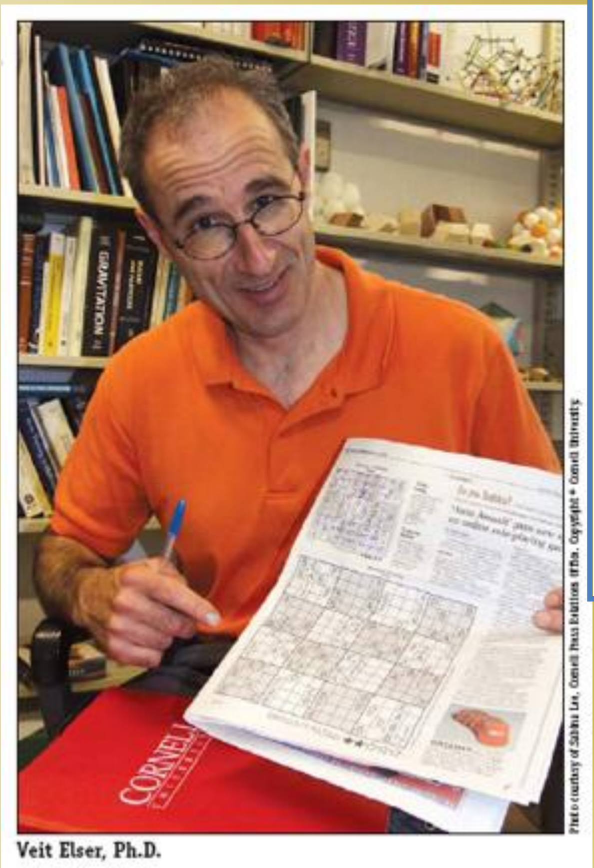


PHASE RECONSTRUCTION

Projectors and Reflectors: $P_A(x)$ is the metric projection or nearest point and $R_A(x)$ reflects in the tangent: x is red



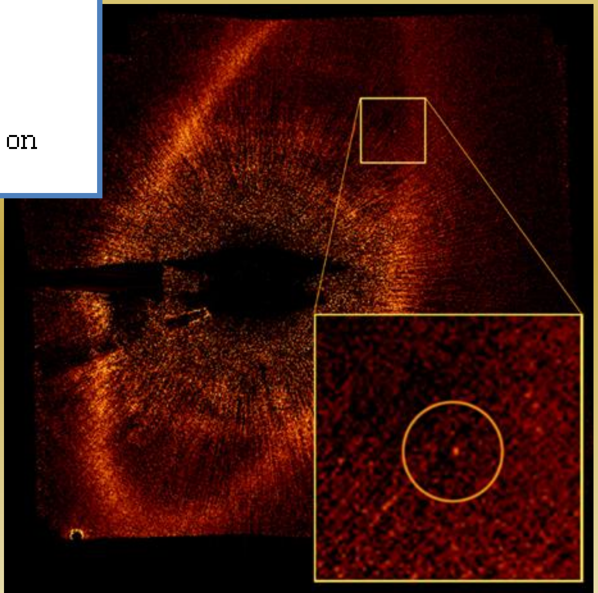
projection (black) and reflection (blue) of point (red) on boundary (blue) of ellipse (yellow)



2007 Elser solving Sudoku with **reflectors**

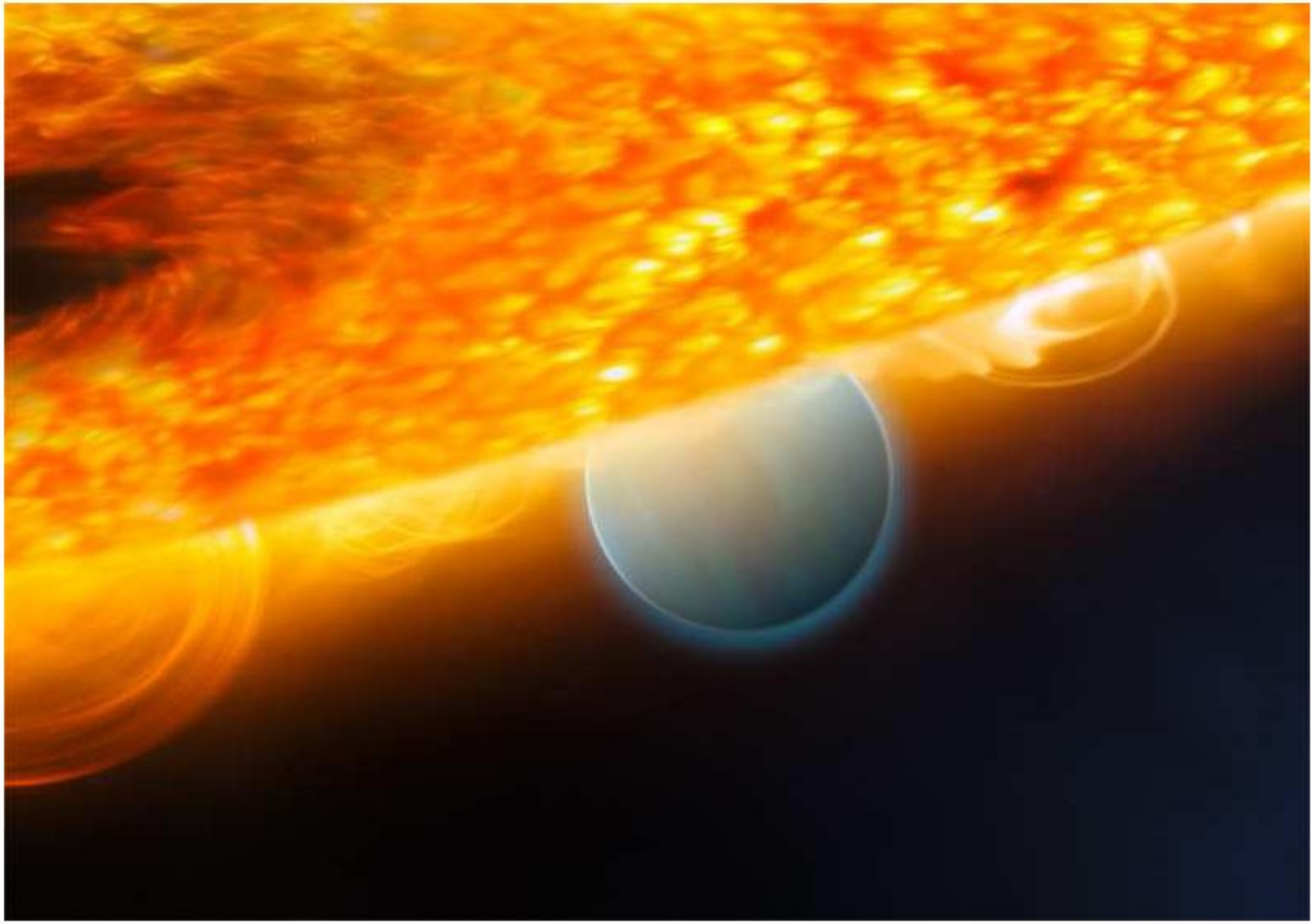
"**All physicists and a good many quite respectable mathematicians are contemptuous about proof**."
G. H. Hardy (1877-1947)

2008 Finding exoplanet Fomalhaut in Piscis with **projectors**



The story of Hubble's 1.3mm error in the "upside down" lens (1990)

And Kepler's hunt for exo-planets (launched March 2009)



Feeling the heat: Kepler scientists justify why some exoplanet data needs to be held back, for now. Image: A "Hot Jupiter" exoplanet close to its host star (ESO).

One of the biggest astronomical stories to unfold over the last decade or so is the story of exoplanets (or "extra-solar planets"). The theory of the formation of our solar system predicts that there should be many more such systems out there. And there certainly are, in fact, 461 at time of writing.

Continuing this epic tale of discovery, the Kepler Mission announced today the discovery of approximately 750 new candidates, but this announcement hasn't come without controversy.

The story of Hubble's 1.3mm error in the "upside down" lens (1990)

And Kepler's hunt for exo-planets (launched March 2009)

A few weeks ago we wrote:

"We should add, however, that many Kepler sightings in particular remain to be 'confirmed.' Thus one might legitimately wonder how mathematically robust are the underlying determinations of velocity, imaging, transiting, timing, micro-lensing, etc.?

http://experimentalmath.info/blog/2011/09/where-is-everybody/

THE CONVERSATION BETA

Academic rigour, journalistic flair

26 September 2011, 8.59am AEST

The exoplanet that wasn't. Or was it?

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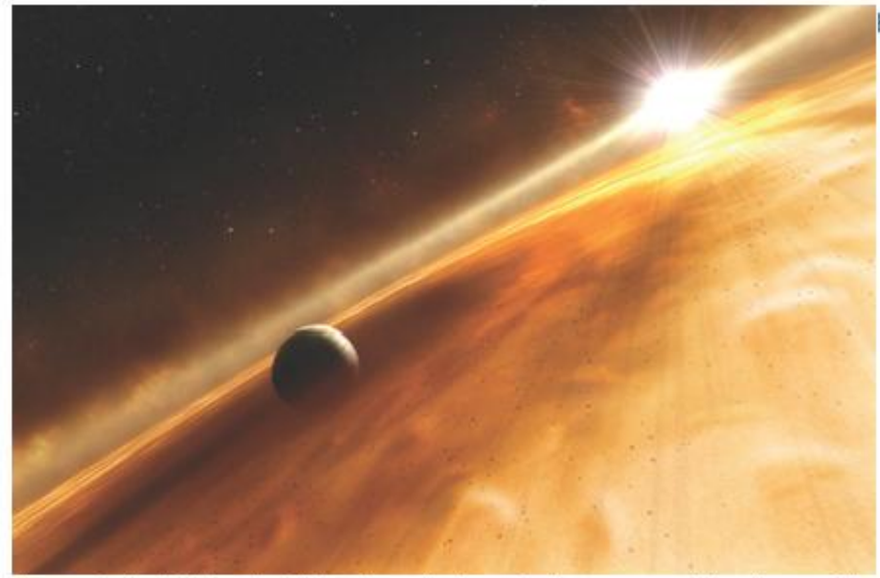
Sunanda Creagh

Editor

DISCLOSURE STATEMENT

Our goal is to ensure the content is not compromised in any way. We therefore ask all authors to disclose any potential conflicts of interest before publication.

LICENCE TO REPUBLISH



An exoplanet called Fomalhaut b has been photographed in an unexpected spot — so is it even an exoplanet at all? NASA/http://www.nasa.gov

A distant planet that made its name as the world's first directly photographed exoplanet is at the centre of an astronomical stoush, after it veered off course and new doubts were raised about its existence.

It was in 2008 that Hubble astronomer Paul Kalas from the University of California at Berkeley and NASA announced that Fomalhaut b had been photographed orbiting a star called Fomalhaut around 25 light years from Earth.

WHY DOES IT WORK?

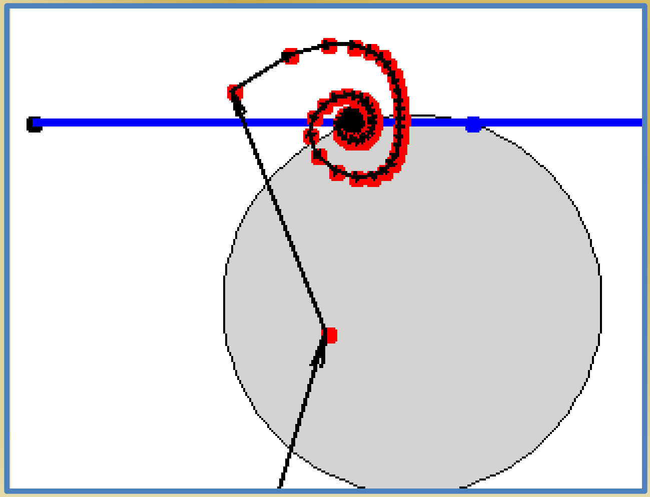
In a wide variety of large hard problems (protein folding, 3SAT, Sudoku) A is non-convex but DR and "divide and concur" (below) works better than theory can explain. It is: $\boxed{R_A(x) := 2\,P_A(x) - x \text{ and } x \to \frac{x+R_A(R_B(x))}{2}}$

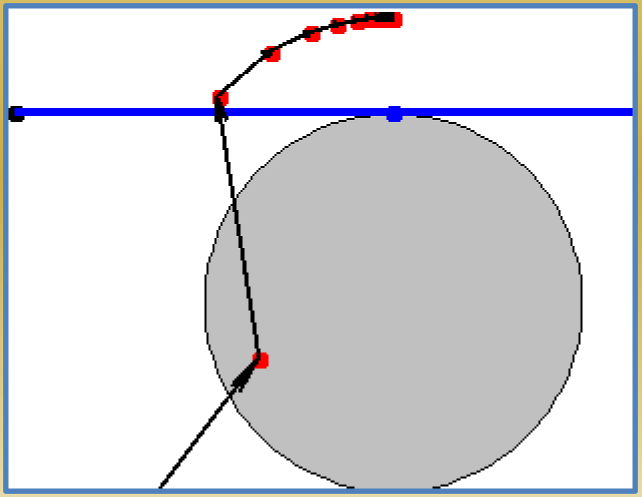
Consider the simplest case of a line B of **height** h and the unit circle A.

With $z_n := (x_n, y_n)$ the iteration becomes

$$\boxed{x_{n+1} := \cos\theta_n, y_{n+1} := y_n + h - \sin\theta_n, \qquad (\theta_n := \arg z_n)}$$

For h=0 We prove convergence to one of the two points in A ∩ B <u>iff</u> we do not start on the vertical axis (where we have chaos). For h>1 (infeasible) it is easy to see the iterates go to infinity (vertically). For h=1 we converge to an infeasible point. For h in (0,1) the pictures are lovely but proofs escaped us for 9 months. Two representative **Maple** pictures follow:





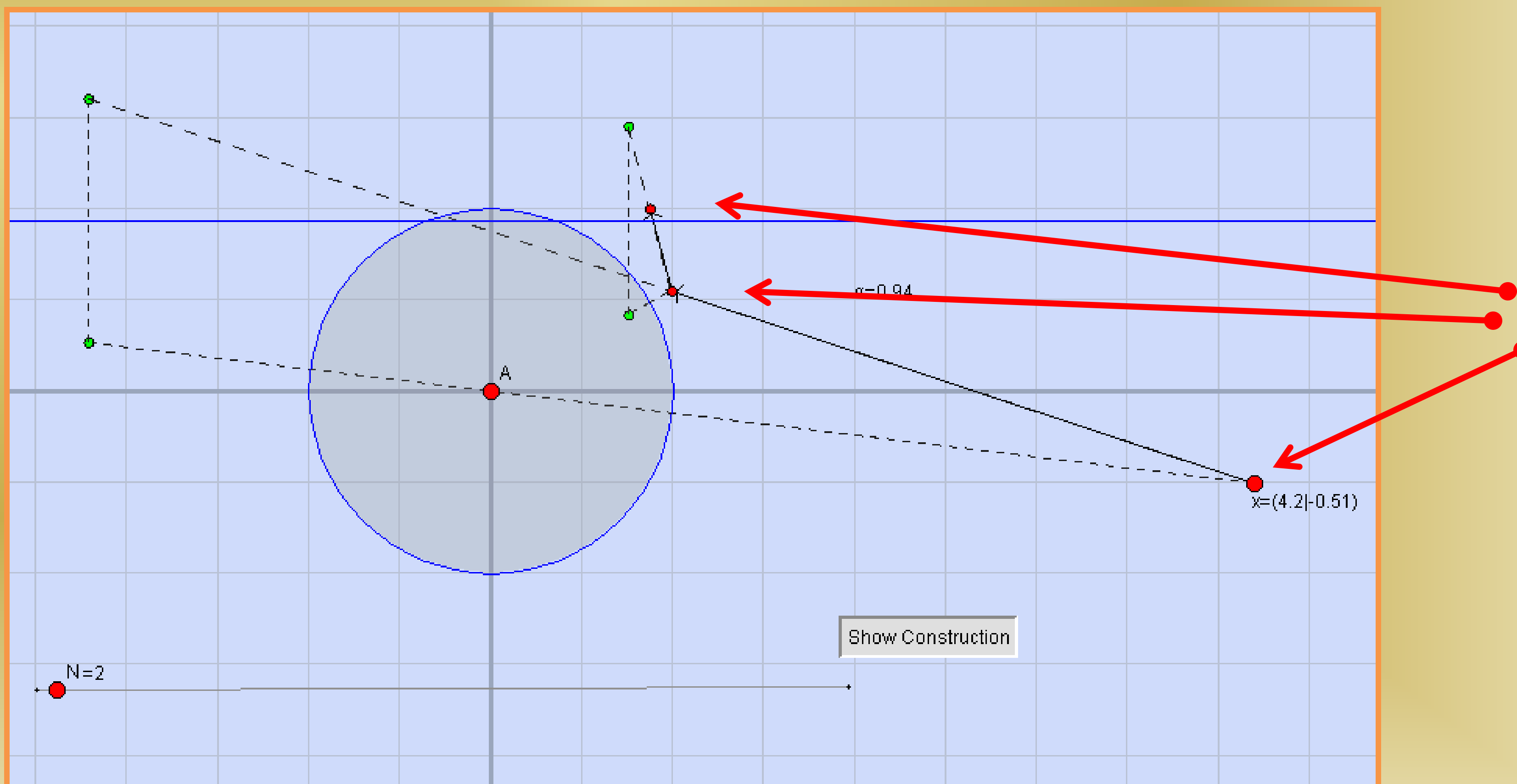
An ideal problem for introducing early under-graduates to research, with many many accessible extensions in 2 or 3 dimensions

INTERACTIVE PHASE RECOVERY in CINDERELLA

Recall the simplest case of a line B of height h and the unit circle A. With $z_n := (x_n, y_n)$ the iteration becomes

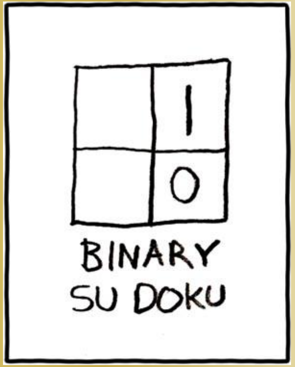
$$\boxed{x_{n+1} := \cos\theta_n, y_{n+1} := y_n + h - \sin\theta_n, \qquad (\theta_n := \arg z_n)}$$

A *Cinderella* picture of two steps from (4.2,-0.51) follows:



DIVIDE AND CONCUR

To find a point in the intersection of M sets A_k in X we can instead consider the subset $A := \prod_{k=1}^{M} A_k$ and the linear subset



$$B := \{x = (x_1, x_2, \ldots, x_M) \colon x_1 = x_2 = \cdots = x_M\}$$

of the product Hilbert space $\tilde{X} := \left(\prod_{k=1}^{M} X\right)$. Then we observe that

$$R_A(x) = \prod R_{A_k}(x_k),$$

Serial (L) and Parallel (R)

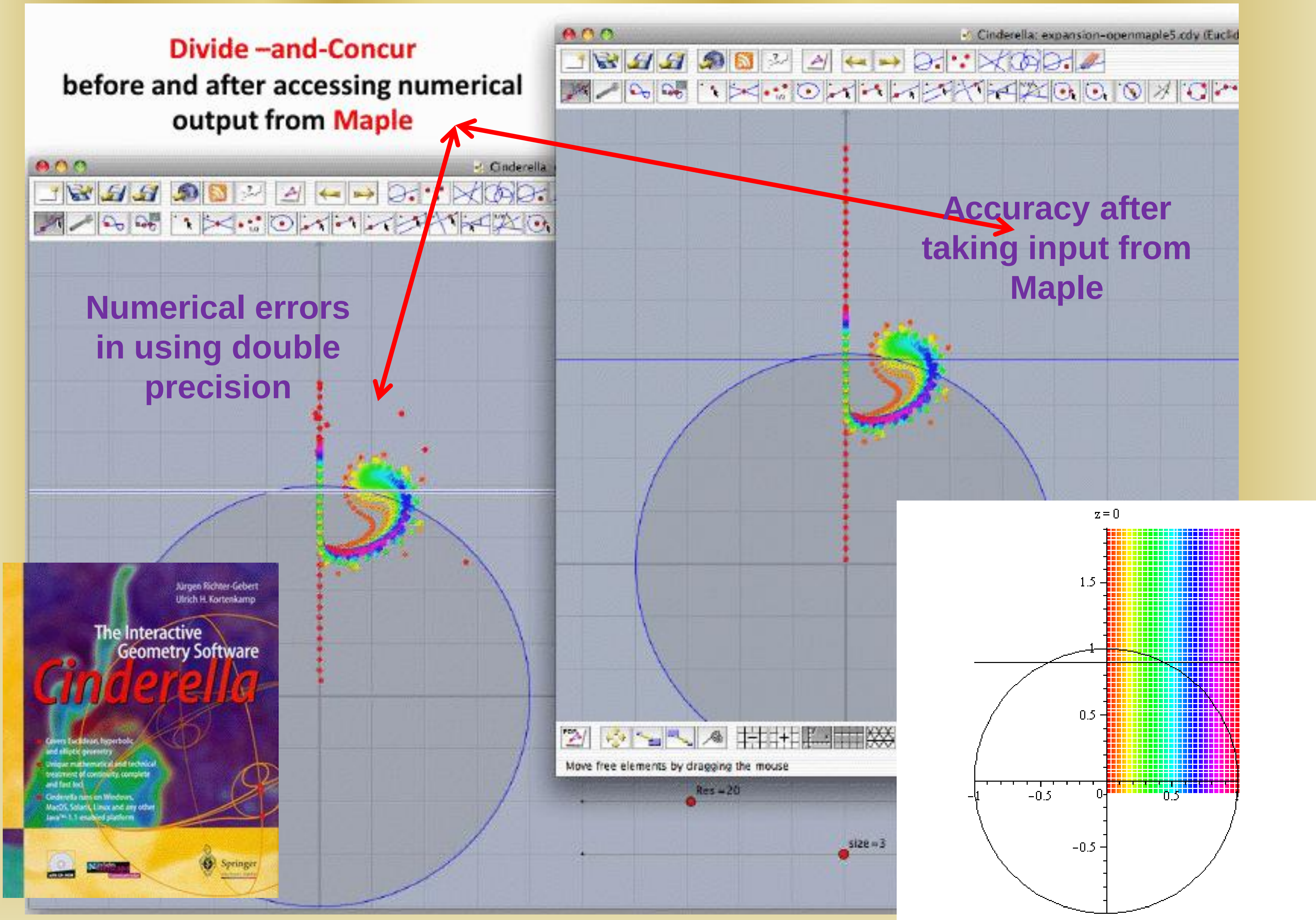
so that the reflections may be 'divided' up and

$$P_B(x) = \left(\frac{x_1 + x_2 + \cdots + x_M}{M}, \ldots, \frac{x_1 + x_2 + \cdots + x_M}{M}\right)$$

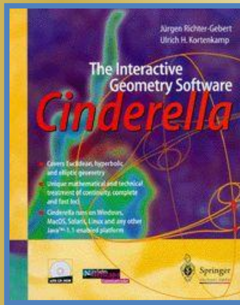
so that the projection and reflection on B are averaging ('concurrences'), hence the name. In this form the algorithm is particularly suited to parallelization.

We can also compose more reflections in serial—we still observe iterates spiralling to a feasible point.

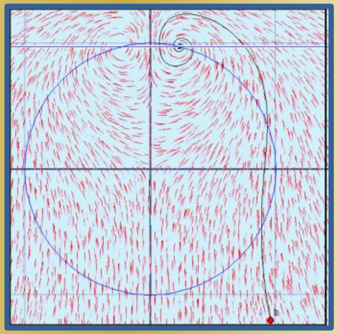
CAS+IGP: THE GRIEF IS IN THE GUI



THE ROUTE TO DISCOVERY



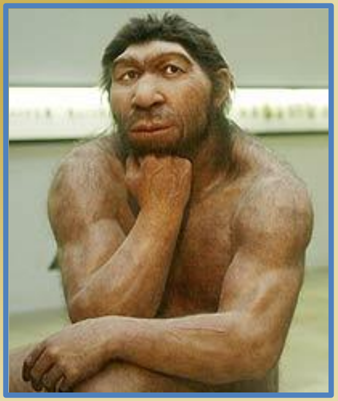
- Exploration first in **Maple** and then in **Cinderella (SAGE)**
  - ability to look at orbits/iterations dynamically is great for insight
  - allows for rapid reinforcement and elaboration of intuition



- Decided to look at **ODE analogues**
  - and their linearizations
  - hoped for **Lyapunov** like results

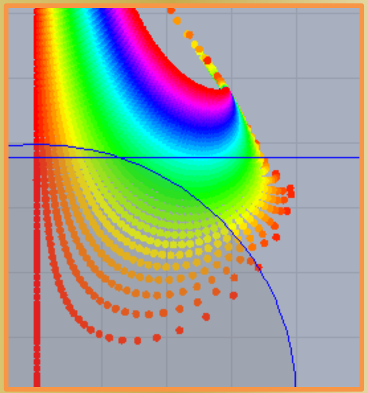
$$x'(t) = \frac{x(t)}{r(t)} - x(t), \quad y'(t) = h - \frac{y(t)}{r(t)},$$

where $r(t) := \sqrt{x(t)^2 + y(t)^2}$, is a reasonable counterpart to the Cartesian formulation—replacing $x_{n+1} - x_n$ by $x'(t)$, etc.—as in the Figure.



- Searched literature for a discrete version
  - found **Perron's work**

THE BASIS OF THE PROOF



Theorem (Perron) If $f : \mathbf{N} \times \mathbf{R}^m \longrightarrow \mathbf{R}^m$ satisfies,

$$\lim_{x \to 0} \frac{\|f(n,x)\|}{\|x\|} = 0,$$

uniformly in n and M is a constant $n \times n$ matrix all of whose eigenvalues lie inside the unit disk, then the zero solution (provided it is an isolated solution) of the difference equation,

$$\boxed{x_{n+1} = M x_n + f(n, x_n),}$$

is *exponentially asymptotically stable*; that is, there exists $\delta > 0$, $K > 0$ and $\zeta \in (0,1)$ such that if $\|x_0\| < \delta$ then $\|x_n\| \leq K \|x_0\| \zeta^n$.

In our case:

$$M = \begin{pmatrix} \alpha^2 & -\alpha\sqrt{1-\alpha^2} & 0 & \cdots & 0 \\ \alpha\sqrt{1-\alpha^2} & \alpha^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Explains spin for height in (0,1)

and the spectrum of the gradient comprises 0, and $\alpha^2 \pm i\alpha\sqrt{1-\alpha^2}$.

WHAT WE CAN NOW SHOW

Theorem [Borwein-Sims 2009] For the case of a sphere in n-space and a line of height α (normalized to $x(2) = \alpha, a = e_1, b = e_2$):

(a) If $0 \leq \alpha < 1$ then the Douglas-Rachford scheme is locally convergent at each of the critical points $\pm\sqrt{1-\alpha^2}a + \alpha b$.

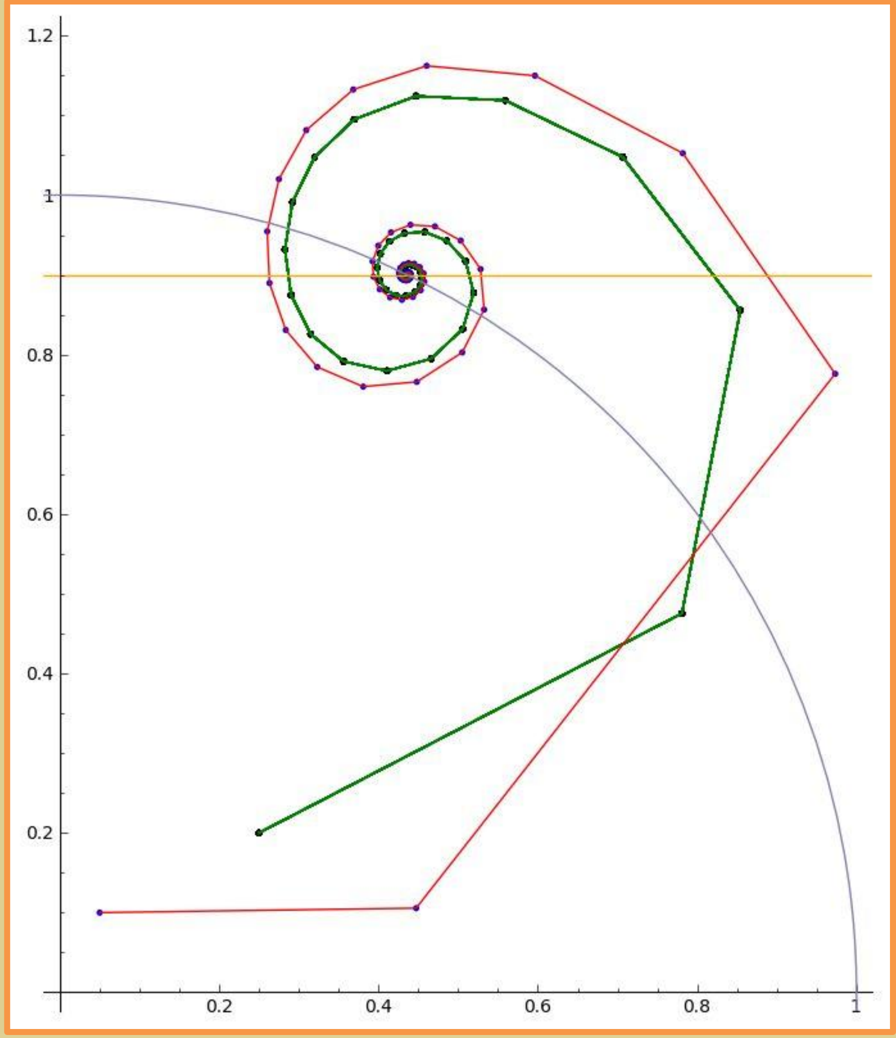
(b) If $\alpha = 0$ and the initial point has $x_0(1) > 0$ then the scheme converges to the feasible point $(1, 0, 0, \cdots, 0)$.

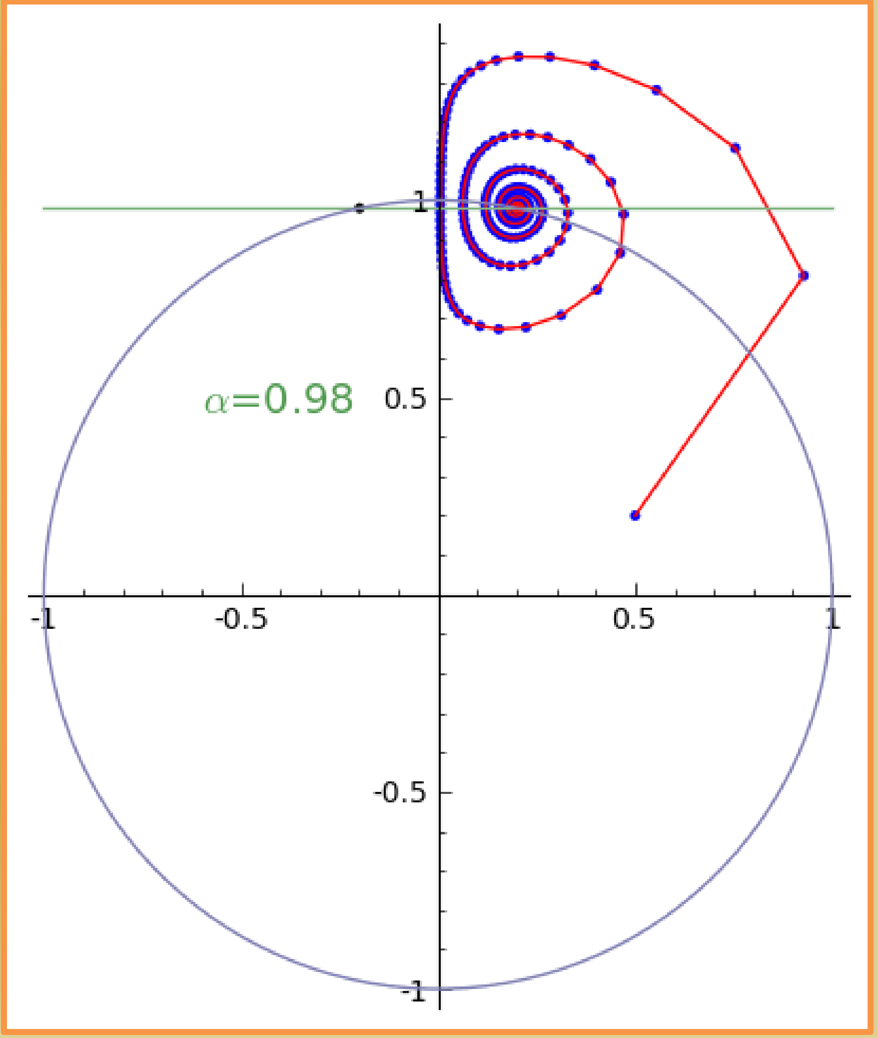
(c) When L is tangential to S at b (that is, when $\alpha = 1$), starting from any initial point with $x_0(1) \neq 0$, the scheme converges to a point yb with $y > 1$.

(d) If there are no feasible solutions (that is, when $\alpha > 1$) then for any non-zero initial point $x_n(2)$ and hence $\|x_n\|$ diverge at at least linear rate to $+\infty$.

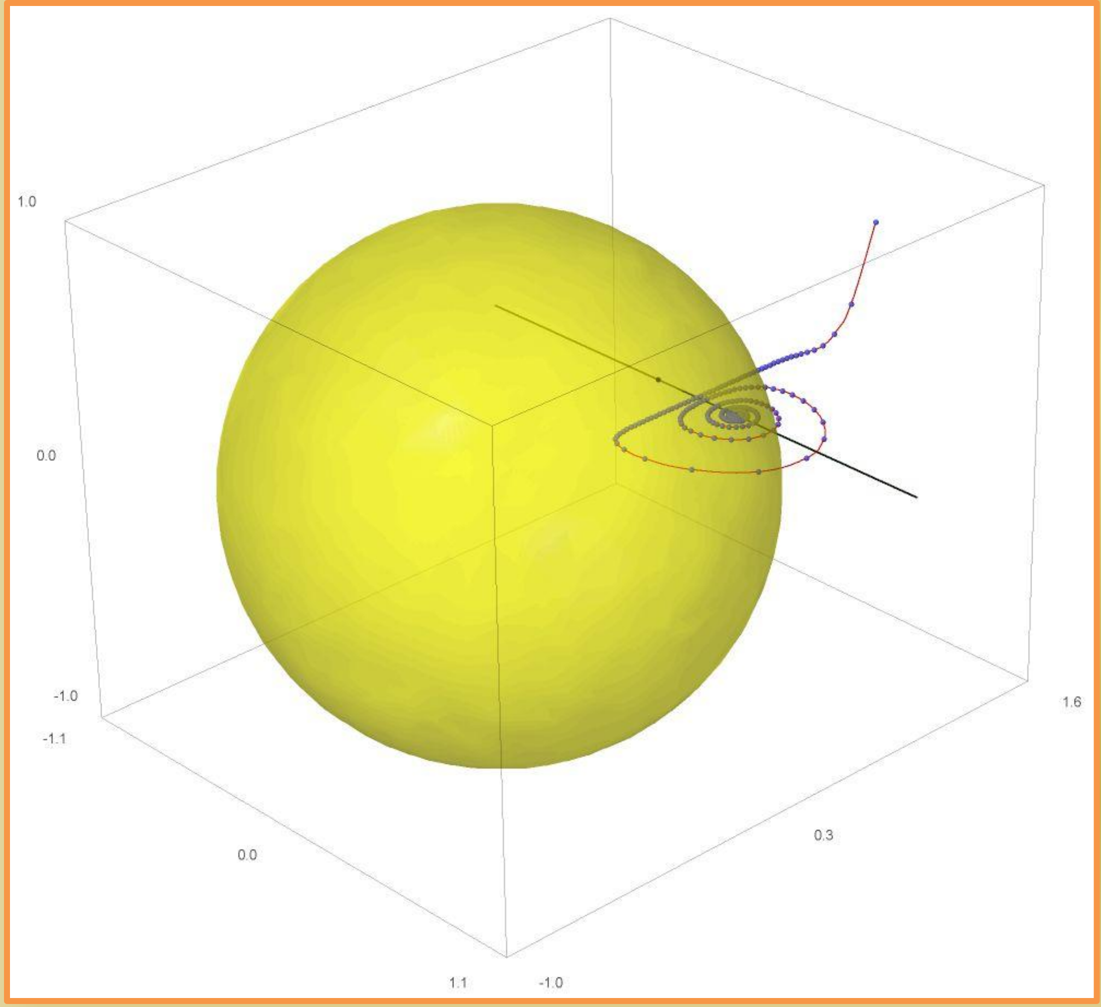
- The same result applies to the sphere S and any *affine* subset B.
- For non-affine B things are substantially more complex—even in the plane.

ALGORITHM *APPEARS* TO BE STABLE





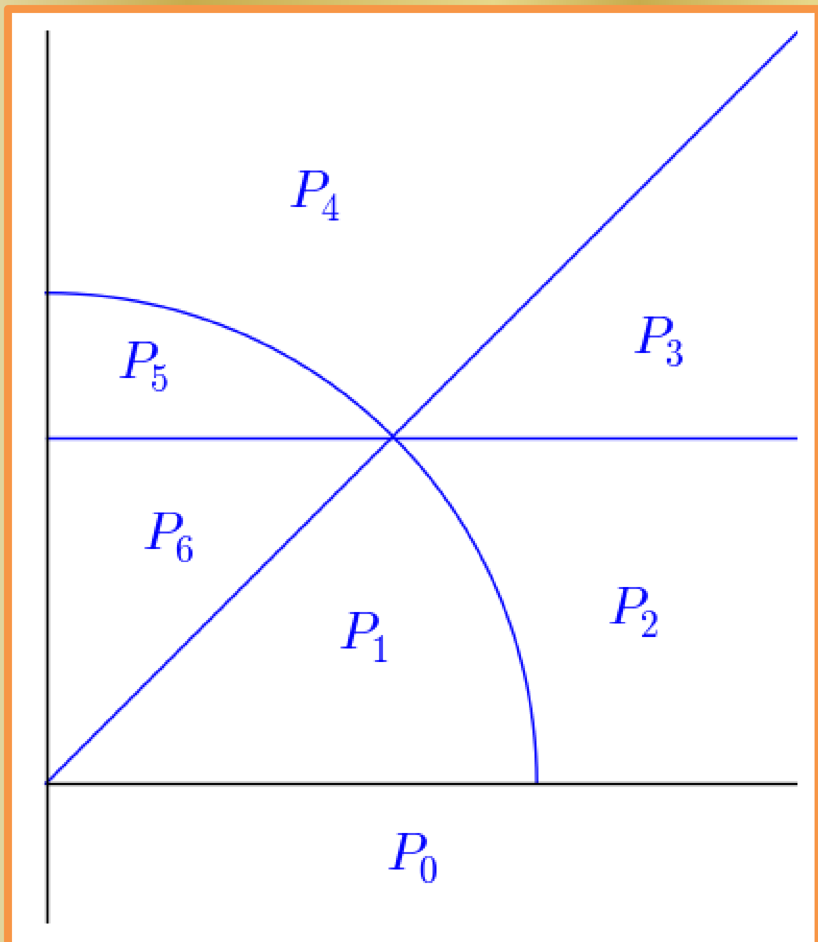
THREE AND HIGHER DIMENSIONS



$$x_{n+1}(1) = x_n(1)/\rho_n,$$
$$x_{n+1}(2) = \alpha + (1 - 1/\rho_n)\, x_n(2), \quad \text{and}$$
$$x_{n+1}(k) = (1 - 1/\rho_n)\, x_n(k), \quad \text{for } k = 3, \cdots, N$$

where $\rho_n := \|x_n\| = \sqrt{x_n(1)^2 + \cdots + x_n(N)^2}$.

AN "EVEN SIMPLER" CASE



Intersection at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

If $(x_n, y_n) \in P_1 \cup P_2 \cup P_3$ *then*

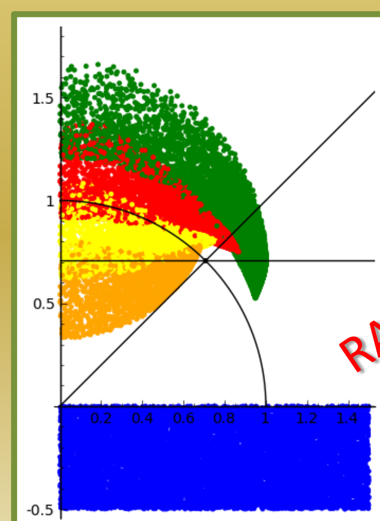
$$|(x_{n+1}, y_{n+1}) - (x^*, y^*)|^2 \leq \frac{1}{2}|(x_n, y_n) - (x^*, y^*)|^2.$$

If $(x_n, y_n) \in P_4$ *then*

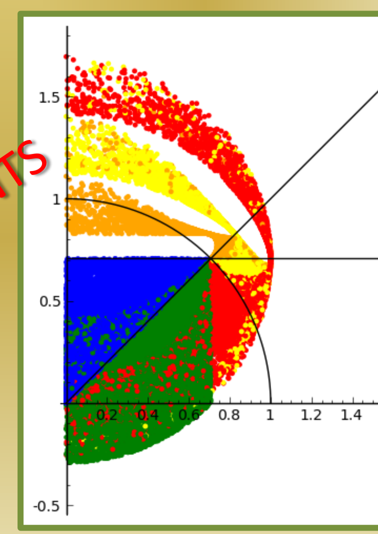
$$|(x_{n+1}, y_{n+1}) - (x^*, y^*)|^2 \leq |(x_n, y_n) - (x^*, y^*)|^2.$$

If $(x_n, y_n) \in P_5 \cup P_6$ *then*

$$|(x_{n+1}, y_{n+1}) - (x^*, y^*)|^2 < \underbrace{\left(\frac{5}{2} - \sqrt{2} + \frac{1}{2}\sqrt{-20\sqrt{2} + 29}\right)}_{\approx 1.51} |(x_n, y_n) - (x^*, y^*)|^2.$$



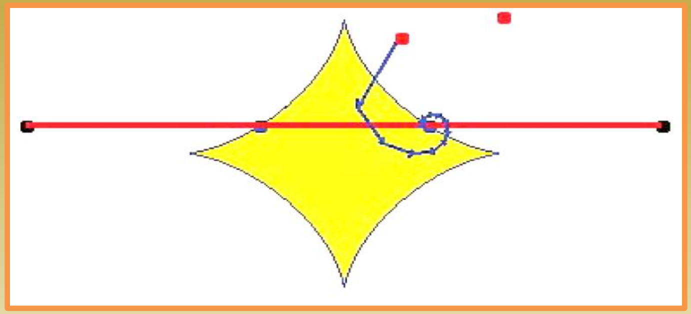
RANDOM POINTS

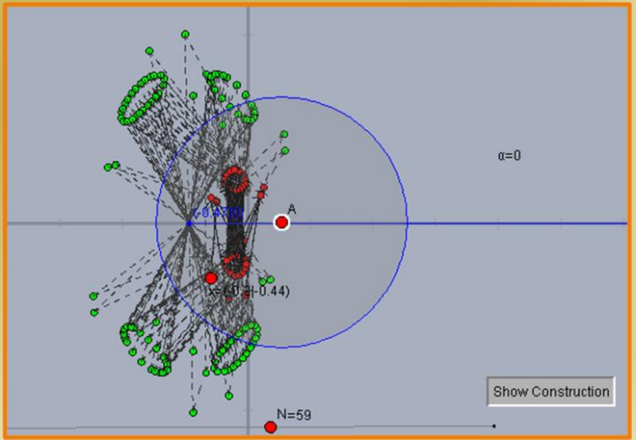


COMMENTS and OPEN QUESTIONS

- As noted, the method parallelizes very well.
- Work out rates in convex case?
- Why does alternating projection (no reflection) work well for **optical aberration** but not **phase reconstruction**?

- Show rigorously **global convergence**
  - in the appropriate basins?
- Extend analysis to more general pairs of sets (and **CAT (0) metrics**)
  - even the half-line case is much more complex
  - as I may now demo.





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http://projects.cs.dal.ca/ddrive/ConvexFunctions/

[3] V. Elser, I. Rankenburg, and P. Thibault, "**Searching with iterated maps**," *Proc. National Academy of Sciences*, **104** (2007), 418–423.

[4] J.M. Borwein and R.L. Luke, ``Duality and Convex Programming," pp. 229–270 in *Handbook of Mathematical Methods in Imaging,* O. Scherzer (Editor-in-Chief), Springer-Verlag. E-pub. 2010.