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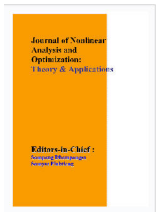

Stability of closedness of convex cones under linear mappings II

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ABSTRACT: In this paper we revisit the question of when the continuous linear image of a fixed closed convex cone K is closed. Specifically, we improve the main result of [3] by showing that for all, except for at most a σ-porous set, of the linear mappings T from $\mathbb{R}^n$ into $\mathbb{R}^m$, not only is $T(K)$ closed, but there is also a neighbourhood around T whose members also preserve the closedness of K.

KEYWORDS: closed convex cone; linear mapping; stability; linear programming.

1. **Introduction**

In [3] we investigated when then continuous linear image of a closed convex cone in $\mathbb{R}^n$ is closed. This was motivated in part by the abstract versions of the Farkas lemma and the Krein-Rutman theorem as given in [2]. The closure of such conical images is central to duality theory in both semi-definite and conical linear programming [2, 3, 9, 10]. Recall that a nonempty set K of a vector space V is a *convex cone* if K is convex and for each $\lambda \in [0, \infty)$ and $x \in K$, $\lambda x \in K$. Although there are simple examples to show that the continuous linear image of a given closed convex cone K in $\mathbb{R}^n$ need not be closed (see [3, Example 1]), it was shown in [3] that in some sense, *for almost all* $T \in L(X, Y)$ —the space of all linear mappings acting between finite dimensional normed linear spaces X and Y, endowed with the operator norm—$T(K)$ is indeed closed in Y.

Specifically, in [3] we showed that for a given closed convex cone K in $\mathbb{R}^n$, $\operatorname{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$ is dense in $L(\mathbb{R}^n, \mathbb{R}^m)$. We also showed that in general

$$\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

is not an open set. However, we did not address the question of the size of the set

$$L(\mathbb{R}^n, \mathbb{R}^m) \setminus \operatorname{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

in terms of measure which, as shown in [7], can be quite distinct from being small in terms of category.

In this paper we remedy this situation by showing that

$$L(\mathbb{R}^n, \mathbb{R}^m) \setminus \operatorname{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$$

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*This paper is dedicated to our colleague Claude Lemarechal on the occasion of his 65th birthday.

is *σ-porous*, a notion which is simultaneously small with regard to both measure and category. We were sure this was so at the time of writing [3] but the details are somewhat subtle. Along the way we shall show in Corollary 2.3 that

$$\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : \operatorname{rank}(T) < \min\{m, n\}\}$$

is σ-porous; a fact that is of independent interest since mappings with gradients of maximal rank admit inverse function theorems [5] and can be used to guarantee metric regularity [1].

2. **Preliminaries**

We start with some notation. For any x in a normed linear space $(X, \|\cdot\|)$ and $r \geq 0$ we shall denote by, $B(x;r)$ the set $\{y \in X : \|y - x\| < r\}$ and for any $M \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $R > 0$ we define $\gamma(x, R, M)$ to be the supremum over all $r \geq 0$ for which there exists $z \in \mathbb{R}^n$ such that $B(z, r) \subseteq B(x, R) \setminus M$. Then we define the *porosity of M at x* as

$$p(M, x) := \liminf_{R \to 0^+} \frac{\gamma(x, R, M)}{R}.$$

Further, we shall say that a set M is *porous at x* if $p(M, x) > 0$ and, moreover, if M is porous at each $x \in M$ then we shall say that M is *porous*. Finally, we shall say that M is *σ-porous* if it is a union of countably many porous sets. Porosity is a very natural geometric notion as the unfamiliar reader may discover by drawing some pictures in the plane.

It is easy to see that σ-porous sets enjoy some permanence properties. For example, if $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on a vector space X then a subset M is σ-porous in $(X, \|\cdot\|)$ if, and only if, it is σ-porous in $(X, |||\cdot|||)$. In fact, an even stronger property is true.

Proposition 2.1. *Suppose If $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ are normed linear spaces are normed spaces and $T : X \to Y$ is a continuous, open linear mapping. Then $T^{-1}(M)$ is σ-porous in $(X, \|\cdot\|)$ whenever $M \subseteq Y$ is σ-porous in $(Y, |||\cdot|||)$, .*

Since the notion of σ-porosity in finite dimensional normed linear spaces is insensitive to the particular choice of norm we shall henceforth (unless otherwise stated) assume that $\mathbb{R}^n$ is equipped with the Euclidean norm and that the space $L(X, Y)$ is equipped with the corresponding operator norm.

Our interest in σ-porosity stems from the fact that σ-porous sets are small in both a measure theoretic sense and in a Baire categorical sense, [11]. More precisely, a Lebesgue measurable set M in $\mathbb{R}^n$ that is σ-porous has Lebesgue measure zero and is at the same time a *first category* set (i.e., a countable union of nowhere dense sets). For further information—old and new—on Baire category the reader could consult [7] and [4].

In order to present our first theorem we need to introduce some matrix notation. For a $m \times n$ matrix A we shall denote by, $[A]_{ij}$ the $(ij)^{\text{th}}$ entry of the matrix A i.e., the entry in the i^{th} row and j^{th} column of the matrix A and by, A_{ij} the sub-matrix of A obtained by deleting the i^{th} row and j^{th} column of A. Finally, we shall denote by $M_{(m,n)}$ the set of all $m \times n$ matrices (over $\mathbb{R}$).

Theorem 2.2. *For each $n \in \mathbb{N}$, the set*

$$\{A \in M_{(n,n)} : \operatorname{rank}(A) = n - 1\}$$

is porous in $M_{(n,n)}$ with respect to any norm on $M_{(n,n)}$.

Proof: Let $\|\cdot\|$ be any norm on $M_{(n,n)}$, $M := \{A \in M_{(n,n)} : \operatorname{rank}(A) = n - 1\}$ and let $B \in M$. It will be sufficient, due to [11, page 517], to show that there is a neighbourhood U of B, subspaces H and F of $M_{(n,n)}$ such that (i) $\operatorname{Dim}(F) = 1$; (ii) $H \oplus F = M_{(n,n)}$ and (iii) a Lipschitz function $f : W \to F$ defined on a nonempty open subset W of H such that $U \cap M = \{x + f(x) : x \in W\}$.

Let $1 \leq i, j \leq n$ be chosen so that $\mathrm{Det}(B_{ij}) \neq 0$. Let $H := \{A \in M_{(n,n)} : [A]_{ij} = 0\}$. Then H is a co-dimension 1 subspace $M_{(n,n)}$. Now define $E_{ij} \in M_{(n,n)}$ by,

$$[E_{ij}]_{i'j'} := \begin{cases} 1 & \text{if } (i,j) = (i',j') \\ 0 & \text{if } (i,j) \neq (i',j') \end{cases}$$

and let $F := \mathrm{span}\{E_{ij}\}$. Finally, let $P_{ij} : M_{(n,n)} \to H$ be defined by, $P_{ij}(A) := A - [A]_{ij}E_{ij}$. Next, let W be any neighbourhood of $P_{ij}(B)$, with respect to the relative topology on H, such that $\mathrm{Det}(A_{ij}) \neq 0$ for all $A \in W$ and let $U := (P_{ij})^{-1}(W)$. Note that for each $A \in U$, $\mathrm{rank}(A) \geq n-1$. If, on the other hand, $A \in U$ and $\mathrm{Det}(A) = 0$ (i.e., if $\mathrm{rank}(A) = n-1$) then

$$\sum_{k=1}^{n} (-1)^{i+k}[A]_{ik}\mathrm{Det}(A_{ik}) = 0$$

and so

$$[A]_{ij} = \frac{1}{\mathrm{Det}(A_{ij})} \sum_{\substack{k=1 \\ k \neq j}}^{n} (-1)^{k+1-j}[A]_{ik}\mathrm{Det}(A_{ik}).$$

Then we define $f : W \to F$ by,

$$f(A) := \left[\frac{1}{\mathrm{Det}(A_{ij})} \sum_{\substack{k=1 \\ k \neq j}}^{n} (-1)^{k+1-j}[A]_{ik}\mathrm{Det}(A_{ik}) \right] E_{ij}$$

and $g : W \to M$ by, $g(A) := A + f(A)$. Since f is C^1 on W, by possibly making W smaller, we can assume that f is Lipschitz on W with respect to $\|\cdot\|$. It is now routine to verify that $M \cap U = \{g(A) : A \in W\}$ since if $A \in M \cap U$ then $P_{ij}(A) \in W$ and $g(P_{ij}(A)) = A$. $\square$

In the following corollary we will repeatedly use the fact that if A' is a sub-matrix of a matrix A, obtained by deleting some rows and/or columns of A, then $\mathrm{rank}(A') \leq \mathrm{rank}(A)$.

We may now prove the result alluded to in the introduction:

Corollary 2.3 (Maximal Rank)**.** *For each $(m,n) \in \mathbb{N}^2$, the set*

$$\{A \in M_{(m,n)} : \mathrm{rank}(A) < \min\{m,n\}\}$$

is σ-porous in $M_{(m,n)}$ with respect to any norm on $M_{(m,n)}$.

Proof: Firstly, we may assume that $m, n \geq 2$. To show that $\{A \in M_{(m,n)} : \mathrm{rank}(A) < \min\{m,n\}\}$ is σ-porous in $M_{(m,n)}$ it is sufficient to show that for each $1 \leq k < \min\{m,n\}$, $\{A \in M_{(m,n)} : \mathrm{rank}(A) = k\}$ is σ-porous. Fix $1 \leq k < \min\{m,n\}$ and let Σ_k denote the set of all strictly increasing functions from $\{1,2,\ldots,k+1\}$ into $\{1,2,\ldots,m\}$ and let Σ_k^* denote the set of all strictly increasing functions from $\{1,2,\ldots,k+1\}$ into $\{1,2,\ldots,n\}$. For each $(\pi,\pi^*) \in \Sigma_k \times \Sigma_k^*$ and $A \in M_{(m,n)}$ let, $A^{(\pi,\pi^*)} \in M_{(k+1,k+1)}$ be the sub-matrix of A defined by, $[A^{(\pi,\pi^*)}]_{ij} := [A]_{\pi(i)\pi^*(j)}$ for each $1 \leq i,j \leq k+1$. Furthermore, let $N_k := \{A \in M_{(k+1,k+1)} : \mathrm{rank}(A) = k\}$.

From Theorem 1 we know that N_k is σ-porous in $M_{(k+1,k+1)}$. For each $(\pi,\pi^*) \in \Sigma_k \times \Sigma_k^*$ let,

$$L_k^{(\pi,\pi^*)} := \{A \in M_{(m,n)} : A^{(\pi,\pi^*)} \in N_k\} = \{A \in M_{(m,n)} : \mathrm{rank}(A^{(\pi,\pi^*)}) = k\}.$$

Since $L_k^{(\pi,\pi^*)}$ is the inverse image of N_k under the linear surjection $A \mapsto A^{(\pi,\pi^*)}$, $L_k^{(\pi,\pi^*)}$ is σ-porous in $M_{(m,n)}$. Now, from linear algebra we can deduce that

$$\{A \in M_{(m,n)} : \mathrm{rank}(A) = k\} \subseteq \bigcup \{L_k^{(\pi,\pi^*)} : (\pi,\pi^*) \in \Sigma_k \times \Sigma_k^*\}$$

as required. $\square$

In order to expedite the proof of our main theorem we shall take this opportunity to record the following prerequisite result. To avoid confusion between scalars and vectors we shall, in the next lemma, denote vectors in bold; and the unit sphere in $\mathbb{R}^n$ by $S_{\mathbb{R}^n}$.

Lemma 2.4. *For any* $\boldsymbol{a} := (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ *and any* $\boldsymbol{x} := (x_1, x_2, \ldots, x_n) \in S_{\mathbb{R}^n}$, *there exists an operator* $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ *such that* $T(\boldsymbol{x}) = \boldsymbol{a}$ *and* $\|T\| = \|\boldsymbol{a}\|$.

Proof: Define $T : \mathbb{R}^n \to \mathbb{R}^m$ by,

$$T(\boldsymbol{y}) := (a_1 \boldsymbol{x} \cdot \boldsymbol{y}, a_2 \boldsymbol{x} \cdot \boldsymbol{y}, \ldots, a_m \boldsymbol{x} \cdot \boldsymbol{y}).$$

Then for any $\boldsymbol{y} \in \mathbb{R}^n$ such that $\|\boldsymbol{y}\| \leq 1$,

$$\begin{aligned} \|T(\boldsymbol{y})\| &= \|(a_1 \boldsymbol{x} \cdot \boldsymbol{y}, a_2 \boldsymbol{x} \cdot \boldsymbol{y}, \ldots, a_m \boldsymbol{x} \cdot \boldsymbol{y})\| \\ &\leq \|(|a_1|, |a_2|, \ldots, |a_m|)\| \quad \text{since } |\boldsymbol{x} \cdot \boldsymbol{y}| \leq 1 \\ &= \|\boldsymbol{a}\|. \end{aligned}$$

Therefore $\|T\| \leq \|\boldsymbol{a}\|$. On the other hand,

$$\|T\| \geq \|T(\boldsymbol{x})\| = \|(a_1 \boldsymbol{x} \cdot \boldsymbol{x}, a_2 \boldsymbol{x} \cdot \boldsymbol{x}, \ldots, a_m \boldsymbol{x} \cdot \boldsymbol{x})\| = \|\boldsymbol{a}\|. \quad \square$$

There are various known sufficient conditions concerning when the continuous linear image of a closed convex cone K is closed. The best known is the classical result that it suffices that K be polyhedral [2]. The following is effectively a specialization of a *recession direction* [2] condition:

Proposition 2.5. [3, Proposition 3] *Let* $T \in L(\mathbb{R}^n, \mathbb{R}^m)$ *and let* K *be a closed cone (not necessarily convex) in* $\mathbb{R}^n$. *If*

$$K \cap \ker(T) = \{0\}$$

then there exists a neighbourhood $\mathcal{N}$ *of* T *in* $L(\mathbb{R}^n, \mathbb{R}^m)$ *such that* $S(K)$ *is closed in* $\mathbb{R}^m$ *for each* $S \in \mathcal{N}$.

For a subset D of a vector space V, the *core* of D, denoted, $\text{cor}(D)$, is the set of all points $d \in D$ where for each $x \in V \setminus \{d\}$ there exists an $0 < r < 1$ such that $\lambda x + (1-\lambda)d \in D$ for all $0 \leq \lambda < r$. Clearly if the affine span $\text{aff}(D) \neq V$ then $\text{cor}(D) = \varnothing$. In this case the following concept is useful.

Given a subset C of a vector space V, the *intrinsic core* of C, denoted $\text{icor}(A)$, is the set of all points $c \in C$ where for each $x \in \text{aff}(C)$ there exists an $0 < r < 1$ such that $\lambda x + (1-\lambda)c \in C$ for all $0 \leq \lambda < r$.

One of the most important properties of the intrinsic core is that if C is a convex subset of a finite dimensional vector space V then $\text{icor}(C) \neq \varnothing$, [6, page 7]. In fact, if V is a finite dimensional topological vector space then $\text{icor}(C)$ is dense in C for each convex subset C of the space V. Another important property of the core is that for a convex subset C of a finite dimensional topological vector space, $\text{cor}(C) = \text{int}(C)$, [2, Theorem 4.1.4].

The reason for our interest in the intrinsic core is that it provides the other sufficient condition that we shall need to apply:

Proposition 2.6 (Intrinsic core)**.** [3, Proposition 5] *Let* Y *be a normed linear space,* $T : \mathbb{R}^n \to Y$ *be a linear transformation and let* K *be a closed cone in* $\mathbb{R}^n$. *If*

$$\ker(T) \cap \text{icor}(K) \neq \varnothing$$

then $T(K)$ *is a finite dimensional linear subspace of* Y *and hence a closed convex cone.*

Corollary 2.7. [3, Corollary 2] *The only way* $T(K)$ *can fail to be closed is if*

$$\ker(T) \cap K \subseteq K \setminus \text{icor}(K)$$

and that at the same time $\ker(T) \cap K$ *is not a linear subspace. In particular,* $\ker(T) \cap K \neq \{0\}$.

3. Main Results

We require one more lemma:

Lemma 3.1. *Let Y be an n-dimensional normed linear space and let K be a closed convex cone in Y. Then $L(Y,\mathbb{R}^m)\setminus \mathrm{int}\{T\in L(Y,\mathbb{R}^m): T(K) \text{ is closed}\}$ is σ-porous in $L(Y,\mathbb{R}^m)$ if, and only if, $L(\mathbb{R}^n,\mathbb{R}^m)\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(\varphi(K)) \text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$ where $\varphi: Y\to\mathbb{R}^n$ is any linear bijection from Y onto $\mathbb{R}^n$.*

Proof: Let $\varphi: Y\to\mathbb{R}^n$ be a linear bijection from Y onto $\mathbb{R}^n$ and let $\varphi^\#: L(\mathbb{R}^n,\mathbb{R}^m)\to L(Y,\mathbb{R}^m)$ be defined by, $\varphi^\#(T):=T\circ\varphi$. Then $\varphi^\#$ is an isomorphism from $L(\mathbb{R}^n,\mathbb{R}^m)$ onto $L(Y,\mathbb{R}^m)$ and

$$\{T\in L(Y,\mathbb{R}^m): T(K)\text{ is closed}\}=\varphi^\#(\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(\varphi(K))\text{ is closed}\}).\quad\Box$$

The next result shows—as promised—that although it is not true that, if $T(K)$ is closed for some closed convex cone K then $S(K)$ is closed for all S in some neighbourhood of T, it is "almost" true, in the sense that *for "almost all" $T\in L(\mathbb{R}^n,\mathbb{R}^m)$ if $T(K)$ is closed then there exists a neighbourhood $\mathcal{W}$ of T such that $S(K)$ is closed for all $S\in\mathcal{W}$*. More precisely:

Theorem 3.2. *Suppose that K is a closed convex cone in $\mathbb{R}^n$ then*

$$L(\mathbb{R}^n,\mathbb{R}^m)\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\}$$

is a σ-porous set in $L(\mathbb{R}^n,\mathbb{R}^m)$.

Proof: Let $Y:=K-K$. We shall consider first the case when $Y=\mathbb{R}^n$. Let $\mathcal{M}$ be the family of all linear mappings $T\in L(\mathbb{R}^n,\mathbb{R}^m)$ with maximal rank (i.e., $\mathrm{rank}(T)=\min\{m,n\}$).

It is easy to verify that $\mathcal{M}$ is an open subset of $L(\mathbb{R}^n,\mathbb{R}^m)$. Let $\varphi: M_{(m,n)}\to L(\mathbb{R}^n,\mathbb{R}^m)$ be defined by, $\varphi(A)(\boldsymbol{x}):=A\boldsymbol{x}$. Then φ is an isomorphism from $M_{(m,n)}$ onto $L(\mathbb{R}^n,\mathbb{R}^m)$. Moreover, $\varphi^{-1}(\mathcal{M})=\{A\in M_{(m,n)}: A \text{ has maximal rank}\}$. Therefore, from Corollary 1, $L(\mathbb{R}^n,\mathbb{R}^m)\setminus\mathcal{M}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$. Hence to show that $L(\mathbb{R}^n,\mathbb{R}^m)\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$ it is sufficient to show that $\mathcal{M}\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\}$ is σ-porous in $L(\mathbb{R}^n,\mathbb{R}^m)$; which is what we shall do. There are two cases:

(i) If $n\le m$ then each member of $\mathcal{M}$ is one-to-one and so $\ker(T)\cap K=\{0\}$ for each $T\in\mathcal{M}$ and thus we are done by Proposition 2.

(ii) Hence we shall assume that $m<n$. We now define $P:\mathcal{M}\to L(\mathbb{R}^n,\mathbb{R}^n)$ by, $P(S):=I-S^*(SS^*)^{-1}S$, where I is the identity mapping on $\mathbb{R}^n$ and S^* is the conjugate of S, i.e., $S^*\in L(\mathbb{R}^m,\mathbb{R}^n)$ and $S^*(\boldsymbol{y})\cdot\boldsymbol{x}=\boldsymbol{y}\cdot S(\boldsymbol{x})$ for all $\boldsymbol{y}\in\mathbb{R}^m$ and all $\boldsymbol{x}\in\mathbb{R}^n$. It is routine to check that:

(i) for each $S\in\mathcal{M}$, $P(S)$ is well-defined, i.e., $(SS^*)^{-1}$ exists;
(ii) P is C^1 on $\mathcal{M}$ and hence locally Lipschitz on $\mathcal{M}$;
(iii) for each $S\in\mathcal{M}$, $P(S)$ is the projection of $\mathbb{R}^n$ onto $\ker(S)$.

For each $n\in\mathbb{N}$, let

$$L_n:=\{S\in\mathcal{M}: \text{ there exists an open neighbourhood } N \text{ of } S \text{ such that } P|_N \text{ is } n\text{-Lipschitz}\}$$

Now each L_n is an open subset of $\mathcal{M}$ and $\mathcal{M}=\bigcup_{n=1}^\infty L_n$. So to show that

$$\mathcal{M}\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\}$$

is σ-porous it is sufficient to show that for each $n\in\mathbb{N}$,

$$E_n:=L_n\setminus \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\}$$

is porous.

To this end, let us fix $n\in\mathbb{N}$ and consider $T\in E_n$. Since

$$T\notin \mathrm{int}\{T\in L(\mathbb{R}^n,\mathbb{R}^m): T(K)\text{ is closed}\},$$

it follows that

$$\{0\}\neq\ker(T)\cap K\subseteq K\setminus \mathrm{icor}(K).$$

Choose $x \in \ker(T) \cap K$ such that $\|x\| = 1$. Note that this is possible since $\ker(T) \cap K$ is a nontrivial cone. Now select $y \in \text{int}(K) = \text{cor}(K) = \text{icor}(K) \neq \varnothing$ such that $\|y\| = 1$. Also choose $0 < r < 1$ such that $B(y; r) \subseteq \text{int}(K)$. We claim that

$$p(E_n, T) \geq \alpha := r/(8n[\|T\| + 1]) > 0.$$

Let $0 < R_0 < 1$ be chosen so that P is n-Lipschitz on $B(T; R_0)$ and for each $0 < R < R_0$ let, $\lambda_R := R/(8[\|T\| + 1])$ and $z_R := \lambda_R y + (1 - \lambda_R)x$. Now for each $0 < \lambda < 1$,

$$B(x + \lambda(y - x); \lambda r) = \lambda B(y, r) + (1 - \lambda)x \subseteq K$$

since K is convex and so $B(x + \lambda(y - x); \lambda r) \subseteq \text{int}(K)$. In particular, $B(z_R; \lambda_R r) \subseteq \text{int}(K)$. Now,

$$\|T(z_R)\| = \|T(x + \lambda_R(y - x)\| = \lambda_R\|T(y - x)\| \leq \lambda_R\|T\|\|y - x\| \leq 2\lambda_R\|T\| < R/4$$

and $1 \geq \|z_R\| = \|x + \lambda_R(y - x)\| \geq \|x\| - \lambda_R\|y - x\| \geq 1 - 2\lambda_R > 1 - 2(1/8) = 3/4$. By Lemma 1 there exists a $S'_R \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that $S'_R(z_R) = T(z_R)$ and $\|S'_R\| < R/3$. Let $S_R := T - S'_R$. Then $\|T - S_R\| < R/3$, $B(S_R; \alpha R) \subseteq B(T; R)$ (since $\alpha \leq 1/8$) and $S_R(z_R) = 0$. To complete the proof in this case we argue below that this holds for all $0 < R < R_0$, $p(E_n, T) \geq \alpha > 0$.

Claim: *For each $0 < R < R_0$, $B(S_R; \alpha R) \subseteq int\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K)$ is closed$\}$.*

Proof of Claim: To see this, suppose that $S'' \in B(S_R; \alpha R)$, that is

$$\|S_R - S''\| < rR/(8n[\|T\| + 1]).$$

To show that

$$S'' \in \text{int}\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K) \text{ is closed}\}$$

it is sufficient to show that $\ker(S'') \cap \text{int}(K) \neq \varnothing$. In fact, since $P(S'')(z_R) \in \ker(S'')$ it is enough to show that $P(S'')(z_R) \in \text{int}(K)$. Now

$$\begin{aligned}
\|P(S'')(z_R) - z_R\| &= \|P(S'')(z_R) - P(S_R)(z_R)\| \quad \text{since } z_R \in \ker(S_R) \\
&\leq \|P(S'') - P(S_R)\|\|z_R\| \\
&\leq \|P(S'') - P(S_R)\| \quad \text{since } \|z_R\| \leq 1 \\
&\leq n\|S'' - S_R\| \quad \text{since } P \text{ is } n\text{-Lipschitz on } B(S_R, \alpha R) \\
&< \frac{nrR}{8n(\|t\| + 1)} = \frac{rR}{8(\|T\| + 1)} = \lambda_R r.
\end{aligned}$$

Therefore, $P(S'')(z_R) \in B(z_R; \lambda_R r) \subseteq \text{int}(K)$. Thus,

$$S'' \in \text{int}\{S \in L(\mathbb{R}^n, \mathbb{R}^m) : S(K) \text{ is closed}\};$$

which concludes the proof of the claim. □

In the case when Y is a proper subspace of $\mathbb{R}^n$, it follows from Lemma 2 and the previous case that $L(Y, \mathbb{R}^m) \setminus \text{int}\{T \in L(Y, \mathbb{R}^m) : T(K) \text{ is closed}\}$ is σ-porous in $L(Y, \mathbb{R}^m)$. To finish the proof we consider the linear surjection $R : L(\mathbb{R}^n, \mathbb{R}^m) \to L(Y, \mathbb{R}^m)$ defined by, $R(T) := T|_Y$. Then by setting $E := L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T(K) \text{ is closed}\}$ we have that:

$$\begin{aligned}
E &= L(\mathbb{R}^n, \mathbb{R}^m) \setminus \text{int}\{T \in L(\mathbb{R}^n, \mathbb{R}^m) : T|_Y(K) \text{ is closed}\} \\
&= R^{-1}(L(Y, \mathbb{R}^m)) \setminus \text{int}[R^{-1}(\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\})] \\
&= R^{-1}(L(Y, \mathbb{R}^m)) \setminus R^{-1}(\text{int}\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\}) \quad (*) \\
&= R^{-1}(L(Y, \mathbb{R}^m) \setminus \text{int}\{S \in L(Y, \mathbb{R}^m) : S(K) \text{ is closed}\}).
\end{aligned}$$

The equality in line $(*)$ follows from the general fact that for any continuous and open mapping $R : X \to Y$ acting between topological spaces and any set $A \subseteq Y$, $R^{-1}(\text{int}(A)) = \text{int}(R^{-1}(A))$. The proof is now completed by appealing to Proposition 1. □

Further results on the images of closed convex cones under linear mappings may be found in [8, 9].

References

[1] H. H. Bauschke, J. M. Borwein and P. Tseng, "Metric regularity, strong CHIP, and CHIP are distinct properties," J. Convex Analysis, 7 (2000), 395–412.

[2] J. M. Borwein and A. S. Lewis, Convex analysis and nonlinear optimization. Theory and examples. Second edition. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 3, Springer, New York, 2006.

[3] J. M. Borwein and W. B. Moors, "Stability of closedness of convex cones under linear mappings," J. Convex Anal. 16 (2009), 699–705.

[4] J. Cao and W. B. Moors, "A survey on topological games and their applications in analysis," RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 100 (2006), no. 1-2, 39–49.

[5] A. Dontchev and R. T. Rockafellar, Implicit Functions and Solution Mappings: A View from Variational Analysis, Springer-Verlag, 2009.

[6] John R. Giles, Convex analysis with applications in the differentiation of convex functions. Research Notes in Mathematics 58 Pitman (Advanced Publishing Program) Boston, Mass-London, 1982.

[7] John C. Oxtoby, Measure and Category. A survey of the analogies between topological and measure spaces, Second edition. Graduate Texts in Mathematics, 2. Springer-Verlag, New York-Berlin, 1980.

[8] Gabor Pataki, "On the closedness of the linear image of a closed convex cone," Math of Oper. Res., **32** (2007), 395–412.

[9] Gabor Pataki and L. Tuncel, "On the Generic Properties of Convex Optimization Problems in Conic Form," Mathematical Programming A, 89 (2001), 449–457.

[10] H. Wolkowicz, R. Saigal, and L. Vandenberghe, Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, International Series in Operations Research & Management Science, 27, Springer-Verlag, 2001.

[11] L. Zajíček, "On σ-porous sets in abstract spaces," Abstract Applied Anal. 2005 (2005), 509–534.

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