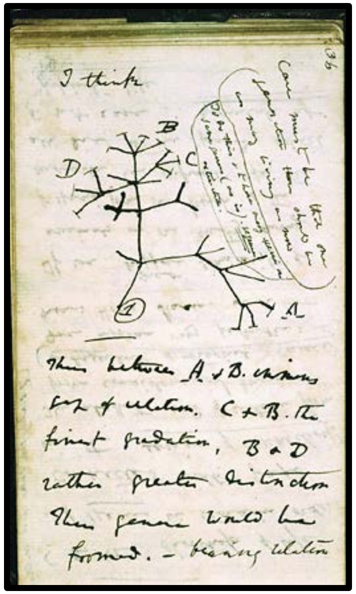
CARMA OANT SEMINAR

Best approximation in (reflexive) Banach space

CARMA

Alan Turing's Enigma

Charles Darwin's notes



March 25, April 8, 15,… of 2013

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Revised 18-3-2013

ABSTRACT

- I will sketch some of the key results known about existence of best approximation in Banach space.
- Nonsmooth analysis, renorming theory and Banach space geometry are crucial tools.
- My main source is

  J.M. Borwein and S. Fitzpatrick, "Existence of nearest points in Banach spaces," *Canadian Journal of Mathematics,* **61** (1989), 702-720.

  which while twenty five years old has largely not been superseded (*porosity* is an exception): see

  J. P. Revalski and N.V. Zhivkov:

  "Small sets in best approximation theory." *J. Global Opt,* **50**(1) (2011), 77-91;

  "Best approximation problems in compactly uniformly rotund spaces," *J. Convex Analysis,* **19**(4) (2012), 1153-1166.

SOME OPEN QUESTIONS

- I will also pose some of the main open questions including

Question 1 *Is every Chebyshev set in Hilbert space convex?*

Question 2 *Is every closed set in Hilbert space with unique farthest points a singleton?*

Question 3 *Is every Chebyshev set in a rotund reflexive Banach space convex?*

Question 4 *Does every closed set in a reflexive Banach space admit a nearest point? What about rotund smooth renormings of Hilbert space?*

Question 5 *Does every closed set in a reflexive Banach space admit proximal normals at a dense set of boundary points?*

OTHER REFERENCES

[1] J.M. Borwein, "Proximality and Chebyshev sets," *Optimization Letters*, **1** , no. 1 (2007), 21-32.

[2] J. M. Borwein, "Future Challenges for Variational Analysis." *Variational analysis and generalized differentiation in optimization and control*, 95-107, *Springer Optim. Appl.*, **47**, Springer, New York, 2010.

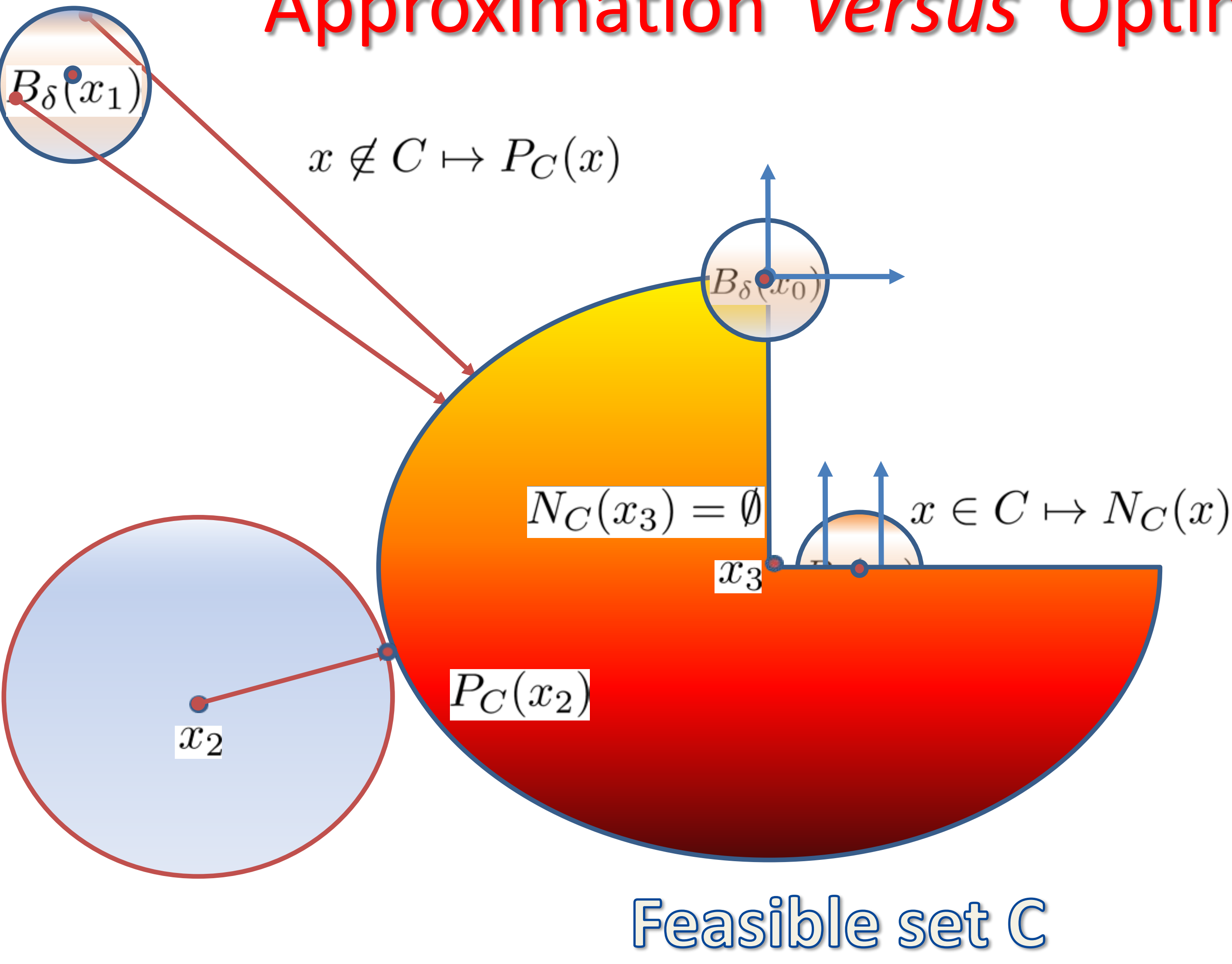
[3] J. M. Borwein, M. Jiménez Sevilla and J. P. Moreno, "Antiproximinal norms in Banach spaces," *J. Approx. Theory*. **114** (2002), 57-69.

[4] Jonathan Borwein, Jay Treiman and Qiji Zhu, "Partially Smooth Variational Principles and Applications," *J. Nonlinear Analysis, Theory Methods Applications*, **38** (1999), 1031-1059.

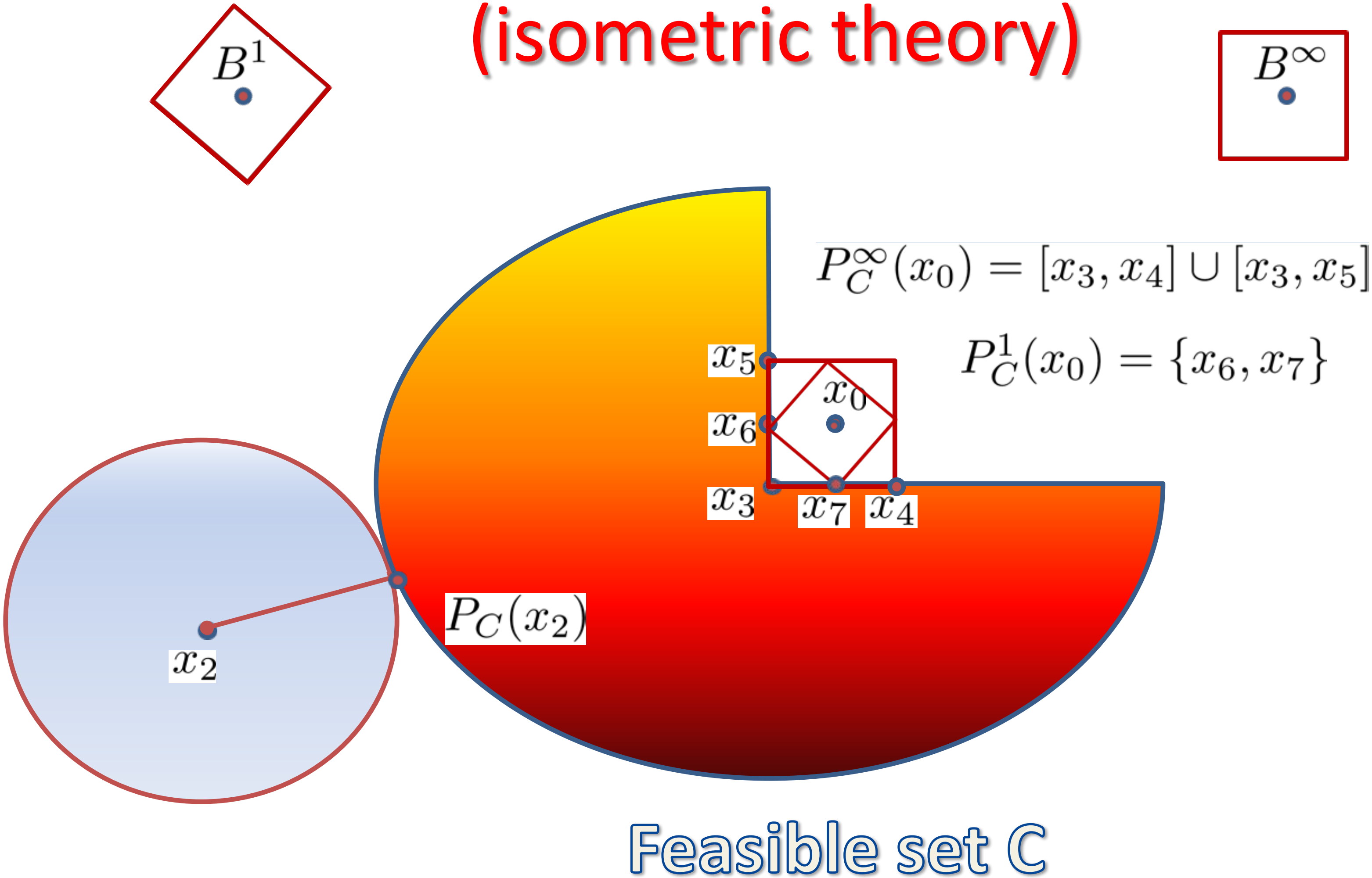
[5] J.M. Borwein and W.B. Moors, "Essentially smooth Lipschitz functions," *Journal of Functional Analysis*, **149** (1997), 305-351.

[6] Stefan Cobzaş "Geometric properties of Banach spaces and the existence of nearest and farthest points," *Abstr. Appl. Anal., no. 3* (2005), 259–285.

Approximation *versus* Optimization



Best approximations rely on the norm (isometric theory)



Simon Fitzpatrick and Regina Burachik (2004)

Other people we shall meet

Edgar Asplund
(1931-74)

Pafnuty Chebysev
(1821-94)

Ivar Ekeland
(1944-)

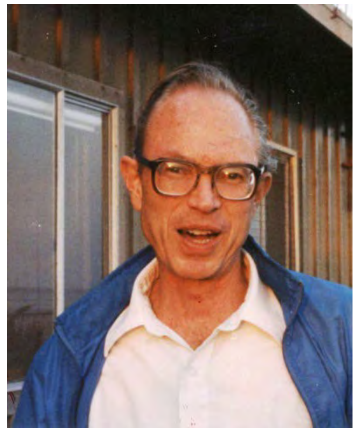
Werner Fenchel
(1905-88)

Victor Klee
(1925-2007)

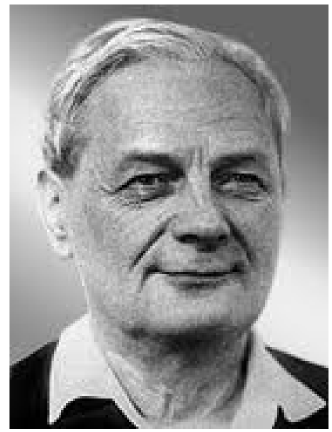
Sergei Konjagin
(1957-)

Ka Sing Lau
(1948-)

Theodore Motzkin
(1908-70)



Bob Phelps
(1926-2013)



Sergei Stechkin
(1920-1995)

PART I

"Best Approximation Problems in (Reflexive) Banach Space"

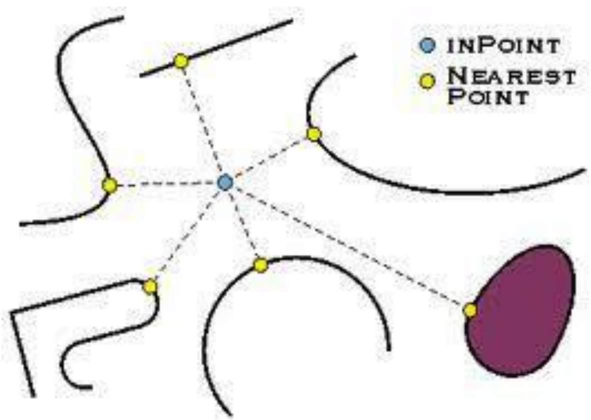
What can we say in terms of x about

$$\operatorname{argmin}_{a\in A}\|x-a\|$$

when A is a non-trivial norm-closed set in a Banach space $(X, \|\cdot\|)$?

In Part I, we shall

- explore the basic structure of the problem
- introduce various analytic tools
- produce some first results

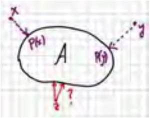


EXISTENCE OF NEAREST POINTS IN BANACH SPACES

JONATHAN M. BORWEIN AND SIMON FITZPATRICK

1. Introduction. This paper makes a unified development of what the authors know about the existence of nearest points to closed subsets of (real) Banach spaces. Our work is made simpler by the methodical use of subderivatives. The results of Section 3 and Section 7 in particular are, to the best of our knowledge, new. In Section 5 and Section 6 we provide refined proofs of the Lau-Konjagin nearest point characterizations of reflexive Kadec spaces (Theorem 5.11, Theorem 6.6) and give a substantial extension (Theorem 5.12). The main open question is: are nearest points dense in the boundary of every closed subset of every reflexive space? Indeed can a proper closed set in a reflexive space fail to have any nearest points? In Section 7 we show that there are some non-Kadec reflexive spaces in which nearest points are dense in the boundary of every closed set.

If E is a real Banach space and C is a closed non-empty subset of E then the distance function d_C is defined by



$$\boxed{d_C(x) := \inf\{\|x - z\| : z \in C\},}$$

and any z in C with $d_C(x) = \|x - z\|$ is a *nearest point* in C to x. If $z \in C$ and there is some $x \in E \backslash C$ with z as its nearest point we call z a *nearest point*. Also $B[x, \alpha]$ and $B(x, \alpha)$ denote respectively the closed and open balls around x of radius $\alpha \geqq 0$.

Definition 1.1. (a) If every $x \in E \backslash C$ has a nearest point in C, we call C *proximinal*. (b) If the set of points in $E \backslash C$ possessing nearest points in C is generic (contains a dense G_δ) we call C *almost proximinal*. (c) A sequence $\{z_n\}$ of elements in C is called a *minimizing sequence* in C for x if

$$d_C(x) = \lim_{n \to \infty} \|x - z_n\|.$$

Definition 1.2. Let f be an extended real valued function f defined on a Banach space with $f(x)$ finite. Then f is *Fréchet subdifferentiable* at x with $x^* \in E^*$ belonging to the *Fréchet subdifferential* at $x, \partial^F f(x)$, provided that

$$\liminf_{y \to 0} \frac{f(x+y) - f(x) - \langle x^*, y \rangle}{\|y\|} \geqq 0.$$

Received September 1, 1988. The research of the first author was partially supported by NSERC. We would like to thank John Giles and the Department of Mathematics at the University of Newcastle, Australia, for their support and hospitality while this work was in progress.

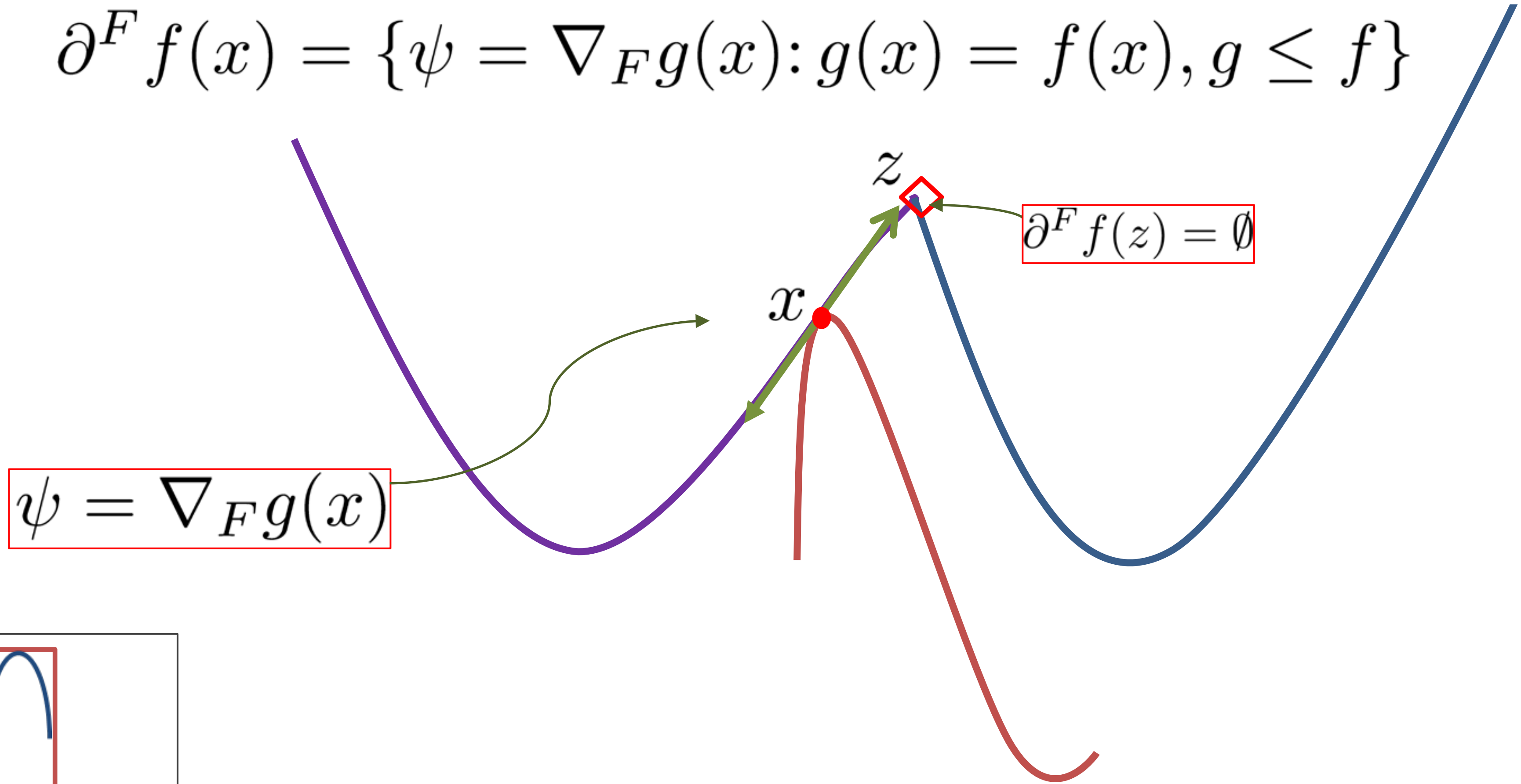
Newcastle July 1988

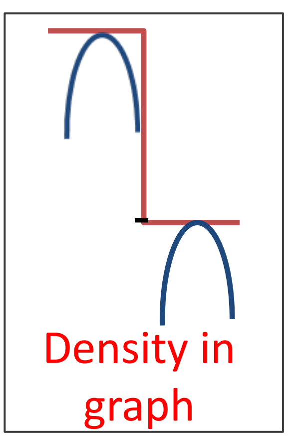
Back: Manash Mukherjee (student) Prof Jack Gray Prof Robin Tucker Dr Gar de Barra
Middle: **JMB**
Front: Prof Jiri Bicak Dr Jim McDougall **Dr Simon Fitzpatrick** Dr Marvin Bishop

Fréchet (viscosity) Subderivatives

(in reflexive spaces or just with Fréchet renorms)

$$\partial^F f(x) = \{\psi = \nabla_F g(x) \colon g(x) = f(x), g \le f\}$$





$f(y)$ above $g(y)$ (locally 'osculating')

Smooth Variational Principle invented for this

THEOREM 1.3. [3] *Let f be a lower semicontinuous function on a Banach space with equivalent Fréchet differentiable norm (in particular, E reflexive will do). Then f is Fréchet subdifferentiable on a dense subset of its graph.*

For distance functions, Fréchet subdifferentiability has the following important consequences.

PROPOSITION 1.4. *Suppose that C is a closed non-empty subset of a Banach space and that $x^* \in \partial^F d_C(x)$ for $x \in E/C$. Then $\|x^*\| = 1$, and for each minimizing sequence $\{z_n\}$ in C for x*

$$d_C(x) = \lim_{n \to \infty} \langle x^*, x - z_n \rangle.$$

Proof. Suppose $\{z_n\}$ is a minimizing sequence in C for x while $0 < t < 1$. We have

$$\begin{aligned} d_C(x + t(z_n - x)) - d_C(x) &\leqq \|x + t(z_n - x) - z_n\| - d_C(x) \\ &\leqq \|x + t(z_n - x) - z_n\| - \|x - z_n\| + [\|x - z_n\| - d_C(x)] \\ &= -t\|x - z_n\| + [\|x - z_n\| - d_C(x)], \end{aligned}$$

and, letting

$$t_n := 2^{-n} + [\|x - z_n\| - d_C(x)]^{1/2},$$

Tends to zero

we have from Fréchet subdifferentiability that

$$\liminf_{n\to\infty} \frac{d_C(x + t_n(z_n - x)) - d_C(x)}{t_n} - \langle x^*, z_n - x\rangle \geqq 0$$

we have from Fréchet subdifferentiability that

$$\boxed{\liminf_{n\to\infty} \frac{d_C(x+t_n(z_n-x))-d_C(x)}{t_n} - \langle x^*, z_n - x\rangle \geqq 0}$$

so that

$$\liminf[-\|x-z_n\| + \langle x^*, z_n - x\rangle + t_n] \geqq 0,$$

and

$$d_C(x) = \lim \|x - z_n\| \leqq \liminf \langle x^*, x - z_n\rangle.$$

Now $\|x^*\| \leqq 1$ since d_C is 1-Lipschitz. It follows that

$$d_C(x) = \lim \|x - z_n\| \geqq \limsup \langle x^*, x - z_n\rangle.$$

Comparison of these last two inequalities shows that $\|x^*\| = 1$ and that

$$\boxed{d_C(x) = \lim_{n\to\infty} \langle x^*, x - z_n\rangle.}$$ ∎

2. Special classes of sets: weak compactness. The first class of closed sets which have many nearest points are those with weak compactness properties.

LEMMA 2.1. *Suppose that C is a closed subset of a Banach space E while $x \in E \backslash C$. If some minimizing sequence $\{z_n\}$ in C for x has a weak cluster point z which lies in C then z is a nearest point to x in C.*

Proof. By the weak lower semicontinuity of the norm we have

$$d_C(x) \leqq \|x - z\| \leqq \liminf \|x - z_n\| \leqq d_C(x),$$

so that z is a nearest point to x. ∎

We say that C is *boundedly weakly compact* provided that $C \cap B[0, r]$ is weakly compact for every $r \geqq 0$.

PROPOSITION 2.2. *If C is non-empty and boundedly weakly compact then C is proximinal.*

Proof. Suppose that $x \in E \backslash C$ and let $\{z_n\}$ be a minimizing sequence in C for x. Then $\{z_n\}$ lies $C \cap B[0, r]$ for some positive r, and so has a weak cluster point z belonging to C. By Lemma 2.1 z is a nearest point to x. ■

As a consequence we have the following.

PROPOSITION 2.3 *Closed non-empty convex subsets of relexive Banach spaces are proximal.*

Proof. $B[0, r]$ is weakly compact and closed convex sets are weakly closed. ■

3. Special classes of sets: "Swiss cheese" in reflexive spaces. In this section we show that the complements of open convex sets in reflexive Banach spaces are not badly behaved, despite being far from weakly closed. The first lemma should be known but we include a proof.

LEMMA 3.1. *If C is a closed non-empty subset of a Banach space E such that $E \backslash C$ is convex then d_C is concave on $E \backslash C$.*

Compare to Klee Caverns

Proof. Let x and y belong to $E \backslash C$ and take $0 < t < 1$. If $x_t := tx + (1-t)y$ and v lies in the open unit ball $B(0,1)$ then $a := x + d_C(x)v$ and $b := y + d_C(y)v$ lie in $E \backslash C$. By convexity $ta + (1-t)b \in E \backslash C$. That is,

$$x_t + [td_C(x) + (1-t)d_C(y)]v \in E \backslash C.$$

Since v is arbitrary in $B(0,1)$,

$$d_C(x_t) \geqq td_C(x) + (1-t)d_C(y),$$

as required. ■

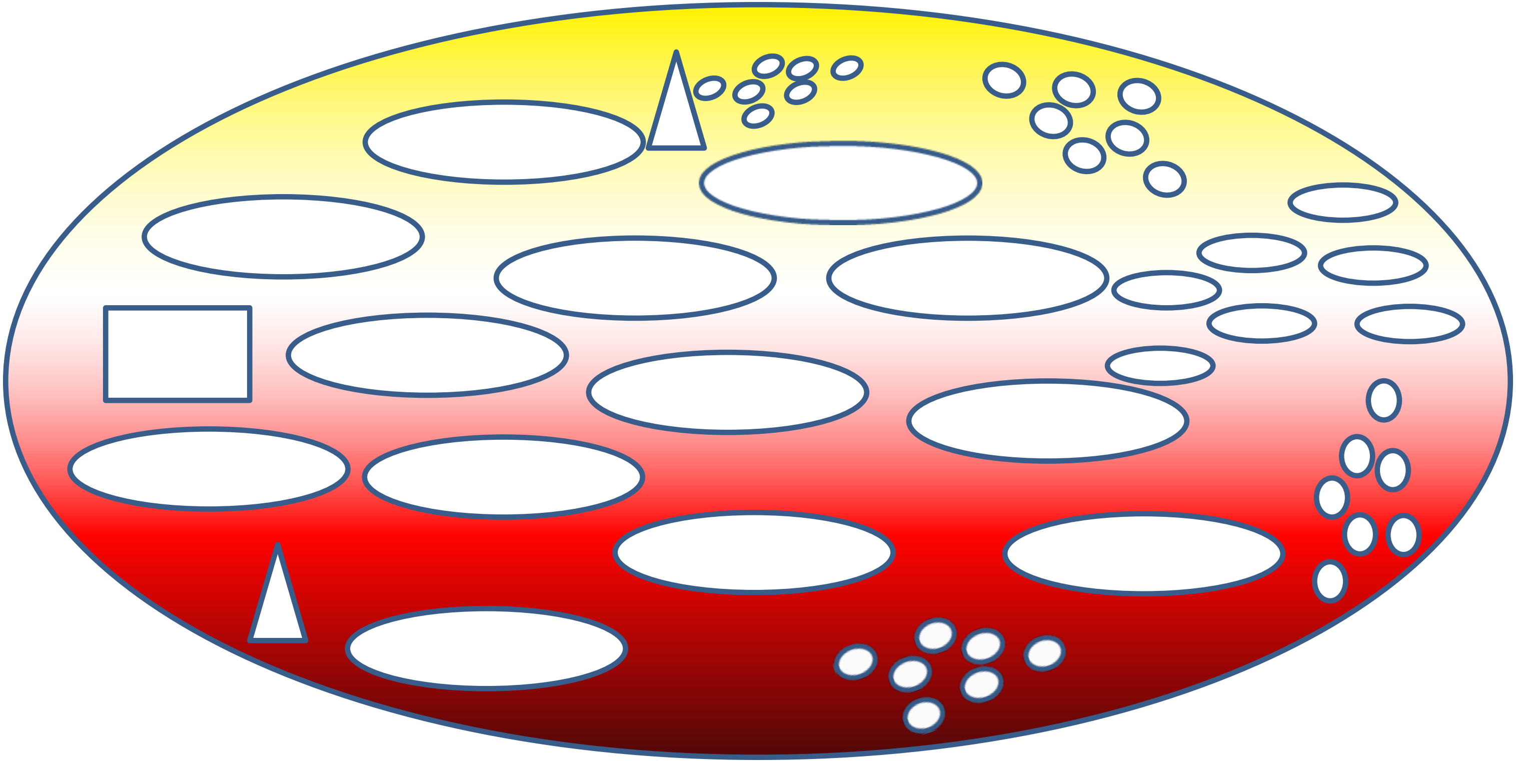
THEOREM 3.2. *If C is a closed non-empty subset of a reflexive Banach space E such that $E\backslash C$ is convex then C is almost proximinal.*

Proof. The lemma shows d_C is concave on $E\backslash C$. Since E is an Asplund space [**1**, **6**] the continuous convex function $-d_C$ is Fréchet differentiable on a dense G_δ subset G of $E\backslash C$. We show that each $x \in G$ has a nearest point in C. Let x^* be the Fréchet (sub-)derivative of d_C at $x \in G$ and let $\{z_n\}$ be any minimizing sequence in C for x. By reflexivity, we may take a weakly convergent subsequence with limit z. If z is in C then z is a nearest point to x by Lemma 2.1. Otherwise, by concavity of d_C on $E\backslash C$

$$d_C(z) - d_C(x) \leqq \langle x^*, z - x\rangle \leqq \limsup \langle x^*, z_n - x\rangle = -d_C(x)$$

where the last equality follows from Proposition 1.4. This shows that $d_C(z) \leqq 0$ and that z is in C after all. ∎

Swiss Cheese



COROLLARY (SWISS CHEESE LEMMA) 3.3. *Let* $\{U_\alpha : \alpha \in A\}$ *be a collection of mutually disjoint open convex subsets of a reflexive Banach space. Then*

$$C := E \backslash \cup \{U_\alpha : \alpha \in A\} \text{ is almost proximinal if it is non-empty.}$$

Proof. Using Theorem 3.2 it suffices to show that if $x \in U_\beta$ has a nearest point y in the closed set $e \backslash U_\beta$ (which contains C) then $y \in C$.

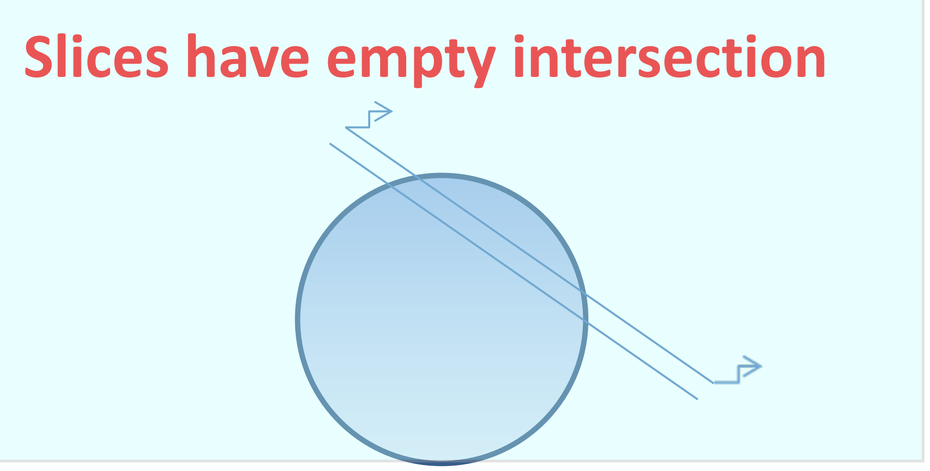
Failing that, $y \in U_\alpha$ with $\alpha \neq \beta$. Since U_α and U_β are disjoint and U_α is open, for small positive t the point $z := tx + (1-t)y$ lies in $U_\alpha \backslash U_\beta$ and so in $E \backslash U_\beta$. But $\|x - z\| < \|x - y\|$, so y was not a nearest point to x in $E \backslash U_\beta$. ∎

James Theorem

James Theorem (1957–1972)
A Banach space X is reflexive iff every $x^* \in S(X^*)$ achieves its norm. That is
$$\|x^*\| = x^*(x) = 1 \text{ for some } x \in B_X.$$

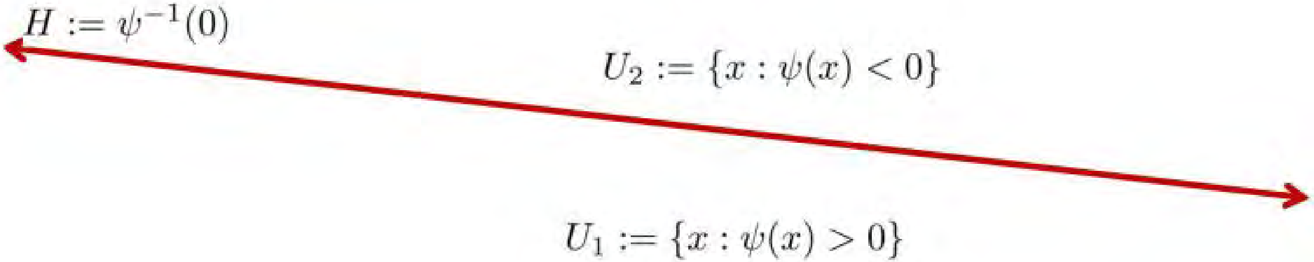
(There is a similar characterization of weak compactness of closed bounded convex sets.)

Example.
If $x^* := (1/2^n)_{n \in N}$, $\|x^*\|_1 = 1$ but $|x^*(x)| < 1$ if $\|x\|_\infty \leq 1$ for $x \in c_0$.



REMARKS 3.4. (i) A closed set is convex if and only if d_C is convex, while an open set C is convex if and only if $d_{X \setminus C}$ is concave on C.

(i) By James' theorem [**6**, p. 63], in any non-reflexive space there are closed hyperplanes H so that no point of $E \setminus H$ has a nearest point in H. (See Theorem 5.10.) This shows that Proposition 2.3 characterizes reflexive spaces. Also the Swiss cheese lemma characterizes reflexive spaces, letting U_1 and U_2 be the open half spaces determined by H.



4. Special classes of Banach spaces: finite dimensional spaces. For any closed non-empty subset C of a finite dimensional Banach space E and any point $x \in E\backslash C$ there is a nearest point in C to x (by Proposition 2.2). Furthermore this characterizes finite dimensional Banach spaces.

THEOREM 4.1. (a) *In any infinite dimensional Banach space there is a closed non-empty set C and a point $x \in E\backslash C$ so that x has no nearest point in C.* (b) *Consequently, a Banach space is finite dimensional if and only if every non-empty closed subset is proximinal.*

Proof. (a) Since the space is infinite dimensional we can find a sequence $\{x_n\}$ of norm one elements with $\|x_n - x_m\| > 1/2$ for $n \neq m$ [**12**]. Let

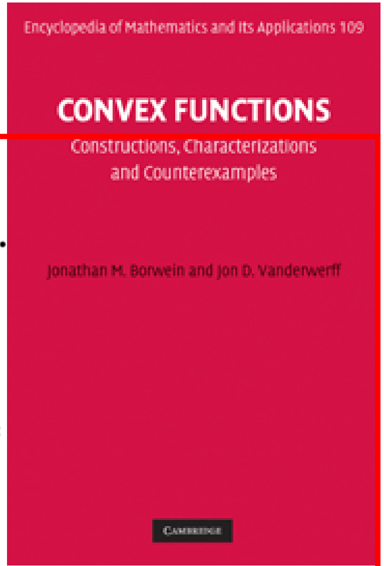
$$C := \{(1+2^{-n})x_n : n \in \mathbf{Z}^+\}.$$

Then C is closed and

$$d_C(0) = 1 < \|0 - (1 + 2^{-n})x_n\| \quad \text{for each } n \in \mathbf{Z}^+.$$

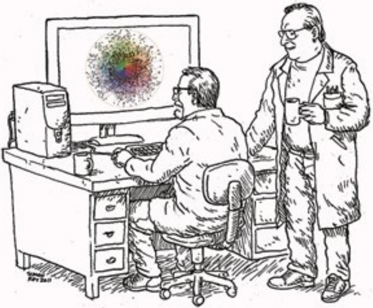
Part (b) now follows. ■

In a similar fashion:



Theorem 8.2.2 [BV Convex functions]
For a Banach space X, the following are (easily) equivalent.

1. X is finite-dimensional.
2. Weak* and norm convergence coincide sequentially in X^* (2 iff 1: Josefson-Neisenzweig).
3. Every continuous convex function on X is bounded on bounded subsets of X.
4. Gâteaux and Fréchet differentiability coincide for continuous convex functions on X.



"Sometimes it is easier to see than to say."

PART II

"Best Approximation Problems in (Reflexive) Banach Space"

What can we say in terms of x about

$$\operatorname{argmin}_{a \in A} \|x - a\|$$

when A is a non-trivial norm-closed set in a Banach space $(X, \|\cdot\|)$?

In Part II, we shall

- characterize norms in which the problem is generically (or densely) solvable
  - The Lau-Konjagin theorem
- discuss norms in which nearest points exist densely
- consider non-reflexive extensions
  - to boundedly weakly locally compact sets
  - including the *Stechkin conjecture*
- consider the case of the *Radon-Nikodym* property

Mikhail Kadets (1923-2011)

5. Reflexive Kadec spaces. We say that a Banach space E is *(sequentially) Kadec* provided that for each sequence $\{x_n\}$ in E which converges weakly to x with $\lim_{n\to\infty} \|x_n\| = \|x\|$ we have

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Warning, *sequential* is often crucial

[Each L_p space $(1 < p < \infty)$ has this property, as does any $l_1(S)$ and any locally uniformly convex Banach space.]

Lau [13] showed that nonempty closed subsets in reflexive Kadec spaces are almost proximinal. Konjagin [14] showed that in any non Kadec space there is a non-empty bounded closed set C such that points in $E\backslash C$ with nearest points in C are not dense in $E\backslash C$. We will develop both of these results in detail.

Definition 5.1. We modify the sets used by Lau so that it is easier to see they are open. This is helpful since we have access to Theorem 1.3. If C is a closed non-empty subset of a Banach space E and $n \in \mathbf{Z}^+$ we define

$$L_n(C) := \{x \in E \backslash C : \text{ for some } \delta > 0 \text{ and some } x^* \in E^* \text{ with } \|x^*\| = 1, \\ \inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta)\} > (1 - 2^{-n}) d_C(x)\}.$$

Also let

$$L(C) := \cap_n L_n(C)$$

and let

Change of quantification

$$\Omega(C) := \{x \in E \backslash C : \text{there exists } x^* \in E^* \text{ with } \|x^*\| = 1, \\ \text{such that for each } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that} \\ \inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta)\} > (1 - \epsilon) d_C(x)\}.$$

LEMMA 5.2. *Each $L_n(C)$ is open in E.*

Proof. Let $x \in L_n(C)$. Then there are $x^* \in E^*$ with $\|x^*\| = 1$ and $\delta > 0$ so that

$$0 < \tau := \inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta)\} - (1 - 2^{-n})d_C(x).$$

Let $\lambda > 0$ be such that $\lambda < \delta/2$ and $\lambda < \tau/2$ and fix y with $\|y - x\| < \lambda$. For $\delta^* := \delta - 2\lambda$ we have

$$C \cap B(x, d_C(x) + \delta) \supseteq A := C \cap B(y, d_C(y) + \delta^*)$$

since d_C is non-expansive. Hence if $z \in A$ then

$$\langle x^*, x - z\rangle \geqq \tau + (1 - 2^{-n}) d_C(x),$$

and

$$\begin{aligned}\langle x^*, y - z\rangle &\geqq \tau + (1 - 2^{-n}) d_C(y) \\ &\quad + \langle x^*, y - x\rangle + (1 - 2^{-n})[d_C(x) - d_C(y)] \\ &\geqq (1 - 2^{-n}) d_C(y) + \tau - 2\|x - y\| \\ &\geqq (1 - 2^{-n}) d_C(y) + \tau - 2\lambda.\end{aligned}$$

Thus

$$\inf\{\langle x^*, y - z\rangle : z \in A\} > (1 - 2^{-n}) d_C(y)$$

and $B(x, \lambda)\backslash C$ lies in $L_n(C)$, which shows $L_n(C)$ is open. ■

LEMMA 5.3. *If $x \in E \backslash C$ and $\partial^F d_C(x) \neq \emptyset$ then $x \in \Omega(C)$.*

Proof. Let $x^* \in \partial^F d_C(x)$. By Proposition 1.4, $\|x^*\| = 1$ and for each minimizing sequence $\{z_n\}$ for x we have $\langle x^*, x - z_n \rangle \longrightarrow d_C(x)$. Thus for each $\epsilon > 0$ there is $\delta > 0$ so that whenever

$$z \in C \cap B(x, d_C(x) + \delta)$$

It follows that

$$\langle x^*, x - z \rangle > (1 - \epsilon/2) d_C(x).$$

it follows that

$$\inf\{\langle x^*, x - z \rangle : z \in C \cap B(x, d_C(x) + \delta)\} > (1 - \epsilon) d_C(x)$$

as required. ■

Next we have:

LEMMA 5.4. *In any Banach space E the set $\Omega(C)$ always lies in $L(C)$.*

Proof. This follows directly from the definitions of the two sets. ■

LEMMA 5.5. *If E has an equivalent Fréchet differentiable renorm then $\Omega(C)$ is dense in $E \backslash C$.*

Proof. By Theorem 1.3 the Lipschitz function $d_C(x)$ is Fréchet subdifferentiable on a dense subset of $E \backslash C$. Now Lemma 5.3 completes the proof. ■

LEMMA 5.6. *When E is reflexive $\Omega(C) = L(C)$.*

Proof. By Lemma 5.4 we need to show that $L(C)$ is contained in $\Omega(C)$. Let $x \in L(C) = \cap_n L_n(C)$. Select x_n^* with $\|x_n^*\| = 1$ and $\delta_n > 0$ so that

$$\inf\{\langle x_n^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta_n)\} > (1 - 2^{-n})d_C(x)$$

and let x^* be any weak* cluster point of $\{x_n^*\}$. Let

$$K_n := \text{weak-cl}[C \cap B(x, d_C(x) + \delta_n)]$$

and observe that each K_n is weakly compact. Thus $\boxed{K := \cap_n K_n}$ is non-empty. For each z in K we have

$$\langle x_n^*, x - z\rangle \geqq (1 - 2^{-n})d_C(x)$$

so that $\langle x^*, x - z\rangle \geqq d_C(x)$. Since $\|x^*\| \leqq 1$ and $\|x - z\| \leqq d_C(x)$ we see that $\|x^*\| = 1$ and

$$\langle x^*, x - z\rangle = d_C(x) = \|x - z\|.$$

Now if $\epsilon > 0$ then K is contained in the weakly open set

$$U(\epsilon) := \{z : \langle x^*, x - z\rangle > (1 - \epsilon/2)d_C(x)\}$$

and as the K_n are nested and weakly compact some K_n lies in $U(\epsilon)$. This implies that

$$\inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta_n)\} > (1 - \epsilon)d_C(x)$$

and x^* is as required. ■

We have now completed the proof of the following result.

THEOREM 5.7. *If C is a closed non-empty subset of a reflexive Banach space E then $\Omega(C) = L(C)$ is a dense G_δ subset of $E \backslash C$.*

COROLLARY 5.8. (Lau) *If E is a reflexive Kadec space then for each closed non-empty set C in E the set of points of $E \backslash C$ with nearest points in C contains the dense G_δ subset $\Omega(C)$ of $E \backslash C$.*

Proof. If $x \in \Omega(C)$ and $\{z_n\}$ is a minimizing sequence in C for x then (by extracting a subsequence if necessary) we may assume that weak-$\lim_{n\to\infty} z_n = z$ exists. If x^* is the norm-1 functional guaranteed by the the definition of $\Omega(C)$ then

$$\|x - z\| \geqq \langle x^*, x - z\rangle = \lim \langle x^*, x - z_n\rangle \geqq d_C(x) = \lim \|x - z_n\|.$$

By weak lower semicontinuity of the norm,

$$\lim \|x - z_n\| \geqq \|x - z\|.$$

It follows that $\|x - z\| = \lim \|x - z_n\|$. Since $x - z_n$ converges weakly to $x - z$ we may deduce from the Kadec property that z_n converges in norm to z; which must then lie in C. Thus z is a nearest point in C for x (by Lemma 2.1). ∎

We turn next to describe Konjagin's construction.

LEMMA 5.9. (i) *If E is not a Kadec space one can find $x_n \in E$ and $x^* \in E^*$ such that*

(a) $x^*(x_n) = \|x^*\| = 1 = \lim_{n\to\infty} \|x_n\|$, *and*

(b) $\inf_{n \neq m} \|x_n - x_m\| > 0$.

(ii) *If* (a) *and* (b) *hold and E is reflexive then E is not Kadec.*

Proof. Suppose E is not Kadec. Select y_n converging weakly to y in E with $\|y_n\| = \|y\| = 1$, but with $y_n - y$ not norm convergent to zero. Relabling if needed we may take $\|y_n - y\| > \epsilon$ for all n. Let x^* be a (norm-1) support functional for the unit ball at y and let

$$z_n := y_n / \langle x^*, y_n \rangle$$

(which may be assumed finite). Then z_n tends weakly to y and

$$x^*(z_n) = 1 = \|x^*\| = \lim_{n \to \infty} \|z_n\|,$$

while

$$\liminf_{m \to \infty} \|z_n - y\| > \epsilon \quad \text{for some } \epsilon > 0.$$

Relabling again if needed we may assume $\|z_n - y\| > \epsilon$ for all n. Now for each n, we have

$$\liminf_{n\to\infty} \|z_n - z_m\| \geqq \|z_n - y\| > \epsilon$$

by weak lower semicontinuity of the norm. Thus for each n there is an integer $m(n) > n$ such that

$$\|z_n - z_m\| > \epsilon \quad \text{for } m \geqq m(n).$$

Set $n(1) := 1$ and $n(k+1) := m(n(k))$ for each k. Then

$$\|z_{n(k)} - z_{n(j)}\| > \epsilon \quad \text{if } j > k.$$

Then x^* and $x_k := z_{n(k)}$ satisfy (a) and (b).

Conversely if E is reflexive and (a) and (b) hold then there is a weakly convergent subsequence of $\{x_n\}$ with limit x. Now (a) shows that we have

$$\|x\| \geqq x^*(x) = 1 = \lim_{n\to\infty} \|x_n\|,$$

while (b) now contradicts the Kadec property. ∎

THEOREM 5.10. (Konjagin) *Suppose that E is a Banach space which is not both reflexive and Kadec. Then there is a closed bounded non-empty set C in E and an open non-empty subset U of $E\backslash C$ such that*

(i) *for each $x \in U$ there is no nearest point in C,*

(ii) *d_C is affine on U;*

in particular

(iii) *d_c is Fréchet differentiable on U.*

Proof. Case 1. E is not reflexive. By James' theorem [11] there is x^* in E^* with $1 = \|x^*\| > \langle x^*, y\rangle$ for each y in the closed unit ball. Let

$$C := B[0,1] \cap \{x \in E \colon \langle x^*, x\rangle = 0\}$$

and

$$U := B[0,1/3] \cap \{x \in E \colon \langle x^*, x\rangle > 0\}.$$

Then $d_C(x) = \langle x^*, x\rangle$ for each $x \in U$. Suppose a point $x \in U$ had a nearest point $z \in C$. Then, since $0 \in C$,

$$d_C(x) = \|x - z\| \leqq \|x - 0\| \leqq 1/3 \quad \text{and} \quad \|z\| \leqq 2/3.$$

In particular z would be a nearest point to x in $\ker x^*$, contradicting the fact that x^* does not attain its norm.

Case 2. E is not Kadec. By (i) of the last lemma we can select $x^* \in E^*$ and $y_n \in E$ so that $\|y_n\| \leqq 2$ and for some $0 < \delta < 1$

$$x^*(y_n) = 1 = \|x^*\| = \lim_{n \to \infty} \|y_n\|, \quad \text{and} \quad \inf_{n \neq m} \|y_n - y_m\| \geqq \delta.$$

Set $z_n := (1 + 2^{-n})y_n$ and define

$$\boxed{C := \cup_n M_n \quad \text{where} \quad M_n := z_n + (B[0, \delta/3] \cap \{x \in E : \langle x^*, x \rangle = 0\}).}$$

Then C is our desired set. First, C is norm closed: if $n \neq m$ and $z \in M_n$, $w \in M_m$ we have

$$\begin{aligned} \|z - w\| &\geqq \|y_n - y_m\| - \|y_m - z_m\| \\ &\quad - \|y_n - z_n\| - \|z_m - z\| - \|z_n - w\| \\ &\geqq \delta - 2^{1-n} - 2^{1-m} - \delta/3 - \delta/3 > \delta/9 \end{aligned}$$

for $m \geqq p$ and $n \geqq p$, p sufficiently large. Since each M_n is closed and since

$$\operatorname{cl}\left(\bigcup_{n \geqq p} M_n\right) = \bigcup_{n \geqq p} M_n,$$

C is norm closed as the finite union of closed sets. Next let $U := B(0, \delta/9)$. For x in U, we will show that $d_C(x) = 1 - \langle x^*, x \rangle$ but x has no nearest point in C. This will conclude the proof. If $x \in U$ set

$$w_n := x + z_n - \langle x^*, x \rangle y_n.$$

Then

$$\|w_n - z_n\| \leqq \|x\| + 2\|x\| < \delta/3$$

while

$$\langle x^*, w_n - z_n \rangle = 0.$$

Thus $w_n \in M_n$ and

$$\begin{aligned} d_C(x) &\leqq \liminf_{n\to\infty} \|w_n - x\| \\ &= \liminf_{n\to\infty} \|z_n - \langle x^*, x\rangle y_n\| \\ &= \liminf_{n\to\infty} [(1+2^{-n}) - \langle x^*, x\rangle]\|y_n\| \\ &= 1 - \langle x^*, x\rangle \end{aligned}$$

since $\langle x^*, x\rangle < 1$. If, however, $z \in C$ then $z \in M_n$ for some n and

$$\langle x^*, z\rangle = \langle x^*, z_n\rangle = (1+2^{-n}) > 1.$$

Thence

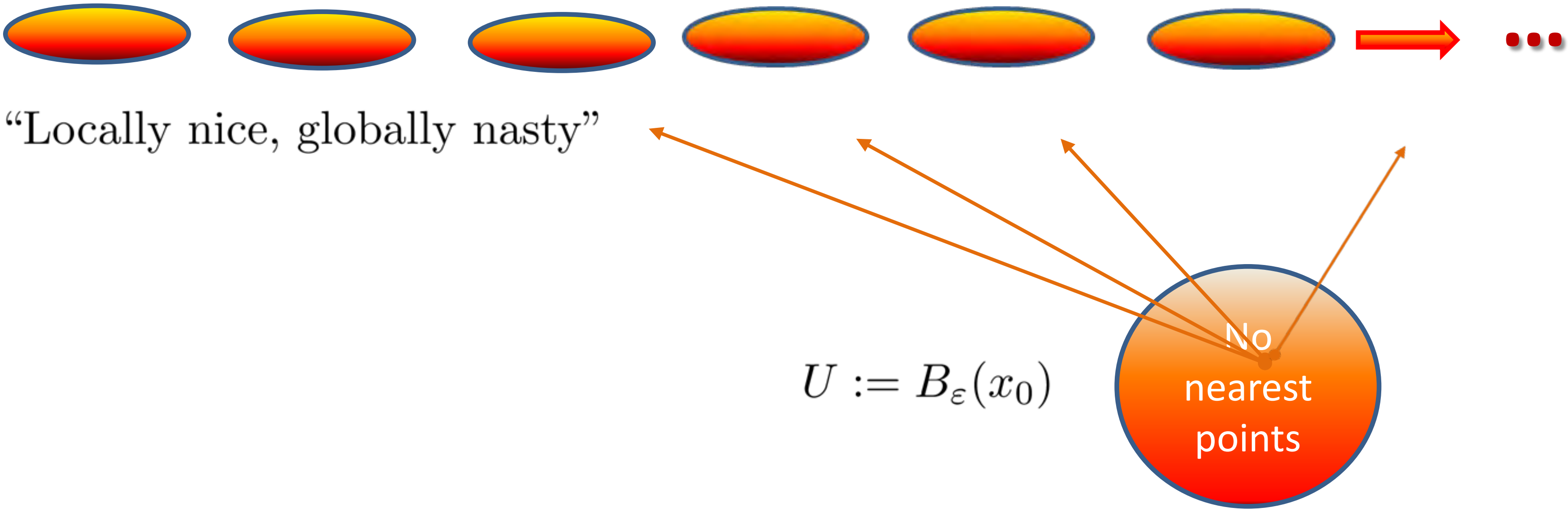
$$\|z - x\| = \|x^*\|\,\|z - x\| \geqq \langle x^*, z\rangle - \langle x^*, x\rangle > 1 - \langle x^*, x\rangle$$

and $d_C(x) = 1 - \langle x^*, x\rangle$ but no nearest point exists in C for U. ■

Konjagin's Set

Let us observe that, in the non-Kadec case, by translation we can arrange for d_C to be linear on U. Also, observe that by taking only the tail of C, C may be supposed locally convex being made up of discrete translates of a fixed convex set. We gather up results as follows.

$$C := \cup_n (z_n + A)$$



"Locally nice, globally nasty"

$$U := B_\varepsilon(x_0)$$

THEOREM 5.11. (Lau-Konjagin) *In any Banach space E the following conditions are equivalent.*

(A) *E is reflexive and Kadec.*

(B) *For each closed non-empty subset C of E, the set of points in $E \backslash C$ with nearest points in C is dense in $E \backslash C$.*

(C) *For each closed non-empty subset C of E, the set of points in $E \backslash C$ with nearest points in C is generic in $E \backslash C$ (i.e., C is almost proximinal).*

One consequence of Theorem 5.11 is that in any reflexive Kadec space there is a workable "proximal normal formula" [**2**]. It is also possible to generalize Lau's result to some sets in non-reflexive Kadec spaces. (See also [**3**].) Recall that a set C in a Banach space E is *boundedly relatively weakly compact* if $B[0,r]\cap C$ has a weakly compact closure for each positive r. It is equivalent to require that each bounded sequence in C possesses a weakly convergent subsequence with limit in E. (This is not entirely obvious.) Clearly every subset of a reflexive space and every subset of a weakly compact set possess this property. The next result is therefore a complete extension of Theorem 5.7.

THEOREM 5.12. *If C is a closed, boundedly relatively weakly compact, non-empty subset of a Banach space E then $\Omega(C) = L(C)$ is a dense G_δ subset of $E\backslash C$.*

From this exactly as in the proof of Corollary 5.8 we obtain a generalization of Lau's theorem.

Corollary 5.13. *Every closed, boundedly relatively weakly compact, non-empty subset of a Kadec Banach space E is almost proximinal. Indeed $\Omega(C)$ is a dense G_δ subset of $E \backslash C$ with nearest points in C.*

To prove Theorem 5.12 we need a replacement for Lemma 5.5 (and the results it depended on). The factorization theorem of Davis, Figiel, Johnson, and Pelczynski provides an avenue. We will use it in the following form.

Theorem 5.14. [7] *Let K be a weakly compact subset of a Banach space Y with $Y =$ closed-span (K). Then there is a reflexive Banach space R and a one to one continuous linear mapping $T: R \longrightarrow Y$ such that $T(B[0,1]) \supseteq K$.*

Now we can show density of $\Omega(C)$.

Lemma 5.15. *If C is a closed, boundedly relatively weakly compact, non-empty subset of a Banach space E then $\Omega(C)$ is dense in $E\backslash C$.*

Proof. Let $x_0 \in E\backslash C$ and suppose $d_C(x_0) > \epsilon > 0$. Fix $N > \|x_0\| + d_C(x_0) + 2\epsilon$ and let

In BTZ [4] a more subtle approach

$$K := \text{weak-cl}[\{(B[0, N] \cap C\} \cup \{x_0\}].$$

Then K is weakly compact and if Y is the closed span of K, we can apply Theorem 5.14 to obtain a reflexive Banach space R and a one to one continuous linear mapping $T: R \longrightarrow Y$ such that $T(B[0, 1]) \supseteq K$. Define $f_C : R \longrightarrow [0, \infty)$ by $\boxed{f_C(u) := d_C(Tu)}$ for each u in R.

By Theorem 1.3 the Lipschitz function f_C is Fréchet subdifferentiable on a dense subset on R. Thus there is a point of subdifferentiability $v \in R$ with $y := Tv \in B(x_0, \epsilon)$. Note that y is in $E \backslash C$. Let $v^* \in \partial^F f_C(v)$ so that

$$\liminf_{h \to 0} \frac{d_C(T(v+h)) - d_C(Tv) - \langle v^*, h \rangle}{\|h\|} \geqq 0$$

and hence

$$\boxed{\liminf_{h \to 0} \frac{d_C(y+Th) - d_C(y) - \langle v^*, h \rangle}{\|h\|} \geqq 0.}$$

Next for $u \in R$, we have

$$\boxed{\langle v^*, u \rangle \leqq \|Tu\|}$$

on substituting tu for h in the previous expression and using the non-expansiveness of d_C. This shows $v^* = T^* y^*$ for some $y^* \in Y^*$ (by the Hahn-Banach theorem). In particular $\boxed{\langle y^*, Tu \rangle \leqq \|Tu\|}$ for each $u \in R$. Since T has dense range this shows that $\|y^*\| \leqq 1$. We extend y^* to $x^* \in E^*$ with $\|x^*\| \leqq 1$ and observe that

$$\liminf_{t\to 0_+, k\in K} \frac{d_C(y+t(k-y)) - d_C(y) - t\langle x^*, k-y\rangle}{t} \geqq 0.$$

(Since $T(B[0,1]-v) \supseteq K-y$.) Suppose now that $\{z_n\}$ is a minimizing sequence in C for y. By the construction of N, $z_n \in K$ for large n. Also we may suppose that

$$\|y - z_n\| < d_C(y) + 4^{-n}.$$

Then

$$\begin{aligned}
0 &\leqq \liminf_{n\to\infty} [d_C(y + 2^{-n}(z_n - y)) - d_C(y)]2^n - \langle x^*, z_n - y\rangle \\
&\leqq \liminf_{n\to\infty} [\|y + 2^{-n}(z_n - y) - z_n\| - \|y - z_n\| - 4^{-n}]2^n - \langle x^*, z_n - y\rangle \\
&= \liminf_{n\to\infty} [-\|z_n - y\| - \langle x^*, z_n - y\rangle].
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} \langle x^*, y - z_n\rangle \geqq \lim_{n\to\infty} \|z_n - y\| = d_C(y),$$

which again shows $\|x^*\| \geqq 1$. Thus

$$\|x^*\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \langle x^*, y - z_n \rangle = d_C(y).$$

As in Lemma 5.3, $y \in \Omega(C)$. Since $\|y - x_0\| < \epsilon$ this establishes our density assertion. ∎

Proof. (of Theorem 5.12) By Lemmas 5.2 and 5.4, $\Omega(C)$ is always contained in the G_δ set $L(C)$. We note that the proof of Lemma 5.6 holds unchanged for C boundedly relatively weakly compact. Thus $\Omega(C) = L(C)$ is a G_δ set in $E \backslash C$. Finally $\Omega(C)$ is dense in $E \backslash C$ by the last lemma. ∎

6. Uniqueness of nearest points. Having constructed the set $\Omega(C)$ we can also use it to prove uniqueness results. The first is a reasonable new partial answer to Stechkin's question whether in every strictly convex Banach space the nearest points to a closed set are generically not multiple. (See also [**3**] and [**10**].)

Revalski-Zhivkov give current status

THEOREM 6.1. *Let E be a strictly convex Banach space and let C be a non-empty, boundedly relatively weakly compact, closed subset of E. Then each point of the dense G_δ subset $\Omega(C)$ of $E\backslash C$ has at most one nearest point.*

Proof. If $x \in \Omega(C)$ and $y, z \in C$ with $\|x - y\| = \|x - z\| = d_C(x) > 0$ then the functional x^* guaranteed by the definition of $\Omega(C)$ has $\|x^*\| = 1$ and

$$x^*(x - y) = x^*(x - z) = d_C(x)$$

and

$$\begin{aligned} \|(x - y) + (x - z)\| &\geqq x^*(x - y) + x^*(x - z) \\ &= 2d_C(x) = \|x - y\| + \|x - z\|. \end{aligned}$$

By strict convexity $y = z$ as required. By Theorem 5.12, $\Omega(C)$ is a dense G_δ subset. ∎

Definition 6.2. A subset C of a Banach space E is *almost Chebyshev* provided there is a generic subset of $E \backslash C$ with unique nearest points in C.

COROLLARY 6.3. *Let E be a Kadec strictly convex Banach space and let C be a non-empty, boundedly relatively weakly compact, closed subset of E. Then each point of the dense G_δ subset $\Omega(C)$ of $E \backslash C$ has exactly one nearest point, and C is almost Chebyshev.*

Proof. Combine Corollary 5.13 and Theorem 6.1. ■

Definition 6.4. A Banach space E is *strongly convex* provided it is reflexive, Kadec, and strictly convex.

COROLLARY 6.5. *Every closed subset of a strongly convex Banach space is almost Chebyshev.*

It is of interest to note that Corollary 6.5 can be turned into various characterizations of strongly convex spaces; many due to Konjagin.

THEOREM 6.6. *Let E be a Banach space. The following statements are equivalent.*

(1) *E is strongly convex.*

(2) *The norm on E^* is Fréchet differentiable.*

(3) *Every closed non-empty subset of E is almost Chebyshev.*

(4) *For every closed non-empty subset C of E there is a dense set of points in $E \backslash C$ possessing unique nearest points.*

Proof. (1) $\Rightarrow$ (3) by Corollary 6.5, while (3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). If E is not strongly convex then either E is not both reflexive and Kadec, or E is not strictly convex. In the first case Theorem 5.11 applies. In the second case, let $[a,b]$ be a closed non-trivial interval in the unit sphere of E. Take $x^* \in E^*$ with $\|x^*\| = 1$ and $\langle x^*, (a+b)\rangle = 2$, so that $\langle x^*, a\rangle = \langle x^*, b\rangle = 1$. Then for $C := \ker x^*$ and $x \in E\backslash C$ there are always multiple nearest points. [Indeed y is a nearest point to x if and only if $\langle x^*, y\rangle = 0$ and $\|x - y\| = |\langle x^*, x\rangle| = d_C(x)$, which holds for $x - \langle x^*, x\rangle c$ whenever $c \in [a,b]$.]

(1) $\Rightarrow$ (2). Since E is reflexive and strictly convex, E^* is smooth. Let x_n^* and $x^* \in E^*\backslash\{0\}$ with $x_n^* \longrightarrow x^*$. Then the corresponding Gateaux derivatives x_n and $x \in E$ of the norm on E^* satisfy $x_n \longrightarrow x$ weakly. $\|x_n\| = \|x\|$ and E is Kadec $x_n \longrightarrow x$ in norm. Thus the norm on E^* is Fréchet differentiable at x^*.

(2) $\Rightarrow$ (1). Here we use the fact that the norm on a Banach space X is Fréchet differentiable at $x \in X$ with derivative x^* if and only if x strongly exposes the unit ball of X^* at x^* [**6**]. (See Definition 8.1.) Now suppose the norm on E^* is Fréchet differentiable. Let F be a norm one support functional so $\langle F, x^* \rangle = \|x^*\| = 1$ for some $x^* \in X^*$. By smoothness F is the Fréchet derivative of the norm at x^*. But then x^* strongly exposes the unit ball of E^{**} at F. Let $\{x_\alpha\}$ be a net converging weak* to F with $x_\alpha \in E$, $\|x_\alpha\| = 1$. Thus

$$\langle x^*, x_\alpha \rangle \longrightarrow \langle F, x^* \rangle = 1 = \|x^*\|,$$

and in consequence x_α converges to F in norm. Thus F lies in E. The Bishop-Phelps theorem shows that the norm one support functionals are dense in the unit sphere. Hence E is reflexive.

Next the smoothness of E^* implies that E is strictly convex. Finally, to settle the Kadec property, let x_n and $x \in E$ satisfy $\|x_n\| = \|x\| = 1$ while $n_n \longrightarrow x$ weakly. There is $x^* \in E^*$, $\|x^*\| = \|x\| = 1 = \langle x^*, x \rangle$. Again x must be the Fréchet derivative of the norm at x^*. But then x^* strongly exposes the unit ball of E at x. Since $\langle x^*, x_n \rangle \longrightarrow \langle x^*, x \rangle = 1$, this forces $x_n \longrightarrow x$ in norm as required.

This completes the proof that (2) implies (1) and so the theorem. ∎

The proof shows that in a strongly convex space the problems are generically *well posed*

Remark 6.7. It is clear that every reflexive locally uniformly convex space is strongly convex. The converse fails since the following renorm $l_2(\mathbf{Z}^+)$ is strongly convex but not locally uniformly convex, as observed by Mark Smith [**16**]. Let $\|\cdot\|$ be the original norm on l_2. Define $|||\cdot|||$ by

Reflexive (WCG) has LUR renorm

$$\boxed{|||x|||^2 := \|Tx\|^2 + (|x_1| + \|Px\|)^2}$$

where

$$\boxed{Tx := (0, x_2/2, x_3/3, \ldots, x_n/n, \ldots) \quad \text{and} \quad Px := (0, x_2, x_3, \ldots, x_n, \ldots).}$$

It is easy to verify that $|||\ \ |||$ is strongly convex. It is not locally uniformly convex since

$$|||e_n||| \longrightarrow |||e_1||| = 1 \quad \text{and} \quad |||e_1 + e_n||| \longrightarrow 2,$$

but $|||e_1 - e_n||| \longrightarrow 2$ not zero.

7. Spaces where nearest points are dense. In this section we show that there are reflexive Banach spaces E which do not have the Kadec property but such that, nevertheless, for each closed non-empty subset C of E the set of nearest points in C to points of $E\backslash C$ is dense in the boundary of C. It is an open question as to whether all reflexive Banach spaces have the latter property.

THEOREM 7.1. *Let X be a reflexive Kadec space, Y a finite dimensional normed space and $|||\cdot|||$ a Riesz (lattice) norm on R^2. Let $E := X \oplus Y$ in the norm*

$$\|(x, y)\| := |||(\|x\|, \|y\|)|||.$$

For each closed non-empty subset C of E the set of nearest points in C to points not in C is dense in the boundary of C.

We will need the following lemma.

LEMMA 7.2. *Suppose E, X, Y, and C are as above. Suppose d_C is Fréchet differentiable at $u \in E \backslash C$ but u has no nearest point in C. Then*

$$\boxed{\{0\} \oplus Y^* \supseteq \partial^F d_C(u).}$$

Proof. Let u be as hypothesised. If $(x^*, y^*) \in \partial^F d_C(u)$ then, by Theorem 1.4, $|||(x^*, y^*)||| = 1$ and for every minimizing sequence $z_n := (x_n, y_n)$ in C for $u = (x, y)$ we have

$$\langle (x^*, y^*), (x_n - x, y_n - y) \rangle \longrightarrow -d_C(u).$$

Thus $|||(\|x^*\|, \|y^*\|)|||^* = 1$ where $|||\cdot|||^*$ is the dual norm on R^2, and

$$\begin{aligned} d_C(u) &= \lim_{n\to\infty} |||(\|x_n - x\|, \|y_n - y\|)||| \\ &= \lim_{n\to\infty} (\langle x^*, x - x_n\rangle + \langle y^*, y - y_n\rangle). \end{aligned}$$

Extracting a subsequence we may and do assume that the sequences $\{\langle x^*, x - x_n\rangle\}$, $\{\|x_n - x\|\}$, and $\{y_n\}$ all converge. Then

$$\begin{aligned} &\lim_{n\to\infty} \langle x^*, x - x_n\rangle + \lim_{n\to\infty} \langle y^*, y - y_n\rangle = d_C(u) \\ &= |||(\lim_{n\to\infty} \|x_n - x\|, \lim_{n\to\infty} \|y_n - y\|)||| \\ &= |||(\|x^*\|, \|y^*\|)|||^* \; ||| \left(\lim_{n\to\infty} \|x_n - x\|, \lim_{n\to\infty} \|y_n - y\|\right) ||| \\ &\geqq \|x^*\| \lim_{n\to\infty} \|x_n - x\| + \|y^*\| \lim_{n\to\infty} \|y_n - y\| \end{aligned}$$

so that

$$\lim_{n\to\infty} \langle x^*, x - x_n \rangle = \|x^*\| \lim_{n\to\infty} \|x_n - x\|$$

approximator looks at C from outside, optimizer looks out from C

and

$$\lim_{n\to\infty} \langle y^*, y - y_n \rangle = \|y^*\| \lim_{n\to\infty} \|y_n - y\|.$$

If $x^* \neq 0$ the Kadec property and reflexivity determine a norm convergent subsequence of $\{x_n\}$ with $\lim x^\#$. Since $\{y_n\}$ converges to some $y^\#$, $(x^\#, y^\#)$ lies in C and is a nearest point to u. This contradiction shows $x^* = 0$ and the conclusion. ■

Proof. (of Theorem 7.1) Suppose $z_0 := (x_0, y_0)$ is in the boundary of C and that $\epsilon > 0$ is such that $U := B(z_0, \epsilon)\backslash C$ contains no points with nearest points in C; this will happen if z_0 is a boundary point not in the closure of nearest points. By Lemma 7.2 we have $\{0\} \oplus Y^* \supseteq \partial^F d_C(u)$ for every u in U (of course $\partial^F d_C(u) = \phi$ is possible). In addition we have by [3], or [15] that

$$\{0\} \oplus Y^* \supseteq \text{weak*cl-conv}\{z^*: z^* \in \partial^F d_C(u), u \in U\} \supseteq \partial d_C(u).$$

Now let (x_2, y) and (x_1, y) lie in $B(z_0, \epsilon)$ with

$$u_t := (tx_1 + (1-t)x_2, y) \in U \quad \text{for all } 0 < t < 1.$$

By Lebourg's Mean-value theorem [5]

$$d_C(x_1, y) - d_C(x_2, y) \in \langle \partial d_C(u_t), (x_1 - x_2, 0) \rangle.$$

But $\partial d_C(u_t)$ annihilates $(x_1 - x_2, 0)$ so that $d_C(x_1, y) = d_C(x_2, y)$.

After some consideration of the case where $(x_1, y) \in C$, it follows that on $B(z_0, \epsilon)$ the distance $d_C(x, y)$ depends only on y. In particular, if $(x, y) \in B(z_0, \epsilon)$ then $d_C(x, y) = d_C(x_0, y)$ where $z_0 = (x_0, y_0)$. Now let $(x, y) \in B(z_0, \epsilon/2)$ have minimizing sequence $\{(x_n, y_n)\}$ from C. Then $z_0 \in C$ so we can assume

$$\|(x_n, y_n) - (x, y)\| \leqq \|z_0 - (x, y)\| < \epsilon/2$$

and $(x_n, y_n) \in B(z_0, \epsilon)$. Thus $0 = d_C(x_n, y_n) = d_C(x_0, y_n)$ so $(x_0, y_n) \in C$ and

$$\begin{aligned} d_C(x_0, y) = d_C(x, y) &= \lim_{n \to \infty} \|(x_n, y_n) - (x, y)\| \\ &\geqq \limsup_{n \to \infty} \|y_n - y\| \geqq \|y^\# - y\| \end{aligned}$$

where $y^\#$ is any cluster point of $\{y_n\}$. Since $(x_0, y^\#) \in C$ we have

$$d_C(x_0, y) \leqq \|(x_0, y^\#) - (x_0, y^\#)\| = \|y^\# - y\| \leqq d_C(x_0, y)$$

and $(x_0, y^\#)$ is a nearest point to (x_0, y), contrary to our assumption. Hence nearest points are dense in the boundary of C. ■

REMARKS 7.3. (a) Choosing

$$|||(s,t)||| := \max\{|s|, |t|\}, Y := R,$$

and any infinite dimensional reflexive Kadec space for X, we obtain a non-Kadec reflexive space E to which Theorem 7.2 applies. If, specifically, $X := l_2(\mathbf{Z}^+)$ it is easy to construct an explicit example of the set promised by the non-Kadec construction of Theorem 5.10.

(b) Choosing $X := l_2(\mathbf{Z}^+)$, $Y := R$ and $|||\cdot|||$ such that the unit ball is

$$B_{|||\cdot|||}[0,1] := \{(s,t) \colon |t| \leqq 1, |s| \leqq 1 + (1-t^2)^{1/2}\}$$

we obtain a uniformly smooth non-Kadec space to which Theorem 7.2 applies.

8. Spaces with the Radon-Nikodym property. We refer the reader to [**4**] for the vast amount known about spaces with the Radon-Nikodym property. All we need here is one definition and one characterization.

Definition 8.1. A functional $x^* \in E^*$ *strongly exposes* a subset C of E at $x \in \operatorname{cl} C$ if $\sup_{z\in C}\langle x^*, z\rangle = \langle x^*, x\rangle$ and

$$\lim_{\alpha\to 0^+} \operatorname{diam}\{y \in C : \langle x^*, y\rangle > \sup_{z\in C}\langle x^*; z\rangle - \alpha\} = 0.$$

A functional $x^* \in E^*$ *strongly exposes* a set C if it strongly exposes some point of the closure of C. This is equivalent to saying that

$$\lim_{\alpha\to 0^+} \operatorname{diam}\{y \in C : \langle x^*, y\rangle > \sup_{z\in C}\langle x^*, z\rangle - \alpha\} = 0.$$

THEOREM 8.2. *A Banach space E has the Radon-Nikodym property (RNP) if and only if for every bounded non-empty subset C of E the set $SE(C)$ of strongly exposing functionals for C is dense in E^*. In particular, reflexive spaces and duals of Asplund spaces have the RNP.*

For unbounded subsets of non-reflexive subspaces there are no general results on nearest points, as shown by the example of the closed hyperplane determined by a non-norm attaining functional (Remark 3.4). For bounded closed sets in spaces with the RNP we have a positive result.

THEOREM 8.3. *Let E be a Banach space with the Radon-Nikodym property and let C be a closed bounded non-empty subset of E. Then C is contained in the closed convex hull of its nearest points to points in $E \backslash C$. In particular C possesses nearest points.*

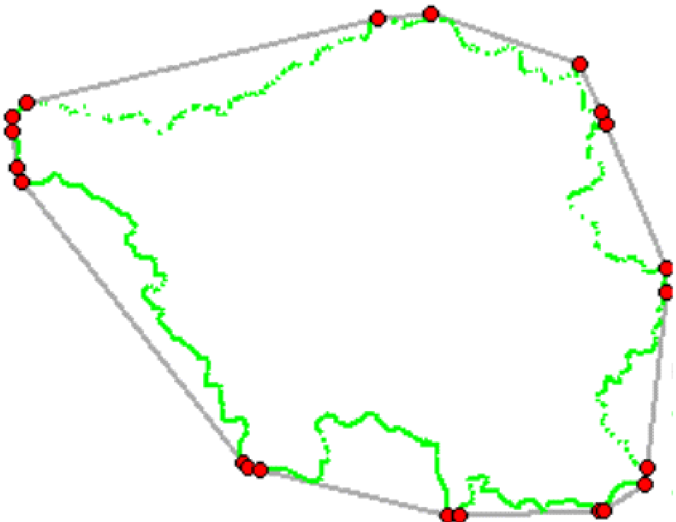
Infinite or C in polytope

Proof. If $x \in C$ does not lie in the convex hull of its nearest points we may separate by $x^* \in E^*$ to obtain

$$\langle x^*, x\rangle > \sup\{\langle x^*, y\rangle : y \text{ is a nearest point in } C\}.$$

Let $\boxed{K := C + B[0, 1]}$ and by Theorem 8.2 find $y^* \in SE(K)$ with $\|y^*\| = 1$ such that

$$\langle y^*, x\rangle > \sup\{\langle y^*, y\rangle : y \text{ is a nearest point in } C\}.$$



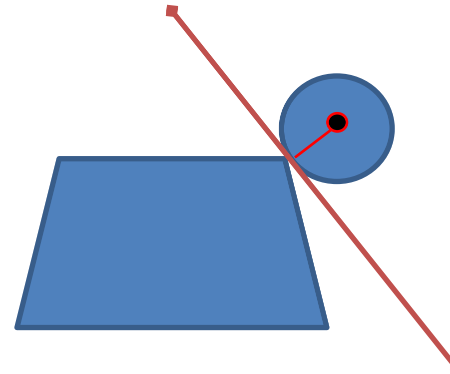
STRONG EXPOSURE

Let A be closed and bounded. Let

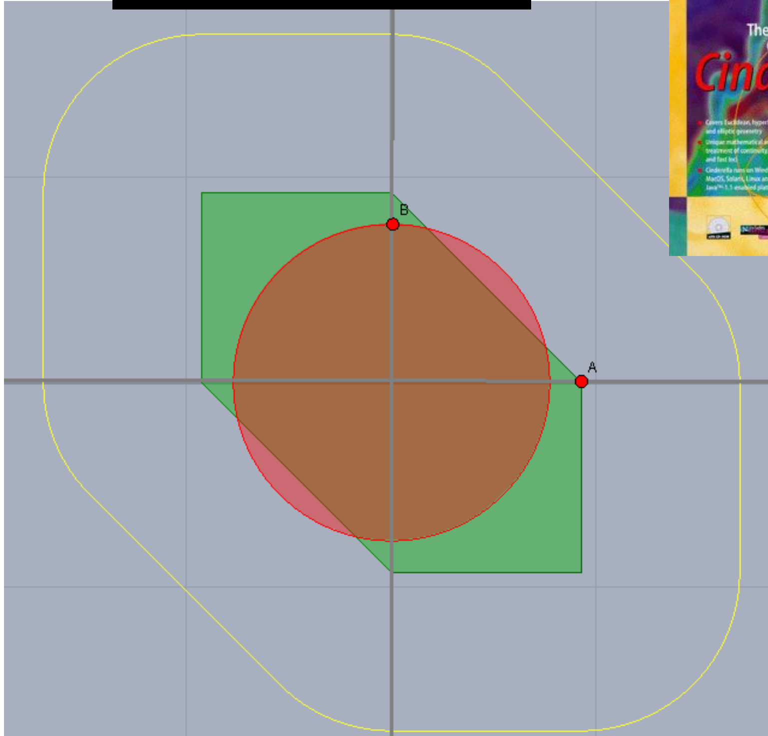
$$C := \overline{\operatorname{conv}} A \text{ and } B := B_X.$$

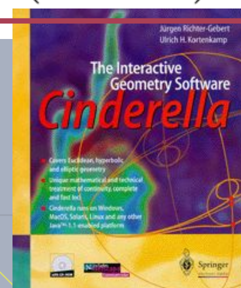
Then

$$SE(\overline{B+C}) \subseteq SE(B) \cap E(C).$$

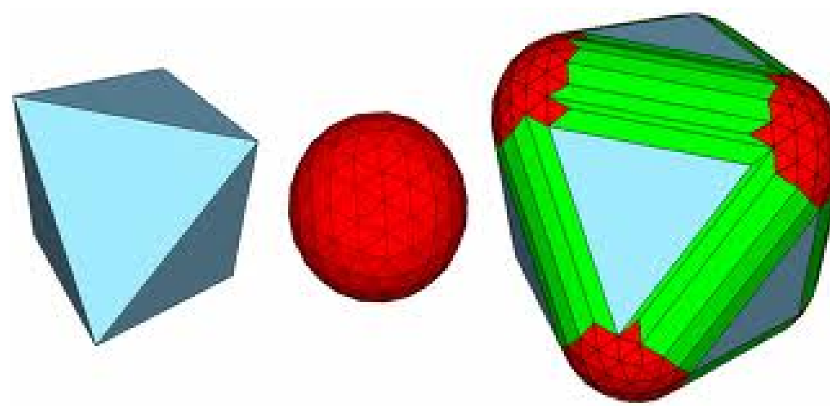


A Cinderella Applet





http://carma.newcastle.edu.au/jon/Minkowski.html



Then, a completeness argument shows that y^* actually both strongly exposes C at $z \in C$ and strongly exposes $B[0,1]$ at u with $\|u\| \leqq 1$. Hence we have

$$\langle y^*, z\rangle = \sup\{\langle y^*, y\rangle : y \in C\} \quad \text{and} \quad \langle y^*, u\rangle = \|y^*\| = 1.$$

Now $z + u$ has a nearest point $z \in C$. Indeed, for $c \in C$

$$\begin{aligned} \|(z+u) - c\| &\geqq \langle y^*, z+u-c\rangle \geqq \langle y^*, u\rangle \\ &= 1 = \|u\| = \|(z+u) - u\|. \end{aligned}$$

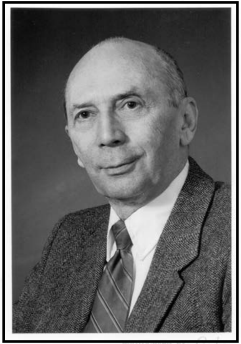
However this contradicts

$$\langle y^*, z\rangle \geqq \langle y^*, x\rangle > \sup\{\langle y^*, y\rangle : y \text{ is a nearest point in } C\}. \qquad \blacksquare$$

For convex sets we state a deeper result of Edelstein [8].

THEOREM 8.4. *Let E be a Banach space with the Radon-Nikodym property and let C be a non-empty closed, convex, bounded subset of E. Then the points in $E \backslash C$ which have nearest points in C are weakly dense in $E \backslash C$.*

Michael Edelstein
(1917-2003)



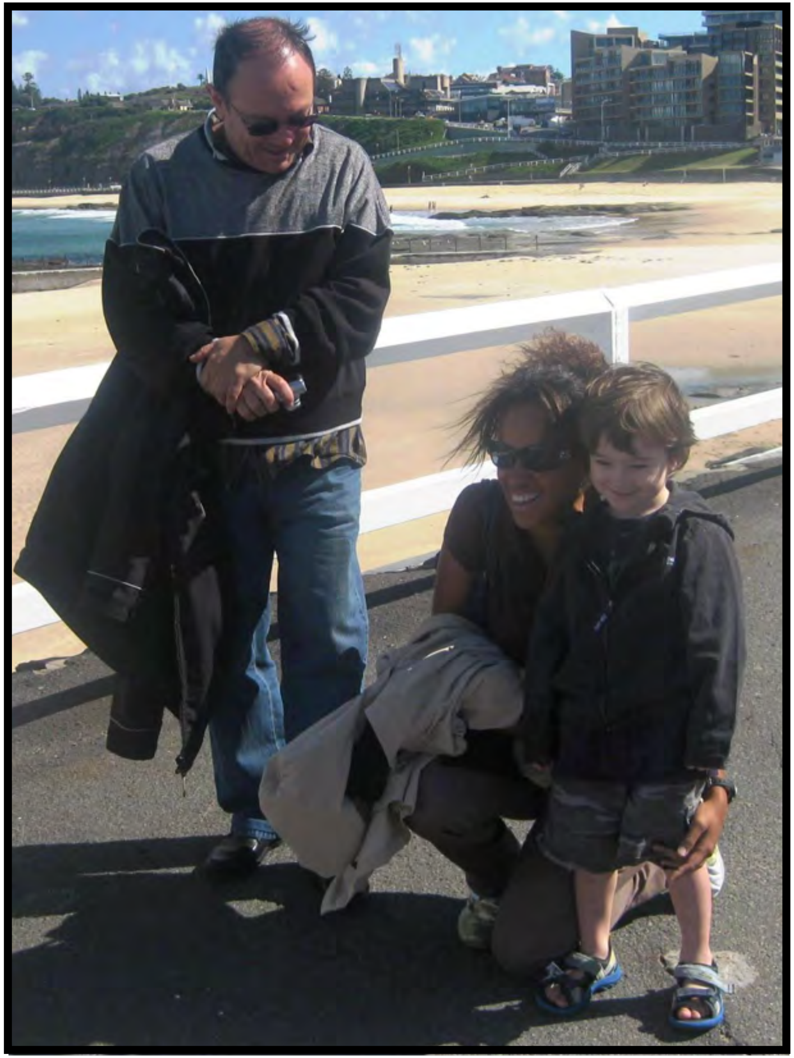
Companion Bodies

THEOREM 8.4. *Let E be a Banach space with the Radon-Nikodym property and let C be a non-empty closed, convex, bounded subset of E. Then the points in $E \backslash C$ which have nearest points in C are weakly dense in $E \backslash C$.*

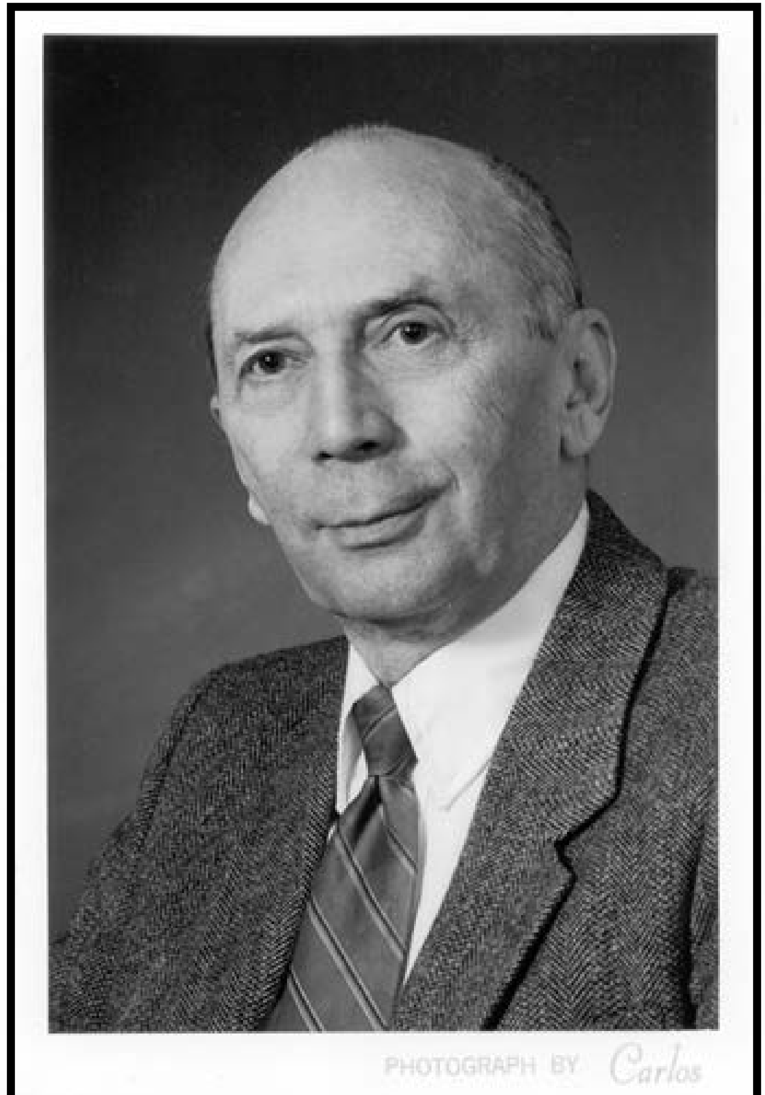
REMARKS 8.5. (i) We observe that outside of a space with the Radon-Nikodym property, Theorem 8.4 can go badly wrong. An example of Edelstein and Thompson [9] shows that in $c_0(\mathbf{Z}^+)$ with the supremum norm there is an equivalent ball B such B has no nearest points in $\| \; \|_\infty$. The sets C and B are called *companion (anti-proximinal) bodies*. ~~The only known examples are in c_0 and its isomorphs.~~ Does the non-existence of companion bodies characterize *RNP* spaces? No see [3] (CCCP)

(ii) Let E be a Banach space with the Radon-Nikodym property and let C be an arbitrary non-empty closed bounded subset of E. Are the points in $E \backslash C$ which have nearest points in C weakly dense in $E \backslash C$?

Two convex bodies are companions if and only if $B + C$ is open. See [3] for examples of companion bodies outside of c_0 isomorphs.



Revalski, Florence Jules and JJ Borwein (Newcastle 2010)



Porosity

See also PART IV

A subset S of a Banach space X is called *porous* if there is a number $\lambda \in (0,1)$ such that for every $x \in S$ and every $\delta > 0$ there is a $y \in X$ such that $0 < \|y - x\| < \delta$ and $S \cap B_r(y) = \emptyset$ where $r = \lambda\|y - x\|$. If S is a countable union of porous sets, then we will say that S is σ-*porous*. The complement of a σ-porous set is said to be a *staunch set*.

In Euclidean space staunch sets are both full measure and generic.

For any closed set C in a Banach space, Revalski and Zhivkov show (more than):

1. If X is LUR the set of points for which minimizing sequences converge to the unique member of $P_C(x)$ (*well posedness*) or $P_C(x)$ is empty is staunch.

2. If X is compactly LUR the set of points for which minimizing sequences converge (*generalised well posedness*) or $P_C(x)$ is empty is staunch.

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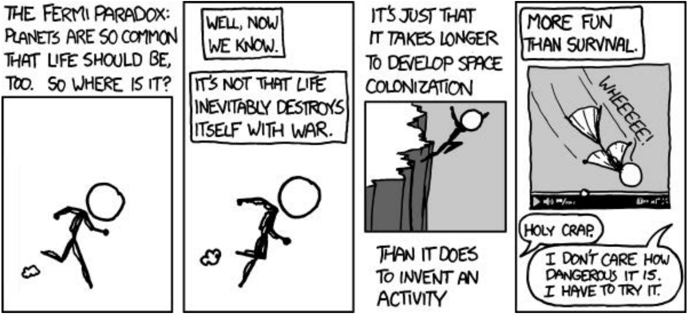
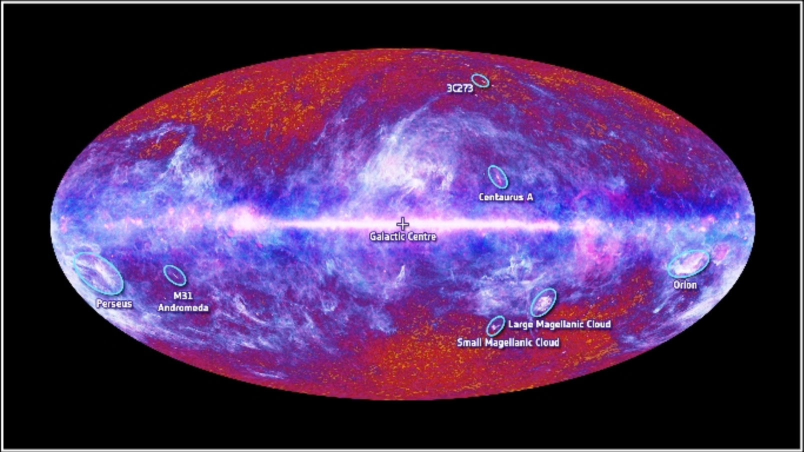

THE FERMI PARADOX: PLANETS ARE SO COMMON THAT LIFE SHOULD BE, TOO. SO WHERE IS IT?
WELL, NOW WE KNOW.
IT'S NOT THAT LIFE INEVITABLY DESTROYS ITSELF WITH WAR.
IT'S JUST THAT IT TAKES LONGER TO DEVELOP SPACE COLONIZATION
THAN IT DOES TO INVENT AN ACTIVITY
MORE FUN THAN SURVIVAL.
WHEEEEE!
HOLY CRAP.
I DON'T CARE HOW DANGEROUS IT IS. I HAVE TO TRY IT.



The Planck one-year all-sky survey

PART III

"Best Approximation Problems in (Reflexive) Banach Space"

A set S is $\boxed{\textit{Cebysev}}$ if every point in the space X has a unique nearest point in the set S.

Klee 1962: Is every $\boxed{\text{Chebysev}}$ set in Hilbert space convex (and closed)?

In Part III, we shall

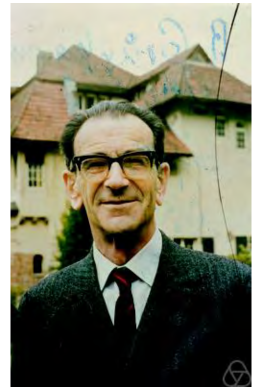
- study Klee's *Chebysev question*
  - discuss norms in which nearest points exist densely
- resolve the Euclidean case
  - giving four proofs of the *Motzkin-Bunt Theorem*
- consider the Hilbert case
  - giving a partial result
  - discuss related examples, extensions and conjectures

Pafnutij Lvovič Čebyšev

Пафнутий Львович Чебышев

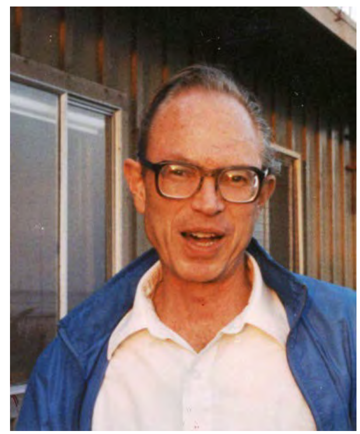
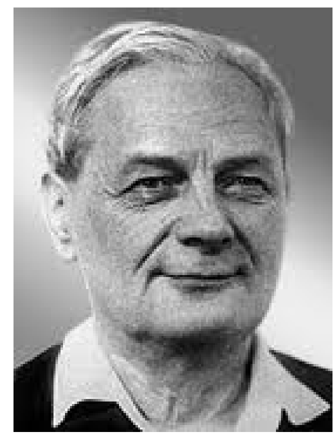
Other people we shall meet




Edgar Asplund (1931-74) | Pafnuty Chebysev (1821-94) | Ivar Ekeland (1944-) | Werner Fenchel (1905-88) | Victor Klee (1925-2007)



Sergei Konjagin (1957-) | Ka Sing Lau (1948-) | Theodore Motzkin (1908-70) | Bob Phelps (1926-2013) | Sergei Stechkin (1920-1995)

Proximality and Chebyshev sets

Jonathan M. Borwein


Marian Fabian and JMB (Paseky, April 2006)

Revised: 3 May 2006 / Accepted: 15 May 2006
Published online: 25 August 2006

Abstract This paper is a companion to a lecture given at the Prague Spring School in Analysis in April 2006. It highlights four distinct variational methods of proving that a finite dimensional Chebyshev set is convex and hopes to inspire renewed work on the open question of whether every Chebyshev set in Hilbert space is convex.

1 Introduction

Let us set some notation and definitions which are for the most part consistent with those in [7,10,13,25]. For a nonempty set A in a Banach space $(X, \|\cdot\|)$ we consider the *indicator function* $\iota_A(x) := 0$ if $x \in A$ and $+\infty$ otherwise. The *distance function* $d_A(x) := \inf_{a \in A} \|x - a\|$ and the *radius function* $r_A(x) := \sup_{a \in A} \|x - a\|$ are our main players. Note that r_A is finite if and only if A is bounded and then $r_A = r_{\overline{\mathrm{co}}\, A}$ is a continuous convex function.

The variational problems we consider are to determine when and if d_A and r_A attain their bounds. Specifically

$$\boxed{P_A(x) := \operatorname{argmin} d_A}$$

and

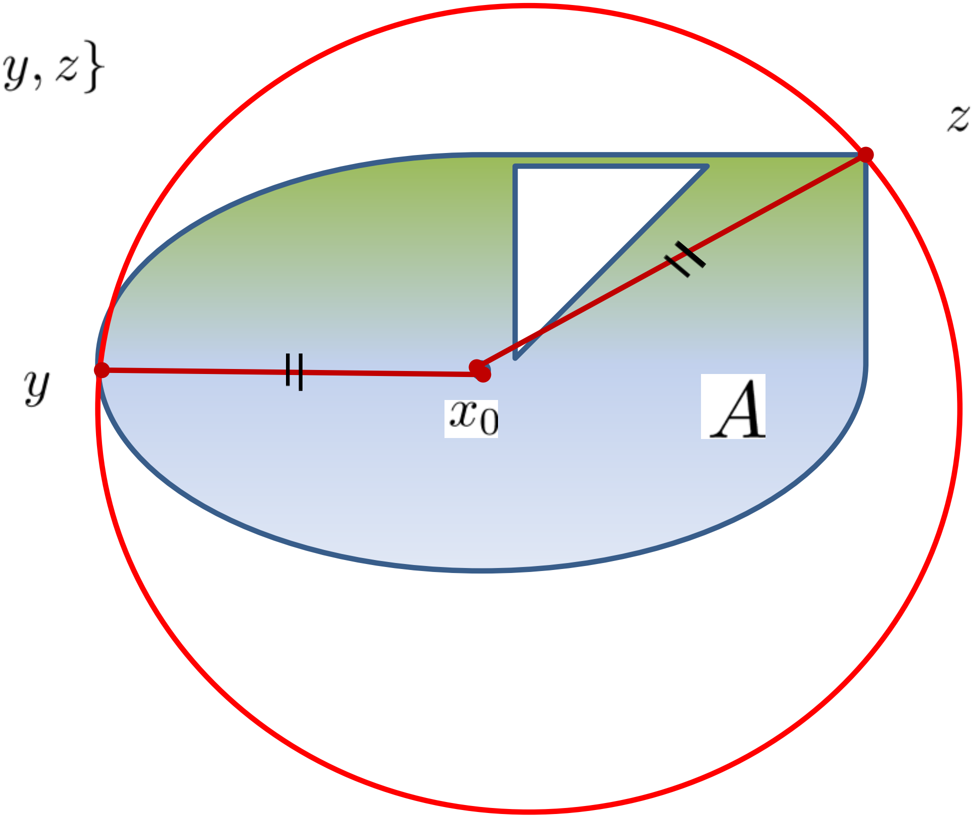
$$\boxed{F_A(x) := \operatorname{argmax} r_A,}$$

define the *nearest point* and *farthest point* operators, respectively. When $P_A(x) \neq \emptyset$ we say x admits *best approximations* or *nearest points* and call the elements of $P_A(x)$ *nearest points* or *proximal points*. *Worst approximation* and *farthest point* are correspondingly defined in terms of F_A. A set is called *proximal* (sometimes *proximinal*) if $D(P_A) = X$ and *Chebyshev* if P_A is both everywhere defined and single-valued. We try to reserve the symbols S for a Chebyshev set and E for a Euclidean space. In that case especially, P_A is often called the *metric projection* on A, and we shall not always distinguish $\{P_A(x)\}$ and $P_A(x)$.

Farthest points are easier

$\boxed{r_A(x) = r_{\overline{\mathrm{co}}A}(x) \text{ is non-expansive and convex.}}$

$F_A(x_0) = \{y, z\}$



As we shall see, these two problems are wonderful testing grounds for nonlinear and convex analysis. A fine variational tool is:

Theorem 1 (Basic Ekeland principle, [7,10,16,18]) *Suppose the function* $f : E \mapsto (g3) - \infty, \infty]$ *is closed and the point* $x \in E$ *satisfies* $f(x) < \inf f + \epsilon$ *for some real* $\epsilon > 0$. *Then for any real* $\lambda > 0$ *there is a point* $v \in E$ *satisfying the conditions*

(a) $\|x - v\| \leq \lambda$,
(b) $f(v) + (\epsilon/\lambda)\|x - v\| \leq f(x)$, *and*
(c) v *minimizes the function* $f(\cdot) + (\epsilon/\lambda)\| \cdot - v\|$.

Usually (b) is decoupled to yield (a) and (b′) $f(v) \leq f(x)$, but we shall need the full power of (b). Sadly, the short finite-dimensional proof in [7,10,18] does not seem to produce (b).

Which we give next

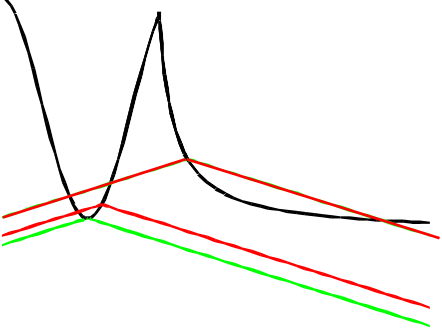


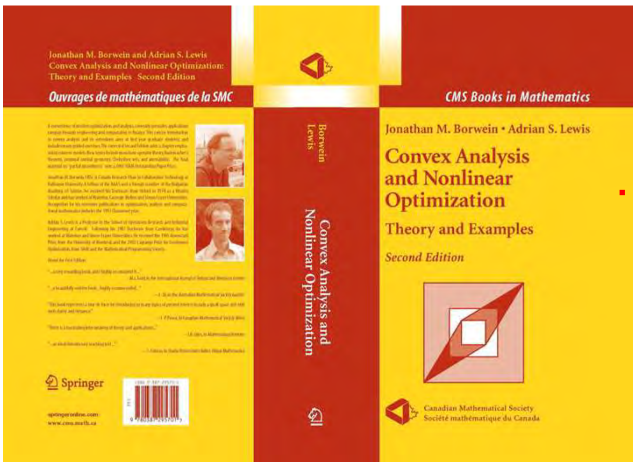
Fig. 2.1. Ekeland variational principle. Top cone: $f(x_0) - \varepsilon|x - x_0|$; Middle cone: $f(x_1) - \varepsilon|x - x_1|$; Lower cone: $f(y) - \varepsilon|x - y|$.

Theorem 7.1.2 (Ekeland variational principle) *Suppose the function $f : \mathbf{E} \to (\infty, +\infty]$ is closed and the point $x \in \mathbf{E}$ satisfies $f(x) \leq \inf f + \epsilon$ for some real $\epsilon > 0$. Then for any real $\lambda > 0$ there is a point $v \in \mathbf{E}$ satisfying the conditions*

(a) $\|x - v\| \leq \lambda$,

(b) $f(v) \leq f(x)$, *and*

(c) *v is the unique minimizer of the function* $f(\cdot) + (\epsilon/\lambda)\|\cdot - v\|$.



Adrian Lewis (1961-)

Proof. We can assume f is proper, and by assumption it is bounded below. Since the function

$$\boxed{f(\cdot) + \frac{\epsilon}{\lambda}\|\cdot - x\|}$$

therefore has compact level sets, its set of minimizers $M \subset \mathbf{E}$ is nonempty and compact. Choose a minimizer v for f on M. Then for points $z \neq v$ in M we know

$$f(v) \leq f(z) < f(z) + \frac{\epsilon}{\lambda}\|z - v\|,$$

while for z not in M we have

$$f(v) + \frac{\epsilon}{\lambda}\|v - x\| < f(z) + \frac{\epsilon}{\lambda}\|z - x\|.$$

Part (c) follows by the triangle inequality. Since v lies in M we have

$$f(z) + \frac{\epsilon}{\lambda}\|z - x\| \geq f(v) + \frac{\epsilon}{\lambda}\|v - x\| \quad \text{for all } z \text{ in } \mathbf{E}.$$

Setting $z = x$ shows the inequalities

$$f(v) + \epsilon \geq \inf f + \epsilon \geq f(x) \geq f(v) + \frac{\epsilon}{\lambda}\|v - x\|.$$

Properties (a) and (b) follow. $\square$

Fact 2 (Projection, [13]) *Let A be a closed set in a Hilbert space. Suppose that $a \in P_A(x)$. Then $P_A(tx + (1-t)a) = \{a\}$ for $0 < t < 1$.*

This clearly holds in any rotund Banach space – that is one with a strictly convex unit ball.

Fact 3 (Chebyshev, [10,13,16]) *Every Chebyshev set is closed and every closed convex set in a rotund reflexive space is Chebyshev. In particular every non-empty closed convex set in Hilbert space is Chebyshev.*

Uniqueness requires only rotundity. A much deeper result is

Proposition 4 (Reflexivity, [13,16]) *A space X is reflexive iff every closed convex set C is proximinal iff every closed convex set has nearest points.*

Proof In reflexive space every closed convex set is boundedly relatively weakly compact. Since the norm is weakly lower semicontinuous the problem $\min_{c \in C} \|x - c\|$ is attained for all $x \in X$.

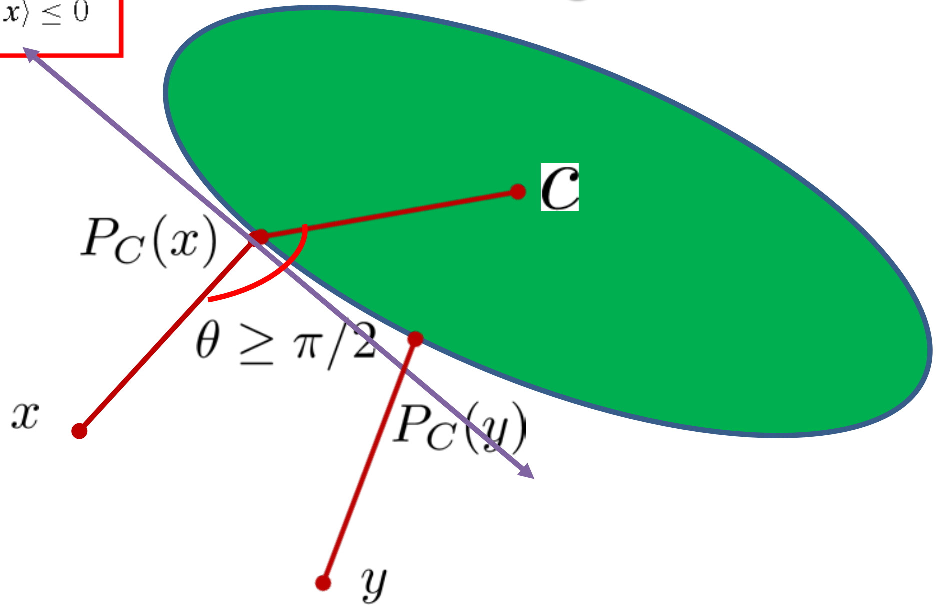
If X is not reflexive, then the *James theorem* [15] guarantees the existence of a norm-one linear functional f such that $f(x) < 1$ for all $x \in B_X$, the unit ball. It is an instructive exercise to determine that $d_{f^{-1}(0)}(x)$ is not attained unless $f(x) = 0$. $\square$

We shall see in Corollary 20 that there are non-reflexive spaces in which each bounded closed set admits proximal points. The non-expansiveness of the metric projection on a closed convex set in Hilbert space is standard and follows from the necessary and sufficient condition

Kolmogorov criterion

$$\boxed{\langle x - P_C(x), c - x\rangle \le 0}$$

for all $x \in C$.



We will now be more precise and interpolate a notion which greatly strengthens the property of Fact 2. We call $S \subset E$ a *sun* if, for each point $x \in E$, every point on the ray $P_S(x) + \mathbf{R}_+(x - P_S(x))$ has nearest point $P_S(x)$.

Proposition 5 (Suns, [7,13,16]) *In Hilbert space* (i) *a closed set C is convex iff* (ii) *C is a sun iff* (iii) *the metric projection P_C is nonexpansive.*

Proof We sketch the proof. It is easy to see that (i) implies (ii); while (iii) implies (i) is usually proved by a mean value argument. It remains to show (ii) implies (iii). Denoting the segment between points $y, z \in E$ by $[y, z]$, one shows that property (ii) implies

$$P_S(x) = P_{[z, P_S(x)]}(x) \quad \text{for all } x \in E,\ z \in S,$$

which quickly yields (iii), [7,13]. □

In three-or-more dimensions, non-expansivity characterizes Euclidean space amongst Banach spaces as do many other fundamental geometric properties (see, for example, [2,13]).

A fundamental result of much independent use is

Proposition 6 (Characterization of Chebyshev sets, [7,13,16]) *If E is Euclidean then the following are equivalent.*

1. *S is Chebyshev.*
2. *P_S is single-valued and continuous.*
3. *d_S^2 is everywhere Fréchet differentiable with $\nabla_F d_S^2/2 = I - P_S$.*
4. *The Fréchet sub-differential $\partial_F(-d_S^2)(x)$ is never empty.*

Proof (1) $\Rightarrow$ (2) follows by a compactness argument. (2) $\Rightarrow$ (3) is nearly immediate since $I - P_S$ is a continuous selection of $\partial d_S^2/2$. (3) $\Rightarrow$ (4). We will see a proof of (4) $\Rightarrow$ (1) in the next section. $\square$

This all remains true assuming only the space to be finite dimensional with a smooth and rotund norm – indeed many of implications remain true in Banach space at least for 'tame' sets. The only really problematic step is (1) $\Rightarrow$ (2).

A more flexible notion than that of a sun is that of an approximately convex set, [7,16]. We call $C \subset X$ *approximately convex* if, for any closed norm ball $D \subset X$ disjoint from C, there exists a closed ball $D' \supset D$ disjoint from C with arbitrarily large radius. Immediate from the definitions, as illustrated in Fig. 1 we have

Proposition 7 *Every sun is approximately convex.*

Proposition 8 (Approximate convexity, [7,16]) *Every convex set in a Banach space is approximately convex. When the space is finite dimensional and the dual norm is rotund every approximately convex set is convex.*

In ℓ^1 or ℓ^∞ norms this clearly fails as the righthand-side of Fig. 1 suggests. In the first case consider $\{(x, y) : y \leq |x|\}$. Vlasov [16, p. 242] shows dual rotundity characterizes the coincidence of convexity and approximate convexity, [16].

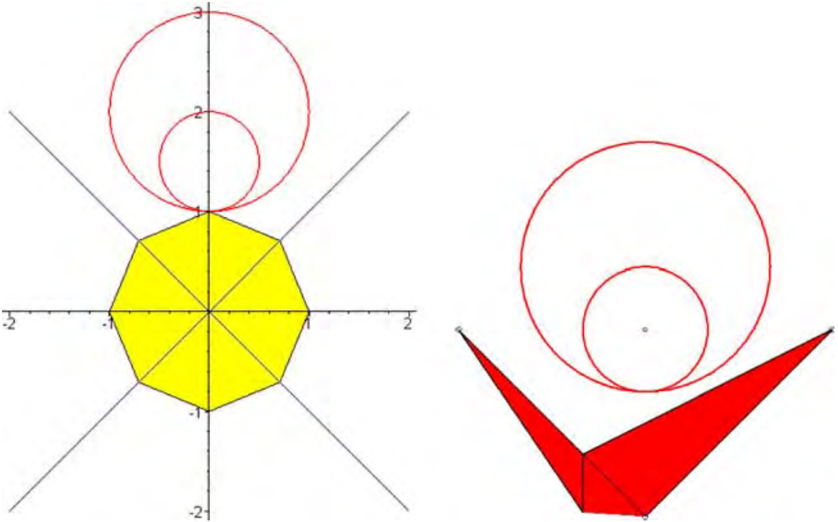


Fig. 1 Suns and approximate convexity

Proposition 8 (Approximate convexity, [7,16]) *Every convex set in a Banach space is approximately convex. When the space is finite dimensional and the dual norm is rotund every approximately convex set is convex.*

Proof The first assertion follows easily from the Hahn–Banach theorem [10, 16, 25].

Conversely, suppose C is approximately convex but not convex. Then there exist points $a, b \in C$ and a closed ball D centered at the point $c := (a+b)/2$ and disjoint from C. Hence, there exists a sequence of points $x_1, x_2, \ldots$ such that the balls $B_r = x_r + rB$ are disjoint from C and satisfy $\boxed{D \subset B_r \subset B_{r+1}}$ for all $r = 1, 2, \ldots$.

The set $\boxed{H := \text{cl} \cup_r B_r}$ is closed and convex, and its interior is disjoint from C but contains c. It remains to confirm that H is a half-space. Suppose the unit vector u lies in the polar set H°. By considering the quantity $\langle u, \|x_r - x\|^{-1}(x_r - x)\rangle$ as $r \uparrow \infty$, we discover H° must be a ray. This means H is a half-space. $\square$

We shall also exploit unexpected relationships between convexity and smoothness properties of d_A and r_A. For this we begin with:

Fact 9 (Fenchel conjugation, [7,16]) *The* convex conjugate *of an extended real-valued function f on a Banach space X is defined by*

$$\boxed{f^*(x^*) := \sup_{x \in X}\{\langle x, x^*\rangle - f(x)\}}$$

and is a convex, closed function (possibly infinite). Moreover, the biconjugate *defined on X^* by*

$$\boxed{f^{**}(x) := \sup_{x^* \in X^*}\{\langle x, x^*\rangle - f^*(x^*)\}}$$

agrees with f exactly when f is convex, proper and lower-semicontinuous.

Fact 9 is often a fine way of proving convexity of a function g by showing g arises as a conjugate, see [7,10,25], even by computer [3]. A particularly good tool is

Proposition 10 (Smoothness and biconjugacy, [20,28]) *If f^{**} is proper in a Banach space and f^* is everywhere Fréchet differentiable then f is convex.*

Proof The general result may be found in [9,28]. Under stronger conditions in a finite dimensional space E we shall prove more [7,19].

We consider an extended real valued function f that is closed and bounded below and satisfies the growth condition

$$\boxed{\lim_{\|x\|\mapsto\infty} \frac{f(x)}{\|x\|} = +\infty,}$$

along with a point $x \in \operatorname{dom} f$. Then *Carathéodory's theorem* [7]; Sect. 1.2] ensures there exist points $x_1, x_2, \ldots, x_m \in E$ and real $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$ satisfying

$$\sum_i \lambda_i = 1, \quad \sum_i \lambda_i x_i = x, \quad \sum_i \lambda_i f(x_i) = f^{**}(x).$$

$$\sum_i \lambda_i = 1, \quad \sum_i \lambda_i x_i = x, \quad \sum_i \lambda_i f(x_i) = f^{**}(x).$$

The definitional *Fenchel–Young inequality*, $\boxed{f(x) + f^*(x^*) \geq \langle x, x^* \rangle}$ valid for all x, x^*, implies that

$$\boxed{\partial(f^{**})(x) = \bigcap_i \partial f(x_i).}$$

Suppose now that the conjugate f^* is indeed everywhere differentiable. If $x \in \operatorname{ri}(\operatorname{dom}(f^{**}))$, we argue that $x_i = x$ for each i. We conclude that $\operatorname{ri}(\operatorname{epi}(f^{**})) \subset \operatorname{epi}(f)$, and use the fact that f is closed to deduce $\boxed{f = f^{**}}$; and so f is convex. $\square$

We illustrate the duality for $W := x \mapsto (1 - x^2)^2$ in Fig. 2. The left hand picture shows W and W^{**}, the right hand shows W^*.

We record next two lovely Hilbertian duality formulas:

Fact 11 (Hilbert duality, [7,19]) *For any closed set A in a Hilbert space*

$$\left(\frac{\iota_A + \|\cdot\|^2}{2}\right)^* = \frac{\|\cdot\|^2 + d_A^2}{2} \tag{1}$$

$$\left(\frac{\iota_{-A} - \|\cdot\|^2}{2}\right)^* = \frac{r_A^2 - \|\cdot\|^2}{2}. \tag{2}$$

Each identity once known is an easy direct computation from the definitions.

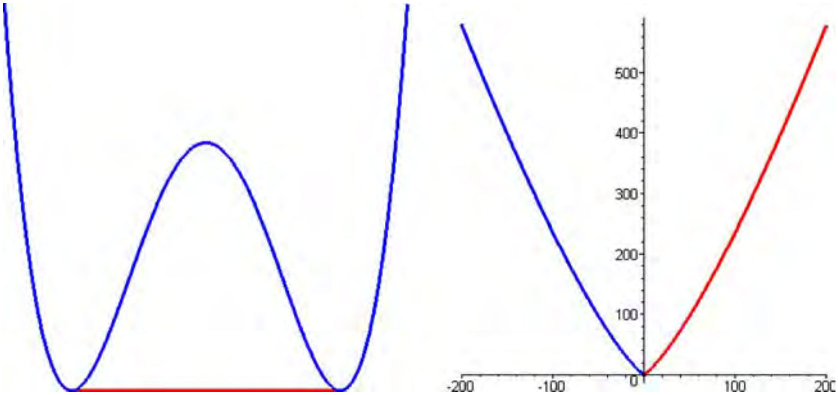


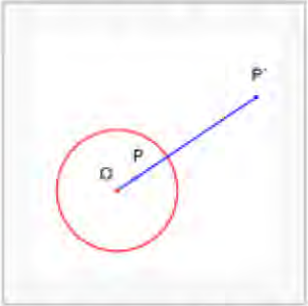
Fig. 2 A smooth nonconvex '*W*' function and its nonsmooth conjugate

We now turn to our final approach via inversive geometry. The self-inverse map $\iota : E \setminus \{0\} \mapsto E$ defined by $\iota(x) = \|x\|^{-2}x$ is called the *inversion in the unit sphere*. While this is meaningful in any Banach space it is nicest in Hilbert space.

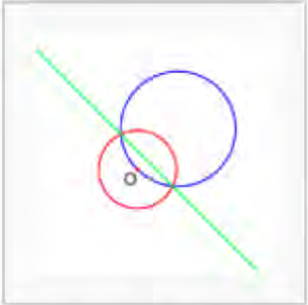
Fact 12 (Preservation of spheres, [1]) *If $D \subset E$ is a ball with $0 \in \operatorname{bd} D$, then $\iota(D \setminus \{0\})$ is a halfspace disjoint from 0. Otherwise, for any point $x \in E$ and radius $\delta > \|x\|$,*

$$\boxed{\iota((x + \delta B) \setminus \{0\}) = \frac{1}{\delta^2 - \|x\|^2}\{y \in E : \|y + x\| \geq \delta\}.}$$

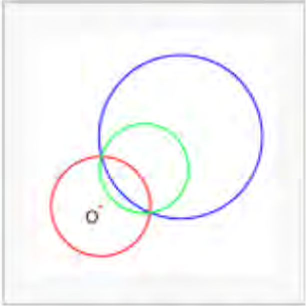
Inverse of a point



P' is the inverse of *P* with respect to the circle.



The inverse, with respect to the red circle, of a circle going through *O* (blue) is a line not going through *O* (green), and vice-versa.



The inverse, with respect to the red circle, of a circle *not* going through *O* (blue) is a circle not going through *O* (green), and vice-versa.

3 Proximality and Chebyshev sets in Euclidean space

We now describe four approaches to the following classic theorem.

Theorem 13 (Motzkin-Bunt, [1,7,13,16,19]) *A finite dimensional Chebyshev set is convex.*

Proof (1, via fixed point theory, [7,13]) By Proposition 5 it suffices to show S is a sun. Suppose S is not a sun, so there is a point $x \notin S$ with nearest point $P_S(x) =: \bar{x}$ such that the ray $L := \bar{x} + \mathbf{R}_+(x - \bar{x})$ strictly contains

$$\boxed{\{z \in L \mid P_S(z) = \bar{x}\}.}$$

Hence by Fact 2 and the continuity of P_S, the above set is a nontrivial closed line segment $[\bar{x}, x_0]$ containing x.

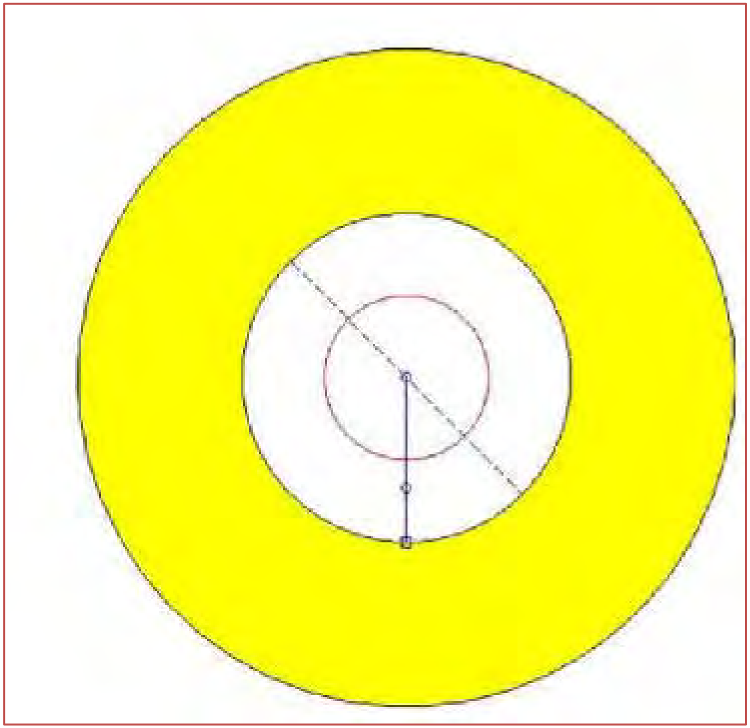
Choose a radius $\epsilon > 0$ so that the ball $x_0 + \epsilon B$ is disjoint from S. The continuous self map of this ball

$$\boxed{z \mapsto x_0 + \epsilon \frac{x_0 - P_S(z)}{\|x_0 - P_S(z)\|}}$$

has a fixed point by Brouwer's theorem. We then quickly derive a contradiction to the definition of the point x_0. We illustrate this construction in Fig. 3. $\square$

Alternatively, via Proposition 8 it suffices to show S is approximately convex. This method is the least coupled to Hilbert space.

Figure 3. Failure of a Sun



Proof (2, via the variational principle, [7,16]) Suppose S is not approximately convex. We claim that for each $x \notin S$

$$\boxed{\limsup_{y \to x} \frac{d_S(y) - d_S(x)}{\|y - x\|} = 1.} \tag{3}$$

This is a consequence of the (Lebourg) mean-value for (Lipschitz) functions [7,12], since all Fréchet (super-)gradients have norm-one off S.

We now appeal to the Basic Ekeland principle of Proposition 1 as follows: Consider any real $\alpha > d_C(x)$. Fix reals $\sigma \in (0,1)$ and ρ satisfying

$$\boxed{\frac{\alpha - d_C(x)}{\sigma} < \rho < \alpha - \beta.}$$

Using approx. convexity

By applying the Basic Ekeland variational principle to the function $-d_C + \delta_{x+\rho B}$, prove there exists a point $v \in E$ satisfying the conditions

$$\boxed{\begin{aligned} &d_C(x) + \sigma\|x - v\| \leq d_C(v) \\ &d_C(z) - \sigma\|z - v\| \leq d_C(v) \ \text{ for all } z \in x + \rho B. \end{aligned}}$$

We deduce $\|x - v\| = \rho$, and hence $x + \beta B \subset v + \alpha B$. Thus, C is approximately convex and Proposition 8 concludes this proof. $\square$

We next consider two theorems that exploit conjugate duality.

Proof (3, via conjugate duality, [7,19]) First, d_S^2 is differentiable by Proposition 6. Now consider formula (1). The righthand side is clearly differentiable and it suffices to appeal to Proposition 10 to deduce that $\boxed{\iota_S + \|\cdot\|^2}$ is convex. A fortiori, so is S. □

We may also deduce a 'dual' result about farthest points that we shall use in our fourth proof.

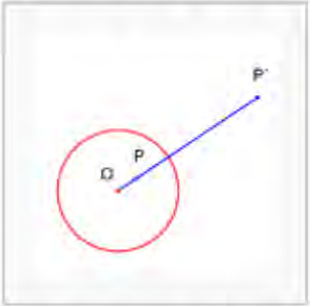
Theorem 14 *Suppose that every point in Euclidean space admits a unique farthest point in a set A. Then A is singleton.*

Proof We leave it to the reader to deduce that r_A^2 is differentiable (and strictly convex), [7, p. 226]. One way is to use the formula for the subgradient of a convex *max-function* over a compact (convex) set [7, p. 129, Exercise 10], or [10,12,20,25]. Uniqueness of the farthest point $F_A(x)$ then implies that

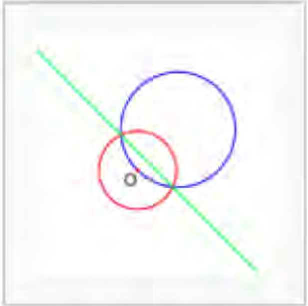
$$\frac{1}{2}\,\partial r_A^2(x) = x - F_A(x) = \frac{1}{2}\,\nabla r_A^2(x).$$

Now consider formula (2). The righthand side is again clearly differentiable and it an to appeal to Proposition 10 to shows that $\iota_{-A} - \|\cdot\|^2$ is convex. As $-\|\cdot\|$ is strictly concave, A can not contain two points. $\square$

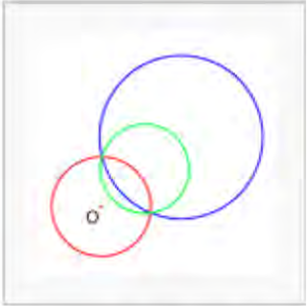
Inverse of a point



P' is the inverse of *P* with respect to the circle.



The inverse, with respect to the red circle, of a circle going through *O* (blue) is a line not going through *O* (green), and vice-versa.



The inverse, with respect to the red circle, of a circle *not* going through *O* (blue) is a circle not going through *O* (green), and vice-versa.

Proof (4, via inversive geometry, [1,7]) Without loss of generality, suppose $0 \notin C$ but $0 \in \operatorname{cl}\operatorname{conv} C$. Consider any point $x \in E$. Fact 12 implies that the quantity

$$\boxed{\rho := \inf\{\delta > 0 \mid \iota C \subset x + \delta B\}}$$

satisfies $\rho > \|x\|$. Now let z denote the unique *nearest* point in C to the point $\boxed{(-x)/(\rho^2 - \|x\|^2)}$, and observe, again via Fact 12, that $\iota(z)$ is the unique *furthest* point in $\iota(C)$ to x. By Theorem 14 the set $\iota(C)$ is a singleton which is not possible. $\square$

4 Proximality and Chebyshev sets in infinite dimensions

In this section we make a discursive look at the subject in infinite dimensions. In 1961, Klee [22] asked whether a Chebyshev set in Hilbert space must be convex? The literature is large but a good start can be made by reading the relevant parts of [13] and [16]. A comprehensive survey up to 1973 is given in [26]. The cleanest partial answer yet known is:

Theorem 15 (Chebyshev sets, [1,9,13,16,22]) *A weakly closed Chebyshev set in Hilbert space is convex.*

Proof Once we establish the Fréchet differentiability of d_S^2 the second and third proofs need no change. To do this it suffices to argue that $I - P_S$ is still norm-weak* continuous while $x \mapsto \|x - P_S(x)\| = d_S(x)$ is continuous. We then appeal to the fact that norm and weak convergence agree on spheres in Hilbert space.

Asplund's proof likewise holds – indeed, this was his proof of the theorem, [1]. The first proof also extends as far as boundedly norm-compact sets via *Schauder's fixed point theorem*, albeit with a little more effort [10, p. 219]. $\square$

Remark 16 (Generalizations) Indeed, the second proof actually shows Vlasov's (1970) result that *in a Banach space with a rotund dual norm any Chebyshev set with a continuous projection is convex* as described in [5,16,17] since (3) will hold under these hypotheses.

Asplund's method [1] also yields the striking result that if there is a non-convex Chebyshev set in Hilbert space there is also one that is the complement of an open convex body – a so called *Klee cavern*. This is both surprising yet consistent with Fig. 3 that we drew for the proof via Brouwer's theorem.

While a sun in a smooth Banach space is known to be convex, [26], the existence in a renorming of $C[0,1]$ of a disconnected non-Chebyshev sun, [23], indicates the limitations of the first approach. □

Remark 17 (Counter-examples) Opinions differ about whether every (norm-closed) Chebyshev set in Hilbert space is convex. Since there are even closed sets of rotund reflexive space with discontinuous projections [11], in that level of generality one must somehow establish the continuity of P_S or avoid the issue to show S is convex.

It is known that any non-convex Chebyshev set in Hilbert space must have a badly discontinuous metric projection [27]. That paper uses monotone operators to show that $H \setminus \{x\colon \nabla_F d_S(x)$ exists$\}$ is the countable union of nonconstant Lipschitz curves. This is based on the fact that P_S is maximal monotone if and only if S is Chebyshev and P_S is continuous. In the separable case Duda [14] shows the the covering can be achieved by *difference-convex* surfaces.

It is also known that there is an example of a bounded non-convex Chebyshev set (actually it can be disconnected *Chebyshev foam*) in an incomplete inner-product

J. Fletcher "Cebysev set problem" Hon. Th., Auck. '13

Recall that a norm is (sequentially) *Kadec–Klee* if weak and norm topologies coincide (sequentially) on norm spheres.

(also called Efimov-Stechkin)

Theorem 18 (Dense and generic proximality) *Every closed set A in a Banach space densely (equivalently generically) admits nearest points iff the norm is Kadec–Klee and the space is reflexive.*

Proof *If* (originally proved by Lau in [24]). We sketch the proof in [4, 10]. Consider a sub-derivative $\boxed{\phi \in \partial_F(-d_A)(x)}$, which by the smooth variational principle exists for a dense set in $X \setminus A$. Let (a_n) be a bounded minimizing sequence, and use reflexivity to extract a subsequence (we use the same name) converging weakly to $z \in X$. Since $\phi \in \partial_F(-d_A)(x)$ it is easy to show that $\|\phi\| = 1$ and that $\boxed{\phi(a_n - x) \to d_A(x)}$. Thus, we see that $\|z - x\| \geq \phi(z - x) = d_A(x) \geq \lim \|a_n - x\|$ and by weak lower-semicontinuity of the norm $\boxed{\|a_n - x\| \to \|z - x\|}$. The Kadec–Klee property then implies that $a_n \to z$ in norm and so $z \in A$. As $\|z - a\| = d_A(x)$ we have shown the set of points with nearest points in A is dense. Showing genericity takes a little more effort.

Only if (originally due to Konjagin). We sketch the proof in [4]. We shall construct a norm closed set A and a neighbourhood U within which no point admits a best approximation in A. If the space is not reflexive we appeal to Proposition 4.

In the reflexive setting, failure of the Kadec–Klee property means there must be a weakly-null sequence (x_n) with $\|x_n\| = 1$ and with $\|x_n - x_m\| \geq 3\varepsilon > 0$ (i.e, the sequence is *3ε-separated*). Let

$$\boxed{A := \cap_n x_n + \varepsilon B_X.}$$

It is routine to verify that in some neighbourhood U of zero there are no points with $P_A(x)$ non-empty. $\square$

Remark 19 (a) An easier version of the 'if' argument exactly proves (4) $\Rightarrow$ (1) of Proposition 6.

(b) Konjagin's construction produces a distance function d_A which is Fréchet differentiable (even affine) in a neighbourhood of zero but induces no best approximations from that neighbourhood. Thus the geometry of the norm is critical even in the presence of Fréchet derivatives. $\square$

Corollary 20 (Existence of proximal points) *A closed set C in a Banach space X has a nonempty set of proximal points under any of the following conditions.*

1. *X is reflexive and the norm is (sequentially) Kadec–Klee,* (Theorem 18).
2. *X has the* Radon Nikodym property [15] *and C is bounded,* [4].
3. *X is norm closed and boundedly relatively weakly compact,* [8].

This list is far from exhaustive. For instance

Example 21 (Norms with dense proximals, [4]) There is a class of reflexive non-Kadec–Klee norms such that every nonempty closed set A densely possesses proximal points. Explicit examples are given in [4]. The counter-example sketched in Theorem 18 is locally weakly-compact and convex and so admits dense proximals. □

Example 22 (Multiple caverns, [4]) Let us call the complement of finitely many disjoint open convex bodies a *multiple cavern*. Using inversive geometry methods as above, one can show that in a reflexive space every multiple Klee cavern admits proximal points. In [4] such sets were called *Swiss cheese*. □

Finally, I discuss two very useful additional properties of the distance function when the norm is *uniformly Gâteaux differentiable* as is the case in Hilbert space and, after renorming, in every super-reflexive and every separable Banach space, [5]. We say that ∂d_A is *minimal* if it contains no smaller w^*-cusco – a norm to w*-upper semicontinuous mapping with non-empty w^*-compact images.

Remark 23 (Some additional properties of d_A, [5]) A Banach space X is uniformly Gâteaux differentiable if and only if $\boxed{\partial d_A}$ is minimal for every closed nonempty set A. This has lovely consequences for proximal normal formulas, [6] (see [10] for the finite dimensional case). It relies on the fact that such norms also characterize those spaces for which

$$\boxed{\partial_-(-d_A)(x) = \partial_\diamond(-d_A)(x) = \partial_o(-d_A)(x),}$$

that is the Dini, Clarke and *Michel-Penot sub-differentials* (see [7]) coincide for all closed sets A, and hence that $-d_A$ is both Clarke and Michel-Penot regular, [5]. □

5 Conclusion

I hope this discussion has whetted some readers' appetites to attempt at least one of the following open questions.

Question 1 *Is every Chebyshev set in Hilbert space convex?*

Question 2 *Is every closed set in Hilbert space with unique farthest points a singleton?*

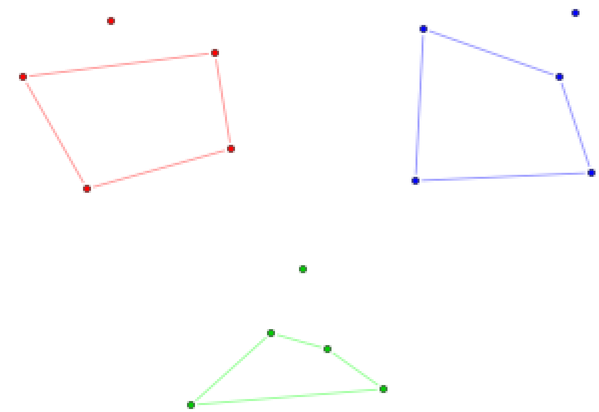
Question 3 *Is every Chebyshev set in a rotund reflexive Banach space convex?*

Question 4 *Does every closed set in a reflexive Banach space admit a nearest point? What about rotund smooth renormings of Hilbert space?*

Question 5 *Does every closed set in a reflexive Banach space admit proximal normals at a dense set of boundary points?*

And finally, I certainly hope I have made good advertisements for the power of variational and nonsmooth analysis.

Happy Ending Problem



Named by Paul Erdős (1913-96) as it led to marriage of George Szekeres (1911- 28-8-2005) and Esther Klein (1910- 28-8-2005). Also Roger Eggelston and John Selfridge (1927-2010)

Theorem. Any set of five points in the plane in general position[1] has a subset of four points that form the vertices of a convex quadrilateral. This was one of the original results that led to the development of Ramsey theory.

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ai-je bien tout dans mon cartable ?
j'ai mon Netbook, mon MP3,
ma clé USB, mon IPAD...
Papa, qu'utilisiez-vous
au lycée ?
notre tête !
Periotti

PART IV

"Small Sets in Banach Space"

A Borel set $S \subset X$ is *Haar null* if there is a *Radon* measure p such that

$$p(S + x) = 0$$

for all $x \in X$.

In Part IV, we shall

– study five concepts of small sets

– all are closed under translation, countable union and inclusion

- and can be used to do infinite dimensional analysis even though no Haar measure exists
- for instance, if X is separable every real valued Lipschitz function is Gateaux differentiable except on a *Haar* null set

Small Sets

This section follows Section 4.6 of *Convex Functions* and outlines five basic types of small sets. These classes of sets will be *closed under translation, countable unions and inclusion.*

A. Countable Sets. This is a familiar notion of smallness. When the domain of a convex function is an open interval in $\mathbf{R}$, there are at most *countably many* points of nondifferentiability (Stromberg).

- By considering $\boxed{f(x, y) := |x|,}$ this result fails in $\mathbf{R}^2$. Therefore, the notion of countability is too small for differentiability results concerning even continuous convex functions beyond $\mathbf{R}$.

See also
J. Lindenstrauss, D. Preiss and J. Tisier, *Fréchet Differentiability of Lipschitz Functions and Porous Sets in Banach Spaces* (AM-179), Princeton University Press, 2012.

Small Sets

B. First category sets. This is a very powerful and widely used notion of smallness. Let X be metric space. A subset S of X is nowhere dense if $\overline{S}$ has empty interior. A subset F of X is of *first category* if it is a union of countably many nowhere dense sets. A set is said to be of the *second category* if it is not of the first category.

The space X is called a *Baire space* if for any set F of first category, the complement $T \setminus F$ is everywhere dense. The complement of a set of first category is said to be *residual* or *fat*.

A first-category set is sometimes also said to be *meager* or *thin*. A set containing a dense G_δ is said to be *generic*.

Small Sets

- On finite-dimensional spaces, the set of points of differentiability of a convex function is a dense G_δ, and in infinite-dimensions the set of points of Fréchet differentiability of a continuous convex function is a possibly empty G_δ-set.

- Moreover, there are infinite-dimensional spaces where even a norm can be nowhere Fréchet or even Gâteaux differentiable.

- Nevertheless, there are wide classes of spaces where the points of nondifferentiability of a continuous convex function are of the first category.

Small Sets

- In addition to its usefulness in questions of Fréchet differentiability of convex functions, another attractive aspect of the notion of category is that the dimension of the Banach space is not an issue in its definition.

- However, on the negative side, the points of a nondifferentiability of a Lipschitz function on $\mathbf{R}$ may be of the second category and as a consequence the points where a continuous convex function on the real line fails to be twice differentiable can be a set of the second category.

Hence another notion of smallness is needed in this context.

Small Sets

C. Null sets.

- On $\mathbf{R}^n$ continuous convex functions are also differentiable a.e.; a harder result (Rademacher 1923) shows this holds for locally Lipschitz functions; additionally, *Alexandrov's theorem* (1940) asserts continuous convex functions on $\mathbf{R}^n$ have second-order Taylor expansions almost everywhere.
- In infinite-dimensions, care must be taken as there is no analogue of *Haar-measure* (the unit ball is not norm compact).

However, classes of *null sets* can be defined with the following properties:

(N1) They are closed under inclusion, translation and countable unions.

(N2) No nonempty open set is null.

(N3) In $\mathbf{R}^n$ they coincide with (or lie inside) Lebesgue null sets.

Small Sets

A *Borel measure* on a topological space X is any measure defined on $\mathcal{B}(X)$ the Borel subsets of X, the σ-algebra generated by the open sets. A *Radon measure* μ on X is a Borel measure which satisfies:

1. $\mu(K) < \infty$ for each compact $K \subset X$;
2. $\mu(A) = \sup\{\mu(K) \subset K \subset A, K \text{ compact}\}$ for each $A \in \mathcal{B}(X)$.

Haar Null sets. A Borel subset N in X is *Haar null* if there exists a (not necessarily unique) Radon probability measure p on X such that

$$p(x + N) = 0 \qquad \text{for each } x \in X.$$

In such a case, we shall call the measure p a *test-measure* for N. More generally we say that an arbitrary subset $N \subset X$ is *Haar-null* if it is contained in a Haar-null Borel set.

Small Sets

Theorem. Let X be a Banach space. Then the following are true.

1. Every subset of a Haar-null set in X is Haar-null.
2. If A is Haar-null, so is $x + A$ for every $x \in X$.
3. If A is Haar-null, then there exists a test-measure for A with compact support.
4. If A is Haar-null, then $X \setminus A$ is dense in X.
5. If $\{A_n\}_{n\in\mathbf{N}}$ are Haar-null sets, then so is $\bigcup_{n=1}^{\infty} A_n$.

Small Sets

Proof. The proofs of(1), (2), (3) are easy.

To prove (4) it is sufficient to show there are no nonempty open Haar-null sets. For this, let U be a nonempty open subset of X and suppose by way of contradiction there exists a test measure p for U on X. Let A denote the support of p.

As A is separable, for some $x_0 \in X$, $(x_0 + U) \cap A \neq \emptyset$. Thus, $p(x_0 + U) \geq p((x_0 + U) \cap A) > 0$ which contracts the fact that p is a test measure for U.

For (5) we may assume each set A_j is Borel. For each $j \in \mathbf{N}$, let p_j be a test measure for A_j on G. Let H be the smallest closed subspace of X that contains the support of each p_j. Since the support of each p_j is provably separable, it is transpires that H is separable.

Let p_j^* be the restriction of p_j to H. Thus, there is a Radon probability measure p^* on H that is a test measure for each set of the form $\bigcup\{B_j \subseteq j \in \mathbf{N}\}$, provided that $B_j \in \mathcal{B}(H)$ and p_j^* is a test measure for B_j.

Small Sets

Let p be the extension of p^* to X. We claim that p is a test measure for $\bigcup\{A_j \subseteq j \in \mathbf{N}\}$. We must show that for each $x \in X$, $p(x + \bigcup\{A_j\colon j \in N\}) = 0$. Now fix $x \in X$. Then

$$p(x + \bigcup_{j=1}^{\infty} A_j) = p^*((x + \bigcup_{j=1}^{\infty} A_j) \cap H) = p^*(\bigcup_{j=1}^{\infty} (x + A_j) \cap H).$$

However, each p_j^* is a test measure for $(x + A_j) \cap H$ since p_j is a test measure for A_j and

$$h + ((x + A_j)) \cap H) = ((h + x) + A_j)) \cap H \subset (h + x) + A_j$$

for each $h \in H$.

Therefore, p^* is a test measure for $\bigcup\{(x + A_j) \cap H\colon j \in \mathbf{N}\}$, and so

$$p^*(\bigcup\{(x + A_j) \cap H\colon j \in N\}) = 0,$$

which implies that $p(x + \bigcup\{A_j\colon j \in N\}) = 0$. **QED**

Small Sets

Proposition. Suppose C is a closed convex set and $d \in X \setminus \{0\}$ is such that C contains no line segment in the direction d. Then C is Haar null.

Proof. Consider

$$\mu(S) := \lambda\{t \in [0,1]\colon td \in S\}$$

where λ is Lebesque measure on $\mathbf{R}$. Then μ defines a Borel probability measure μ such that $\mu(x + C) = 0$ for all $x \in X$. QED.

- From this, it follows that all compact sets in infinite-dimensional Banach space are Haar null.

Small Sets

Aronszajn null sets. Another important class of *negligible* sets is defined follows. For $y \in X \setminus \{0\}$, let $\mathcal{A}(y)$ be the family of Borel sets A in X which intersect each line parallel to y in a set of one-dim Lebesgue measure 0.

If $\{x_n\}$ is an at-most-countable collection in $Z \setminus \{0\}$, we let $\mathcal{A}(\{x_n\})$ denote the collection of Borel sets A that can be written as $A = \bigcup A_n$, where $A_n \in \mathcal{A}(x_n)$ for every n.

A set is *Aronszajn null* if for each nonzero sequence $\{x_n\}$ whose linear span is dense in X, A can be decomposed into a union of Borel sets $\{A_n\}$ such that $A_n \in \mathcal{A}(x_n)$ for every n.

- Every Aronszajn null set is Haar null, but there are Haar null sets that are not Aronszajn null.

- Another well known class, the *Gaussian null* sets, which we will not define here, is known to coincide with the Aronszajn null sets. (See also the recent book by Lindenstrauss-Preiss-Tisier.)

Small Sets

D. Porosity. A subset S of a Banach space X is *porous* if there is a number $\lambda \in (0,1)$ such that for every $x \in S$ and every $\delta > 0$ there is a $y \in X$ such $0 < \|y - x\| < \delta$ and $S \cap B_r(y) = \emptyset$ where $r = \lambda \|y - x\|$.

If S is a countable union of porous sets, we will that S is σ-*porous*.

The complement of a σ-porous set is said to be a *staunch set*.

Small Sets

E. Angle small sets. Let X be a Banach space. For $x^* \in X^*$ and $\alpha \in (0,1)$ consider the cone

$$K(x^*, \alpha) := \{x \in X \colon \alpha \|x\| \|x^*\| \leq \langle x^*, x \rangle\}.$$

For fixed $\alpha \in (0,1)$, a subset S of X is *α-cone meager* if for every $x \in S$ and $\epsilon > 0$ there exists $z \in B_\epsilon(x)$ and $x^* \in X^* \setminus \{0\}$ such that

$$S \cap [z + \operatorname{int} K(x^*, \alpha)] = \emptyset.$$

The set S is said to be *angle small* if for every $\alpha \in (0,1)$ it can be expressed as a countable union of α-cone meager sets.

- Clearly both σ-porous and angle small sets are first category
- They satisfy (N1) , (N2), and (N3) and so are reasonable classes of null sets.
- if X^* is separable, all continuous convex functions are Fréchet differentiable except on an angle small set.

Application to Distance Functions

In Hilbert space Asplund observed

$$\boxed{\begin{aligned}\frac{1}{2}\,d_C(x)^2 &= \frac{1}{2}\,\inf_{c\in C}\|x-c\|^2\\ &= \frac{1}{2}\,\|x\|^2 - \sup_{c\in C}\langle x, c\rangle - \frac{1}{2}\,\|c\|^2.\end{aligned}} \tag{1}$$

This, and a clever mean value theorem to deal with points on bd C, allows all analysis of d_C to be reduced to convex analysis.

- This shows d_C^2 is the difference of two very nice convex functions(one is C^∞).
- Thus d_C^2 is generically Fréchet differentiable,
- and when H is separable $\frac{1}{2}\,d_C(x)^2$ is Fréchet differentiable except on an angle-small set.