Maximum Entropy-type Methods, Projections, and (Non-)Convex Programming

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BRAGG and BLAKE

"I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate. ··· The spoken word and the written word are quite different arts. ··· I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."

Sir Lawrence Bragg (1890-1971)
Nobel Crystallographer (Adelaide)

Songs of Innocence and Experience (1825)
(We are both.)

SANTAYANA

"If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician. But they were overworked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation."

(**George Santayana**)

Persons and Places, 1945, 238–9.

TWO FINE REFERENCES:

1. J.M. Borwein and Qiji Zhu, *Techniques of Variational Analysis*, CMS/Springer-Verlag, New York, 2005.

2. J.M. Borwein and A.S Lewis, *Convex Analysis and Nonlinear Optimization*, CMS/Springer-Verlag, 2nd expanded edition, New York, 2005.

OUTLINE

I shall discuss in "tutorial mode" the formalization of inverse problems such as signal recovery and option pricing as (convex and non-convex) optimization problems over the infinite dimensional space of signals. I shall touch on* the following:

1. The impact of the choice of "entropy" (e.g., Boltzmann-Shannon, Burg entropy, Fisher information) on the *well-posedness* of the problem and the form of the solution.

2. Convex programming duality: what it is and what it buys you.

3. Algorithmic consequences.

4. Non-convex extensions: life is hard. But sometimes more works than should.

♠ Related papers at `http://docserver.cs.dal.ca/`

*More is an unrealistic task!

THE GENERAL PROBLEM

• Many applied problems reduce to "**best**" solving (under-determined) systems of linear (or non-linear) equations $\boxed{Ax = b}$, where $b \in I\!R^n$, and the unknown x lies in some appropriate function space.

Discretization reduces this to a finite-dimensional setting where A is now a $m \times n$ matrix.

◇ In many cases, I believe it is better to address the problem in its function space home, discretizing only as necessary for computation.

• Thus, the problem often is *how do we estimate x from a finite number of its 'moments'*? This is typically an under-determined linear inversion problem where the unknown is most naturally a function, not a vector in $I\!R^m$.

EXAMPLE 1. AUTOCORRELATION

• Consider, extrapolating an *autocorrelation function* $R(t)$ given sample measurements.

◇ The Fourier transform $S(z)$ of the autocorrelation is the power spectrum of the data.

> Fourier moments of the power spectrum are the same as samples of the autocorrelation function, so by computing several values of $R(t)$ directly from the data, we are in essence computing moments of $S(z)$.

• We compute a finite number of moments of S, and estimate S from them, and may *compute more moments* from the estimate $\widehat{S}$ by direct numerical integration.

• Thereby *extrapolating* R, without directly computing R from potentially noisy data.

THE ENTROPY APPROACH

• Following (B-Zhu) I sketch a maximum entropy approach to under-determined systems where the unknown, x, is a function, typically living in a *Hilbert space*, or more general space of functions.

> This technique picks a "best" representative from the infinite set of *feasible* functions (functions that possess the same n moments as the sampled function) by minimizing an integral functional, f, of the unknown.

http://projects.cs.dal.ca/ddrive

⋄ The approach finds applications in countless fields including:

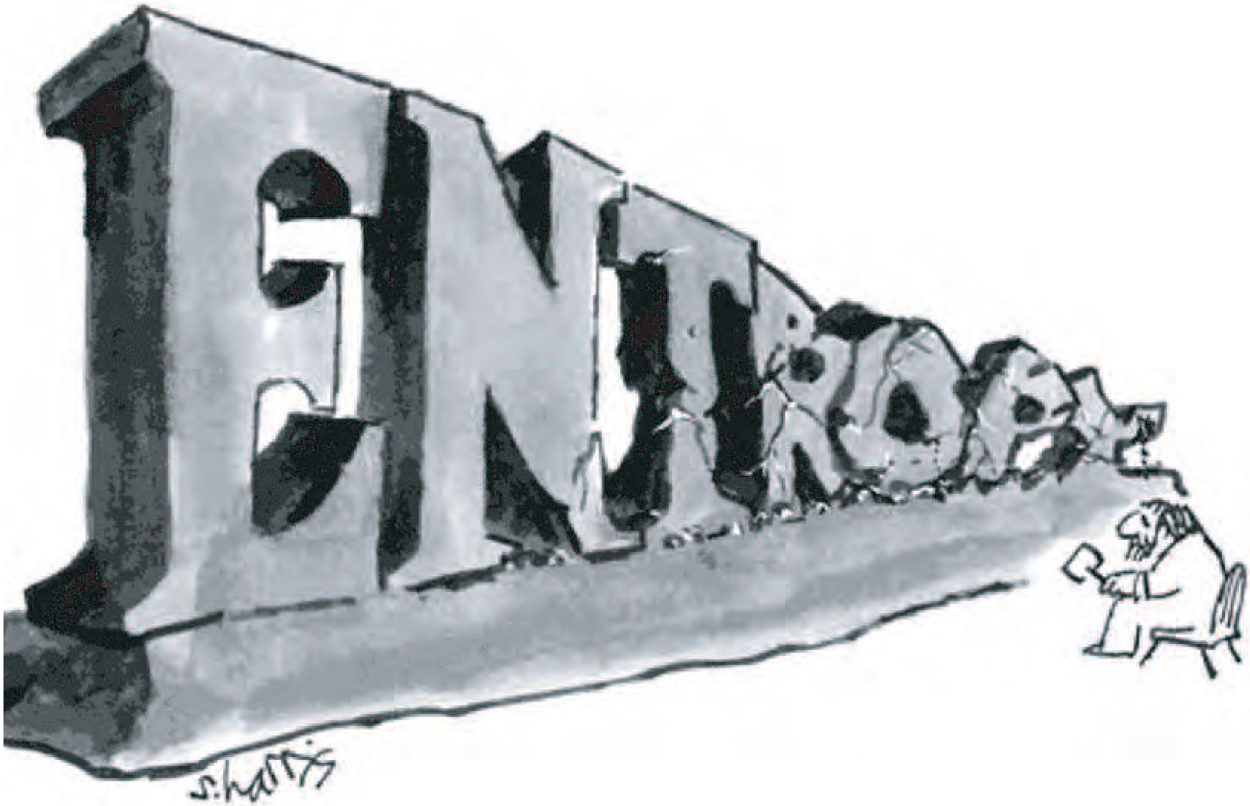
Acoustics, constrained spline fitting, image reconstruction, inverse scattering, optics, option pricing, multidimensional NMR, tomography, statistical moment fitting, and time series analysis, etc.

(Many thousands of papers)

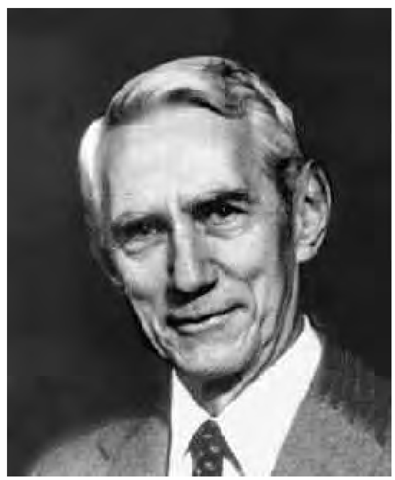
• However, the derivations and mathematics are fraught with subtle errors.

I will discuss some of the difficulties inherent in infinite dimensional calculus, and provide a simple theoretical algorithm for correctly deriving maximum entropy-type solutions.

WHAT is



Boltzmann (1844-1906)



Shannon (1916-2001)

WHAT is ENTROPY?

Despite the narrative force that the concept of entropy appears to evoke in everyday writing, in scientific writing entropy remains a thermodynamic quantity and a mathematical formula that numerically quantifies disorder. When the American scientist Claude Shannon found that the mathematical formula of Boltzmann defined a useful quantity in information theory, he hesitated to name this newly discovered quantity entropy because of its philosophical baggage.

The mathematician John von Neumann encouraged Shannon to go ahead with the name entropy, however, since "no one knows what entropy is, so in a debate you will always have the advantage."

- **19C**: **Boltzmann**—thermodynamic *disorder*
- **20C**: **Shannon**—information *uncertainty*
- **21C**: **JMB**—potentials with *superlinear growth*

CHARACTERIZATIONS of ENTROPY

- Information theoretic characterizations abound. A nice one is:

Theorem $H(\overrightarrow{p}) = -\sum_{k=1}^{N} p_k \log p_k$ is *the unique continuous function* (up to a positive scalar multiple) on finite probabilities such that

I. Uncertainty grows:

$$H\left(\overbrace{\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}}^{n}\right)$$

increases with n.

II. Subordinate choices are respected: for distributions $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ and $0 < p < 1$,

$$H\left(p\,\overrightarrow{p_1}, (1-p)\,\overrightarrow{p_2}\right) = p\,H(\overrightarrow{p_1}) + (1-p)\,H(\overrightarrow{p_2}).$$

ENTROPIES FOR US

• Let X be our *function space*, typically Hilbert space $L^2(\Omega)$, or the function space $L^1(\Omega)$ (or a Sobolev space).

◇ For $p \geq 1$,

$$L^p(\Omega) = \left\{ x \text{ measurable } : \int_\Omega |x(t)|^p dt < \infty \right\}.$$

It is well known that $L^2(\Omega)$ is a Hilbert space with *inner product*

$$\langle x, y \rangle = \int_\Omega x(t) y(t) dt,$$

(with variations in Sobolev space).

• A *bounded linear map* $A : X \to \mathbb{R}^n$ is determined by

$$(Ax)_i = \int x(t) a_i(t)\, dt$$

for $i = 1, \ldots, n$ and $a_i \in X^*$ the 'dual' of X (L^2 in the Hilbert case, L^∞ in the L^1 case).

• To pick a solution from the infinitude of possibilities, we may freely define "**best**".

$\otimes$ The most common approach is to find the minimum norm solution*, by solving the *Gram system*

$$\boxed{AA^T\lambda = b}\ .$$

$\oplus$ The solution is then $\hat{x} = A^T\lambda$. This recaptures all of *Fourier analysis*!

• This actually solved the following *variational problem*:

$$\inf\left\{\int_\Omega x(t)^2 dt : Ax = b \ \ x \in X\right\}.$$

*Even in the (realistic) infeasible case.

• We generalize the norm with a *strictly convex functional* f as in

$$\min\{f(x) : Ax = b, \ \ x \in X\}, \qquad (P)$$

where f is what we call, an *entropy functional*, $f : X \to (-\infty, +\infty]$. Here we suppose f is a strictly convex integral functional* of the form

$$f(x) = \int_\Omega \phi(x(t))dt.$$

The functional f can be used to include other constraints†.

For example, the constrained L^2 norm functional ('positive energy'),

$$f(x) = \begin{cases} \int_0^1 x(t)^2\, dt & \text{if } x \geq 0 \\ +\infty & \text{else} \end{cases}$$

is used in constrained *spline fitting*.

Protter and Arun use this model

• Entropy constructions abound: *Bregman* and *Csizar distances* model statistical divergences.

*Essentially $\phi''(t) > 0$.

†Including nonnegativity, by appropriate use of $+\infty$.

• Two popular choices for f are the *Boltzmann-Shannon* entropy (in image processing)

$$f(x) = \int x \log x,$$

and the *Burg entropy* (in time series analysis),

$$f(x) = -\int \log x.$$

◇ *Both implicitly impose a nonnegativity constraint* (positivity in Burg's non-superlinear case).

• There has been much information-theoretic debate about which entropy is best.

This is more theology than science!

• More recently, the use of *Fisher Information*

$$f(x, x') = \int_\Omega \frac{x'(t)^2}{2x(t)} \mu(dt)$$

has become more usual as it *penalizes* large derivatives; and can be argued for physically ('hot' over past five years).

WHAT 'WORKS' BUT CAN GO WRONG?

• Consider solving $Ax = b$, where, $b \in \mathbb{R}^n$ and $x \in L^2[0,1]$. Assume further that A is a continuous linear map, hence represented as above.

• As L^2 is infinite dimensional, so is $N(A)$: if $Ax = b$ is solvable, it is under-determined.

We pick our solution to *minimize*

$$f(x) = \int \phi(x(t))\,\mu(dt)$$

$\odot$ $\phi(x(t), x'(t))$ in Fisher-like cases [BN1, BN2, B-Vanderwerff (*Convex Functions*, CUP 2009)].

• We introduce the *Lagrangian*

$$L(x,\lambda) := \int_0^1 \phi(x(t))dt + \sum_{i=1}^n \lambda_i \left(b_i - \langle x, a_i\rangle\right),$$

and the associated *dual problem*

$$\max_{\lambda \in \mathbb{R}^n} \min_{x \in X} \{L(x,\lambda)\}. \qquad (D)$$

• So we formally have a "dual pair" (BL1)

$$\min\left\{f(x) : Ax = b,\ \ x \in X\right\}, \qquad (P)$$

and

$$\max_{\lambda \in I\!R^n} \min_{x \in X} \{L(x, \lambda)\}. \qquad (D)$$

• Moreover, for the solutions $\widehat{x}$ to (P), $\hat{\lambda}$ to (D), the derivative (w.r.t. x) of $L(x, \widehat{\lambda})$ should be zero, since $L(\widehat{x}, \widehat{\lambda}) \leq L(x, \widehat{\lambda}), \forall\, x$.

This implies

$$\begin{aligned} \widehat{x}(t) &= (\phi')^{-1}\left(\sum_{i=1}^{n} \widehat{\lambda}_i a_i(t)\right) \\ &= (\phi')^{-1}\left(A^T \widehat{\lambda}\right). \end{aligned}$$

- We can now reconstruct the primal solution (qualitatively and quantitatively) from a presumptively easier dual computation.

A DANTZIG ANECDOTE

"George wrote in "Reminiscences about the origins of linear programming," 1 and 2, *Oper. Res. Letters*, April 1982 (p. 47):

> *"The term Dual is not new. But surprisingly the term Primal, introduced around 1954, is. It came about this way. W. Orchard-Hays, who is responsible for the first commercial grade L.P. software, said to me at RAND one day around 1954: 'We need a word that stands for the original problem of which this is the dual.'*
> *I, in turn, asked my father, Tobias Dantzig, mathematician and author, well known for his books popularizing the history of mathematics. He knew his Greek and Latin. Whenever I tried to bring up the subject of linear programming, Toby (as he was affectionately known) became bored and yawned.*

But on this occasion he did give the matter some thought and several days later suggested Primal as the natural antonym since both primal and dual derive from the Latin. It was Toby's one and only contribution to linear programming: his sole contribution unless, of course, you want to count the training he gave me in classical mathematics or his part in my conception."

A lovely story. I heard George recount this a few times and, when he came to the "conception" part, he always had a twinkle in his eyes. (Saul Gass, Oct 2006)

- In a Sept 2006 *SIAM book review*, I asserted George assisted his father—for reasons I believe but cannot reconstruct.

  I also called Lord Chesterfield, Chesterton (*gulp*!).

PITFALLS ABOUND

There are 2 major problems to this approach.*

1. *The assumption that a solution $\hat{x}$ exists*. For example, consider the problem

$$\inf_{x \in L^1[0,1]} \left\{ \int_0^1 x(t)dt : \int_0^1 tx(t)\, dt = 1, x \geq 0 \right\}.$$

$\diamond$ The optimal value is not attained. Similarly, existence can fail for the Burg entropy with trig moments. Additional conditions on ϕ are needed to insure solutions exist.† (BL2)

2. *The assumption that the Lagrangian is differentiable*. In the above, f is $+\infty$ for every x negative on a set of positive measure.

$\diamond$ This implies the Lagrangian is $+\infty$ on a dense subset of L^1, the set of functions *not* nonnegative a.e.. The Lagrangian is *nowhere continuous*, much less differentiable.

*A third, the existence of $\hat{\lambda}$, is less difficult to surmount.

†The solution is actually the *absolutely continuous part of a measure* in $C(\Omega)^*$.

FIXING THE PROBLEM

• One approach to circumvent the differentiability problem, is to pose the problem in $L^\infty(\Omega)$, or in $C(\Omega)$, the space of essentially bounded, or continuous, functions. However, in these spaces, even with additional side qualifications, we are not necessarily assured solutions to (P) exist.

◇ In (BL2), an example is given of a one parameter problem on the torus in $\mathbb{R}^3$, using the first four Fourier coefficients, and Burg's entropy, where solutions fail to exist for certain feasible data values.

• Alternatively, Minerbo poses the problem of tomographic reconstruction in $C(\Omega)$ with the Boltzmann-Shannon entropy. Unfortunately, the functions a_i are characteristic functions of strips across Ω, and the solution is piecewise constant, not continuous.

CONVEX ANALYSIS (AN ADVERT)

We prepare to state a theorem that guarantees that the form of solution found in the above faulty derivation $\boxed{\widehat{x} = (\phi')^{-1}(A^T\widehat{\lambda})}$ is, in fact, correct. A full derivation is given in (BL2) and (BZ05).

• We introduce the *Fenchel (Legendre) conjugate* (see BL1) of a function $\phi : I\!R \to (-\infty, +\infty]$:

$$\phi^*(u) = \sup_{v \in I\!R} \{uv - \phi(v)\}.$$

• Often this can be (pre-)computed explicitly, using Newtonian calculus. Thus,

$$\phi(v) = v \log v - v, -\log v \text{ and } v^2/2$$

yield

$$\phi^*(u) = \exp(u), -1 - \log(-u) \text{ and } u^2/2$$

respectively. The red is the *log barrier* of interior point fame!

• The Fisher case is similarly explicit.

EXAMPLE 2. CONJUGATES & NMR

The *Hoch and Stern information measure*, or *neg-entropy*, is defined in complex $n-$space by

$$H(z) = \sum_{j=1}^{n} h(z_j/b),$$

where h is convex and given (for scaling b) by:

$$\boxed{h(z) \triangleq |z| \log\left(|z| + \sqrt{1+|z|^2}\right) - \sqrt{1+|z|^2}}$$

for *quantum theoretic* (NMR) reasons.

- Recall the *Fenchel-Legendre conjugate*

$$f^*(y) := \sup_x \langle y, x \rangle - f(x).$$

- Our *symbolic convex analysis* package (stored at `www.cecm.sfu.ca/projects/CCA/`, also in Chris Hamilton's package at Dal) produced:

$$h^*(z) = \cosh(|z|)$$

⋄ Compare the *Shannon entropy*:

$$(z \log z - z)^* = \exp(z).$$

COERCIVITY AND DUALITY

- We say ϕ possess *regular growth* if either $d = \infty$, or $d < \infty$ and $k > 0$, where $d = \lim_{u\to\infty} \phi(u)/u$ and $k = \lim_{v\uparrow d}(d-v)(\phi^*)'(v)$.*

- The *domain* of a convex function is $\operatorname{dom}(\phi) = \{u : \phi(u) < +\infty\}$; ϕ is *proper* if $\operatorname{dom}(\phi) \neq \emptyset$. Let $\imath = \inf \operatorname{dom}(\phi)$ and $\sigma = \sup \operatorname{dom}(\phi)$.

- Our *constraint qualification*,† (CQ), reads

$$\exists\, \overline{x} \in L^1(\Omega), \text{ such that } A\overline{x} = b,$$
$$f(\overline{x}) \in \mathbb{R}, \quad \imath < \overline{x} < \sigma \quad a.e.$$

◇ *In many cases, (CQ) reduces to feasibility*, (e.g., spectral estimation) and trivially holds.

- In this language, the *dual problem* for (P) is

$$\sup\left\{\langle b, \lambda\rangle - \int_\Omega \phi^*(A^T\lambda(t))dt\right\}. \qquad (D)$$

*-log does nor possess regular growth; $v \to v \log v$ does.

†The standard Slater's condition fails; this is what guarantees dual solutions exist.

Theorem 1 (BL2) *Let Ω be a finite interval, μ Lebesgue measure, each a_k continuously differentiable (or just locally Lipschitz) and ϕ proper, strictly convex with regular growth.*

Suppose (CQ) holds and also
(1)

$$\exists\, \tau \in \mathbb{R}^n \text{ such that } \sum_{i=1}^{n} \tau_i a_i(t) < d \quad \forall t \in [a,b],$$

then the unique solution to (P) is given by

(2)
$$\widehat{x}(t) = (\phi^*)'(\sum_{i=1}^{n} \widehat{\lambda}_i a_i(t))$$

where $\widehat{\lambda}$ is any solution to dual problem (D) *(and such $\widehat{\lambda}$ must exist).*

• This theorem generalizes to cover $\Omega \subset \mathbb{R}^n$, and more elaborately in Fisher-like cases. These results can be found in (BL2, BN1).

◇ 'Bogus' differentiation of a discontinuous function becomes the delicate $\boxed{(\int_\Omega \phi)^*(x^*) = \int_\Omega \phi^*(x^*)}$.

• Thus, the form of the maximum entropy solution can be legitimated simply by validating the *easily* checked conditions of Theorem 1.

♠ Also, any solution to $Ax = b$ of the form in (2) is automatically a solution to (P).

So, solving (P) is equivalent to finding $\lambda \in I\!R^n$ with

$$(3) \qquad \langle (\phi^*)'(A^T\lambda), a_i \rangle = b_i, \quad i = 1, \ldots, n,$$

a *finite dimensional* set of non-linear equations.

> One can then apply a standard 'industrial strength' nonlinear equation solver, like Newton's method, to this system, to find the optimal λ.

• Often, $\boxed{(\phi')^{-1} = (\phi^*)'}$ and so the 'dubious' solution agrees with the 'honest' solution.

Importantly, we may tailor $(\phi')^{-1}$ to our needs.

• **Note** that discretization is only needed to compute terms in (3). Indeed, *these integrals can sometimes be computed exactly* (e.g., in some tomography and option estimation problems). This is the gain of *not discretizing* early.

> By waiting to see the form of dual problem, one can customize one's integration scheme to the problem at hand.

• For European option pricing the constraints are based on 'hockey-sticks' of the form

$$a_i(x) := \max\{0, x - t_i\}$$

so the dual can be computed *exactly* and leads to a relatively small and explicit nonlinear equation to solve (BCM).

◇ Even when this is not the case one can often use the shape of the dual solution to fashion very*efficient heuristic reconstructions* that avoid any iterative steps (see BN2).

$\mathcal{M}$om$\mathcal{E}$nt+

- $\mathcal{M}$om$\mathcal{E}$nt+ (`www.cecm.sfu.ca/interfaces/`) has code for entropic reconstructions as above. Moments (including wavelets), entropies and dimension are easily varied. It also allows for adding noise and relaxation of the constraints.

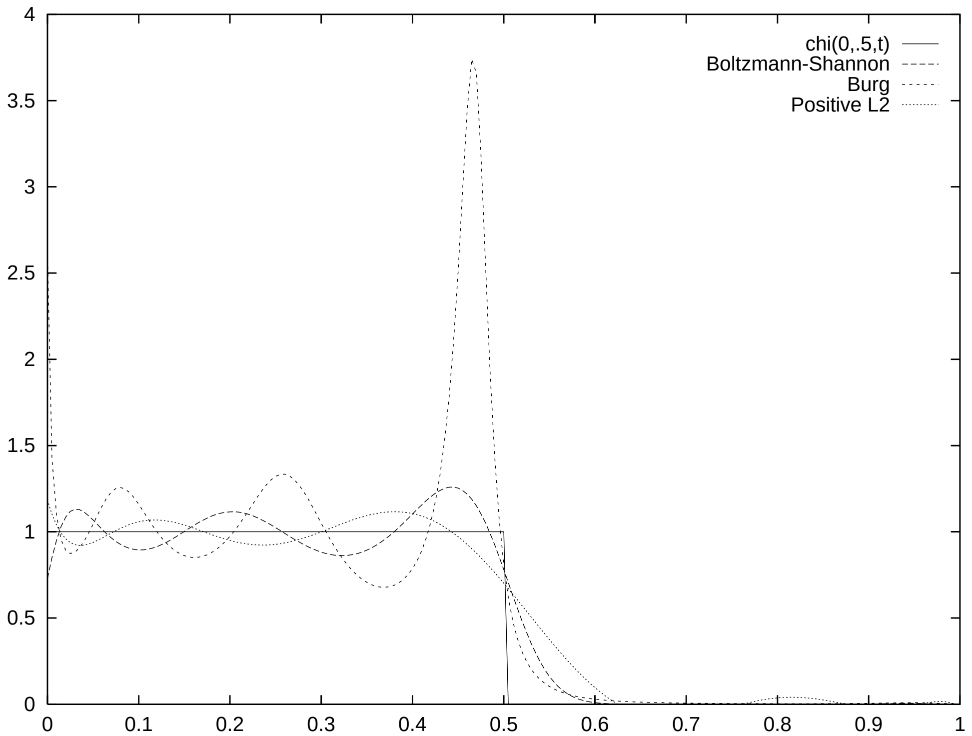
> Several methods of solving the dual are possible, including *Newton and quasi-Newton methods (BFGS, DFP), conjugate gradients*, and the suddenly sexy *Barzilai-Borwein line-search free method*.

- For iterative methods below, I recommend:

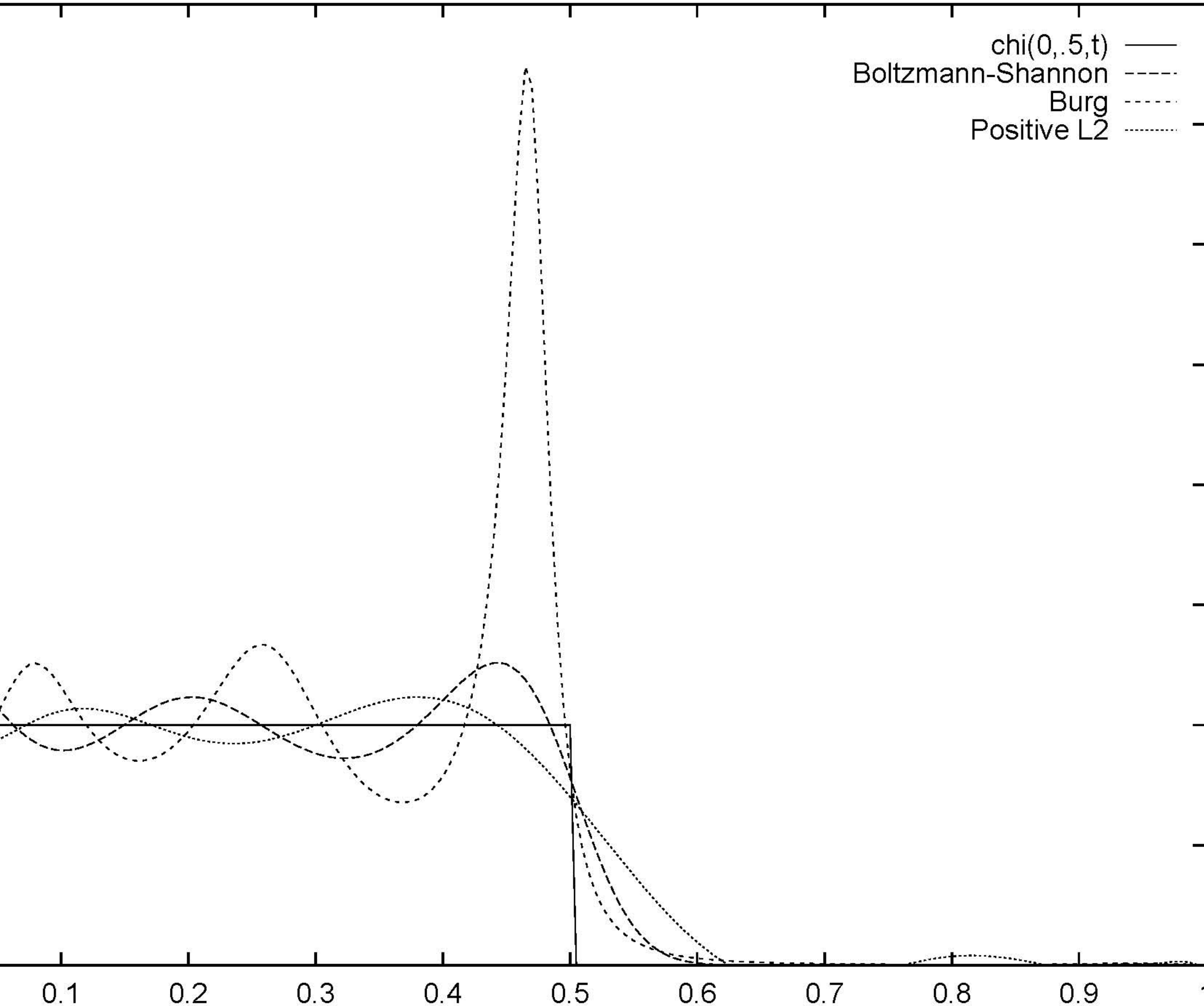
H.H. Bauschke and J.M. Borwein, "On projection algorithms for solving convex feasibility problems," *SIAM Review*, **38** (1996), 367–426 (cited over 100 time by MathSciNet, 215 times in ISI, 350 in Google!), and a forthcoming CMS-Springer book written by *Bauschke and Combettes*.

COMPARISON OF ENTROPIES

• The positive L^2, Boltzmann-Shannon and Burg entropy reconstruction of the **characteristic function** of $[0,1/2]$ using *10 algebraic moments* $(b_i = \int_0^{1/2} t^{i-1}dt)$ on $\Omega = [0,1]$.



• **Solution**: $\widehat{x}(t) = (\phi^*)'(\sum_{i=1}^{n} \widehat{\lambda}_i t^{i-1})$.
Burg over-oscillates since $(\phi^*)'(t) = 1/t$.


chi(0,.5,t)
Boltzmann-Shannon
Burg
Positive L2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

THE NON-CONVEX CASE

• In general non-convex optimization is a much less satisfactory field. We *can usually hope only to find critical points* ($f'(x) = 0$) or local minima. Thus, problem-specific heuristics dominate.

• **Crystallography**: We of course wish to estimate x in $L^2(\mathbb{R}^n)$* Then the modulus $c = |\widehat{x}|$ is known ($\widehat{x}$ is the Fourier transform of x).†

Now $\{y : |\widehat{y}| = c\}$, is not convex. So the issue is to find x given c and other convex information. An appropriate optimization problem extending the previous one is

$$\min \{f(x) : Ax = b, \|Mx\| = c, \;\; x \in X\}, \quad (NP)$$

where M models the modular constraint, and f is as in Theorem 1.

*Here $n = 2$ for images, 3 for holographic imaging, etc.

†Observation of the modulus of the diffracted image in crystallography. Similarly, for optical aberration correction.

EXAMPLE 3: CRYSTALLOGRAPHY

• My Parisian collaborator Combettes is expert on optimization perspectives of cognates to (NP) and related feasibility problems.

◇ Most methods rely on a two-stage (easy convex, hard non-convex) decoupling schema—the following from Decarreau et al. (D). They suggest solving

$$\min\left\{f(x) : Ax = y, \|B_k y\| = b_k,\ \ x \in X\right\}, \tag{NP^*}$$

where $\|B_k y\| = b_k,\ k \in K$ encodes the hard modular constraints.

• They solve formal *first-order Kuhn-Tucker conditions* for a relaxed form of (NP^*). The easy constraints are treated by Thm 1.

I am obscure, largely because the results were largely negative:

• They applied these ideas to **a prostaglandin molecule (25 atoms)**, with known structure, using quasi-Newton (which could fail to find a local min), truncated Newton (better) and trust-region (best) numerical schemes.

◇ They observe that the "*reconstructions were often mediocre*" and highly dependent on the amount of prior information – a small proportion of unknown phases to be satisfactory.

> "**Conclusion**: It is fair to say that the entropy approach has limited efficiency, in the sense that it requires a good deal of information, especially concerning the phases. *Other methods are wanted when this information is not available.*"

• Thus, I offer this part of my presentation largely to illustrate the difficulties.

EXAMPLE 4. HUBBLE TELESCOPE

The basic setup—more details follow.

- **Electromagnetic field:** $u : \mathbb{R}^2 \to \mathbb{C} \in L^2$

- **DATA:** Field intensities for $m = 1, 2, \ldots, M$:

$$\psi_m : \mathbb{R}^2 \to \mathbb{R}_+ \in L^1 \cap L^2 \cap L^\infty$$

- **MODEL:** Functions $\mathcal{F}_m : L^2 \to L^2$, are *modified Fourier Transforms*, for which we can measure the modulus (intensity)

$$|\mathcal{F}_m(u)| = \psi_m \quad \forall m = 1, 2, \ldots, M.$$

$\bigoplus$ **INVERSE PROBLEM:** For the given transforms $\mathcal{F}_m$ and **measured** field intensities ψ_m (for $m = 1, \ldots, M$), find a **robust estimate** of the underlying u.

... AND SOME HOPE FROM HUBBLE

• The (human-ground) lens with a micro-asymmetry was mounted upside-down. The perfect back-up (computer-ground) lens stayed on earth!

◇ NASA challenged ten teams to devise algorithmic fixes.

• **Optical aberration correction**, using the *Misell algorithm*, a *method of alternating projections*, works much better than it should—given that it is being applied to find a member of a version of

$$\Psi := \bigcap_{k=1^M} \{x : Ax = b, \|M_k x\| = c_k, \ \ x \in X\}, \tag{NCFP}$$

which is a **non-convex feasibility problem** as on the next page.

Is there **hidden convexity**?

HUBBLE IS ALIVE AND KICKING

Hubble reveals most distant planets yet

Last Updated: Wednesday, October 4, 2006 | 7:21 PM ET
CBC News

Astronomers have discovered the farthest planets from Earth yet found, including one with a year as short as 10 hours — the fastest known.

Using the Hubble space telescope to peer deeply into the centre of the galaxy, the scientists found as many as 16 planetary candidates, they said at a news conference in Washington, D.C., on Wednesday.

The findings were published in the journal Nature.

Looking into a part of the Milky Way known as the galactic bulge, 26,000 light years from Earth, Kailash Sahu and his team of astronomers confirmed they had found two planets, with at least seven more candidates that they said should be planets.

The bodies are about 10 times farther away from Earth than any planet previously detected.

A light year is the distance light travels in one year, or about 9.46 trillion kilometres.

- From *Nature* Oct 2006. Hubble has since been reborn twice and exoplanets have become quotidian. There were 228 exoplanets listed at `www.exoplanets.org` in Sept 08 and March 09.

330 now?

5 Facts About Kepler (launch March 6)

-- Kepler is the world's first mission with the ability to find true Earth analogs -- planets that orbit stars like our sun in the "habitable zone." The habitable zone is the region around a star where the temperature is just right for water -- an essential ingredient for life as we know it -- to pool on a planet's surface.

-- By the end of Kepler's three-and-one-half-year mission, it will give us a good idea of how common or rare other Earths are in our Milky Way galaxy. This will be an important step in answering the age-old question: Are we alone?

-- Kepler detects planets by looking for periodic dips in the brightness of stars. Some planets pass in front of their stars as seen from our point of view on Earth; when they do, they cause their stars to dim slightly, an event Kepler can see.

-- Kepler has the largest camera ever launched into space, a 95-megapixel array of charge-coupled devices, or CCDs, as in everyday digital cameras.

-- Kepler's telescope is so powerful that, from its view up in space, it could see one person in a small town turning off a porch light at night.

NASA 05.03.2009

TWO MAIN APPROACHES

I. Non-convex (in)feasibility problem: Given $\psi_m \neq 0$, define $\mathbb{Q}_0 \subset L^2$ convex, and

$\mathbb{Q}_m := \left\{u \in L^2 \mid |\mathcal{F}_m(u)| = \psi_m \ a.e.\right\}$ (nonconvex)

we wish to find $u \in \bigcap_{m=0}^{M} \mathbb{Q}_m = \emptyset$.

⊙ via an *alternating projection method*: e.g., for two sets A and B, **repeatedly compute**

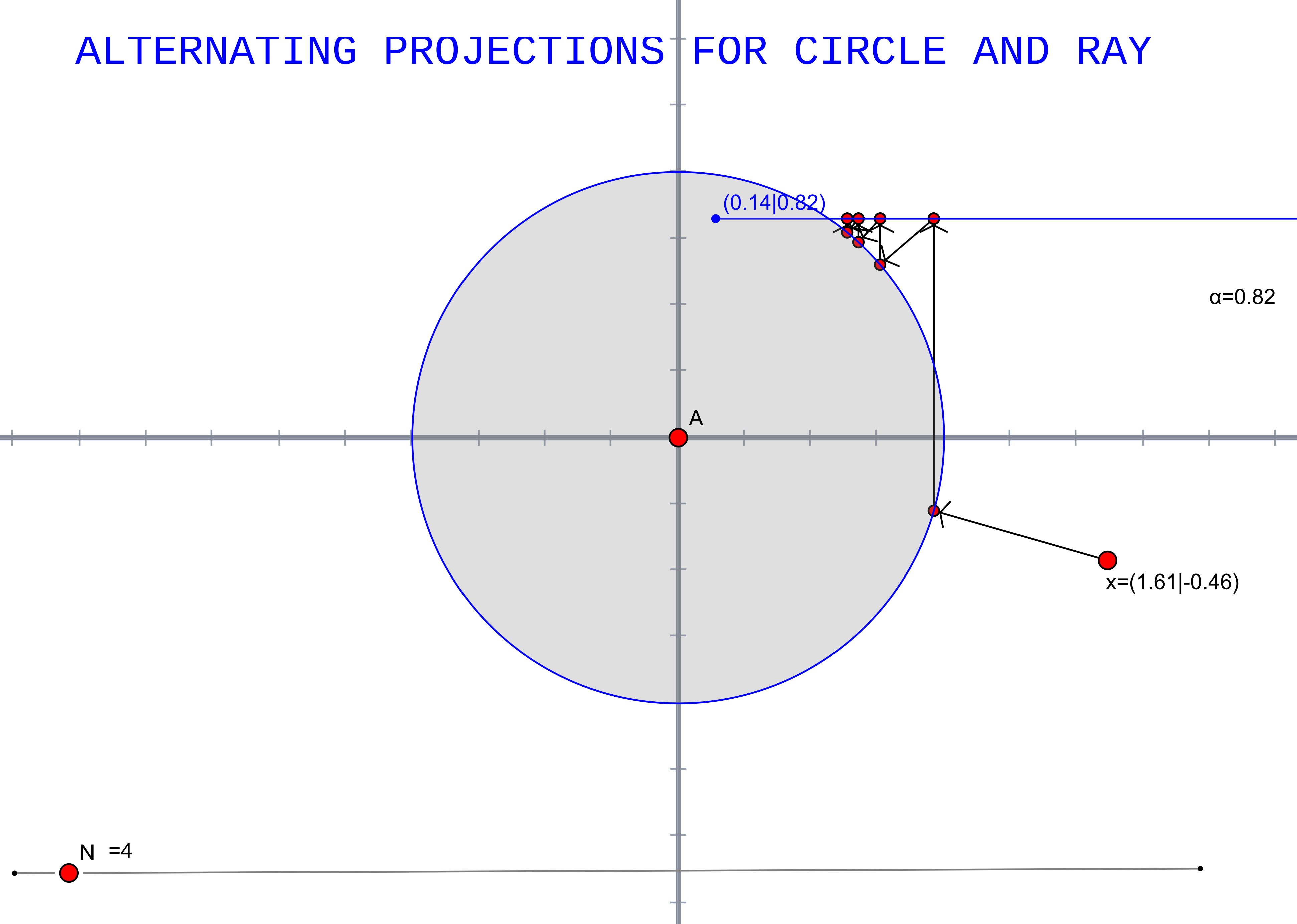
$$x \to P_B(x) =: y \to P_A(y) =: x.$$

II. Error reduction of a nonsmooth objective ('entropy') : for fixed $\beta_m > 0$

⊙ we attempt to solve

$$\text{minimize} \quad E(u) := \sum_{m=0}^{M} \frac{\beta_m}{2} \text{dist}^2(u, \mathbb{Q}_m)$$
$$\text{over} \quad u \in L^2.$$

ALTERNATING PROJECTIONS FOR CIRCLE AND RAY

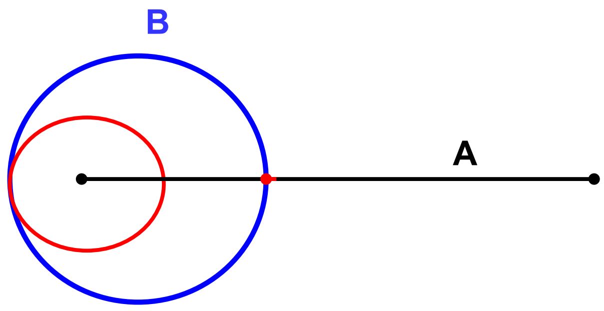


I: NON-CONVEX PROJECTION CAN FAIL

• If $A \cap B \neq \emptyset$ and A, B are closed convex then weak convergence (only 2002) is assured—von Neumann (1933) for subspaces, Bregman (1965).

⊙ Consider the **alternating projection method** to find the unique red point on the **line-segment A** (convex) and the blue **circle B** (non-convex).

• The **method is 'myopic'**.



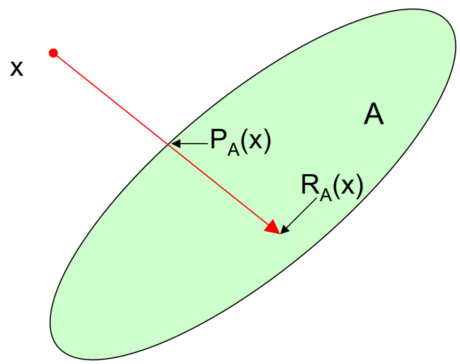
• Starting on line-segment outside the *red circle*, we converge to the unique feasible solution.

• Starting inside the red circle leads to a period-two locally 'least-distance' solution.

I: PROJECTION METHOD OF CHOICE

• For optical abberation correction this is the **alternating projection** method:

$$x \to P_A\left(P_B(x)\right)$$



• For crystallography it is better to use (HIO) **over-relax and average**: *reflect* to $R_A(x) := 2\,P_A(x) - x$ and use

$$x \to \frac{x + R_A\left(R_B(x)\right)}{2}$$

• Both parallelize neatly: A :=diag, $B := \prod_i C_i$.

• Both are nonexpansive *in the convex case*.

APPROACH I: NAMES CHANGE . . .

• **The optics community** calls projection algorithms "*Iterative Transform Algorithms*" .

> Hubble used *Misell's Algorithm*, which is just averaged projections. The best projection algorithm Luke* found was *cyclic projections* (with no relaxation).

• For the **crystallography problem** the best known method is called the *Hybrid Input-Output algorithm* in the optical setting. Bauschke-Combettes-Luke (JMAA, 2004) showed HIO, *Lions-Mercier* (1979), *Douglas-Rachford*, *Feinup*, and *divide-and-concur* coincide.

• When $u(t) \geq 0$ is imposed, Feinup's no longer coincides, and LM ('HPR') is still better.

*My former PDF, he was a *Hubble Graduate student.*

ELSER, QUEENS and SUDOKU

2006 Veit Elser at Cornell has had huge success (and press) using divide-and-concur on protein folding, sphere-packing, 3SAT, **Sudoku** ($\mathbb{R}^{2916}$), and more. Bauschke and Schaad likewise study **Eight queens problem** ($\mathbb{R}^{256}$) and image-retrieval (Science News, 08).

Veit Elser, Ph.D.

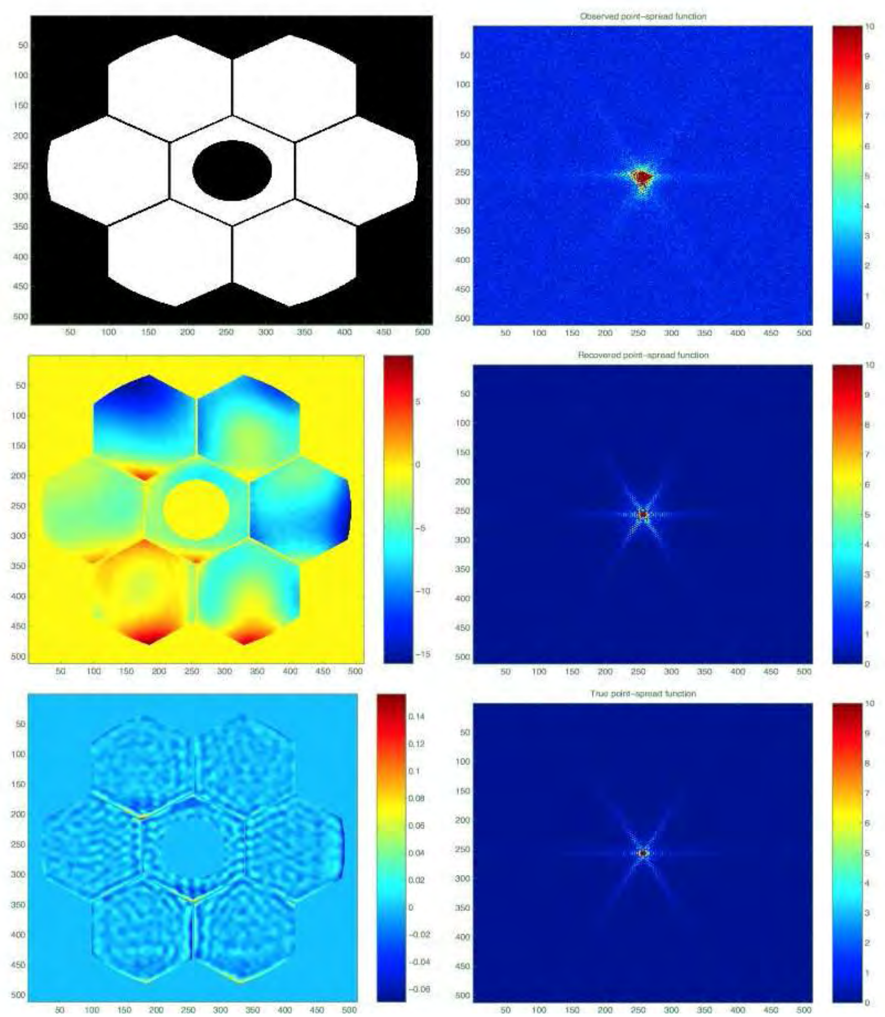
Given a partially completed grid, fill it so that each column, each row, and each of the nine 3×3 regions contains the digits from 1 to 9 only once.

	7	5		9				6
	2	3		8			4	
8					3			1
5			7		2			
	4		8		6		2	
			9		1			3
9			4					7
	6			7		5	8	
7				1		3	9	

• This success (a.e.?) is not seen with alternating projections and cries out for explanation.

A SAMPLE RECONSTRUCTION (via II)

- The object and its spectrum



Top row: data
Middle: reconstruction
Bottom: truth and error

EXAMPLE 5. INVERSE SCATTERING

• **Central problem:** determine the location and shape of buried objects from measurements of the *scattered field* after illuminating a region with a known *incident field.*

• **Recent techniques:** determine if a point z is inside or outside of the scatterer by determining *solvability* of the linear integral equation

$$\mathcal{F} g_z \stackrel{?}{=} \varphi_z$$

where $\mathcal{F} \rightarrow X$ is a compact linear operator constructed from the observed data, and $\varphi_z \in X$ is a known function parameterized by z.

• $\mathcal{F}$ has *dense range*, but if z is on the exterior of the scatterer, then $\varphi_z \notin$Range$(\mathcal{F})$.

• Since $\mathcal{F}$ is compact, any numerical implementation to solve the above integral equation will need some *regularization scheme*.

• If *Tikhonov regularization* is used—in a restricted physical setting—the solution to the regularized integral equation, $g_{z,\alpha}$, has the behaviour

$$||g_{z,\alpha}|| \to \infty \quad \text{as} \quad \alpha \to 0$$

if and only if z is a point outside the scatterer.

• **An important open problem** is to determine the behavior of regularized solutions $g_{z,\alpha}$ under different regularization strategies.

<u>In other words, when can these techniques fail?</u> (On going joint work with Russell Luke for a 2009 IMA Summer School: also in *Experimental Math in Action*, AKP, 2007).

FINIS: REFLECTIONS IN THE CIRCLE

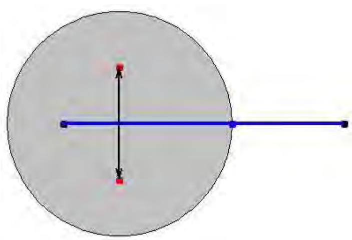
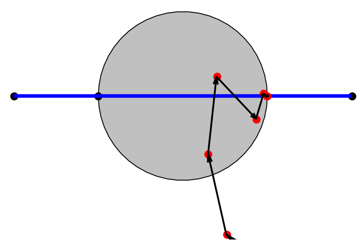
- **Dynamics** when B is the unit circle and A is the blue horizontal line at height $\alpha \geq 0$ are already fascinating. Steps are for

$$T := \frac{I + R_A \circ R_B}{2}:$$

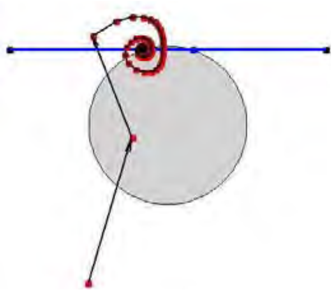
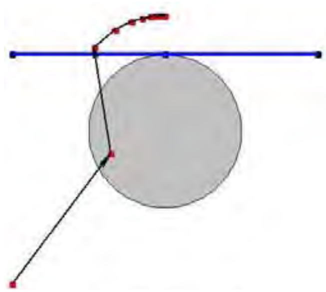
with θ_n the argument this becomes set

$$\boxed{x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n.}$$

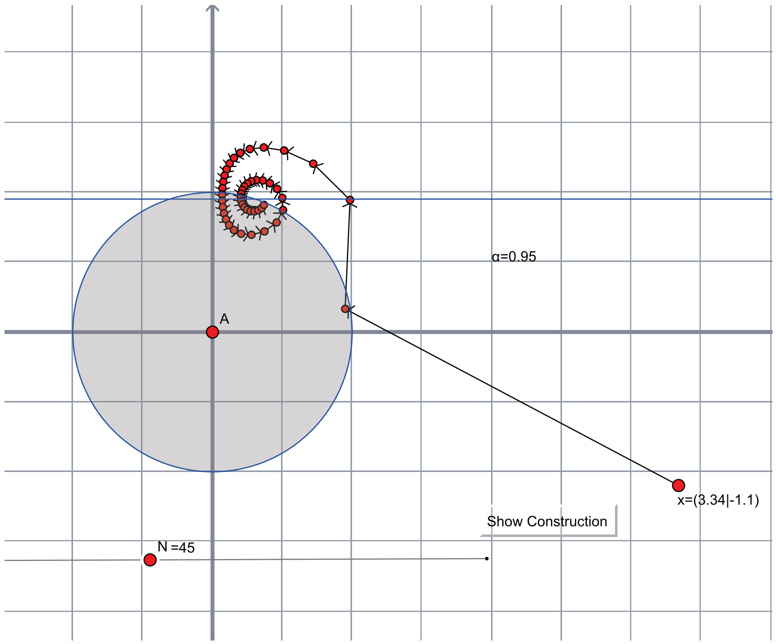
- $\alpha = 0$: converge **iff** start off y-axis ('chaos'):

- $\alpha > 1 \Rightarrow y \to \infty$, while $\alpha = 0.95$ $(0 < \alpha < 1)$ (**unproven**) and $\alpha = 1$ respectively produce:

DYNAMIC GEOMETRY



- I finish with a *Cinderella* demo developed with Chris Maitland.

- Next week a proper introduction to the package will be given by Ulli Kortenkamp ...

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How to Maximize Surprise

Australian Mathematical Sciences Institute

Australian and New Zealand Industrial and Applied Mathematics

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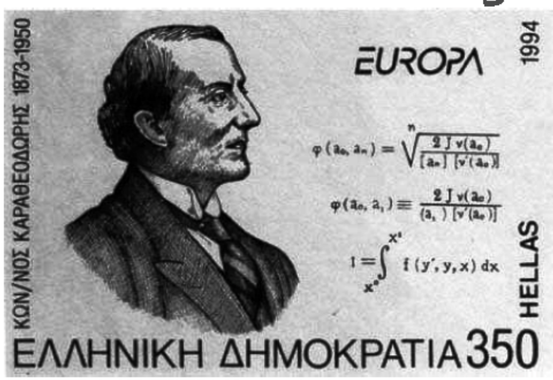


MAA 1936

I'll be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science.

The *Surprise Examination* or *Unexpected Hanging* Paradox has long fascinated mathematicians and philosophers, as the number of publications devoted to it attests.

> For an exhaustive bibliography on the subject, the reader is referred to [1].

Herein, the optimization problems arising from an information theoretic *avoidance* of the *Paradox* are examined and solved.

They provide a very satisfactory application of both the Kuhn-Tucker theory and of various classical inequalities and estimation techniques.

$\triangleright$ Although the necessary convex analytic concepts are recalled in the course of the presentation, some elementary knowledge of optimization is assumed.

REFERENCE. D. Borwein, J. M. Borwein and P. Marchal, "Surprise maximization," *The American Mathematical Monthly*, **107** June-July 2000, 527–537. [CECM Preprint 98:116].

`www.cecm.sfu.ca/preprints/1998pp.html`

Open Question. How does one quantify *average multiple surprise*?

INFORMATION MEASURE OF SURPRISE

Tim Chow's [3] version of the *Paradox*:

> *A teacher announces in class that an examination will be held on some day during the following week, and moreover that the examination will be a surprise. The students argue that a surprise exam cannot occur. For suppose the exam were on the last day of the week. Then on the previous night, the students would be able to predict that the exam would occur on the following day, and the exam would not be a surprise. So it is impossible for a surprise exam to occur on the last day.*

But then a surprise exam cannot occur on the penultimate day, either, for in that case the students, knowing that the last day is an impossible day for a surprise exam, would be able to predict on the night before the exam that the exam would occur on the following day. Similarly, the students argue that a surprise exam cannot occur on any other day of the week either. Confident in this conclusion, they are of course totally surprised when the exam occurs (on Wednesday, say). The announcement is vindicated after all. Where did the students' reasoning go wrong?

In this work, we study two optimization problems arising from an entropic approach to maximizing surprise. Such an approach was proposed in outline by Karl Narveson [3, p. 49].

We do not discuss here the various approaches to the logical resolution of the paradox itself; one may consult [1,3].

▷ Rather we ask the question:

> *What should be the probability distribution of an event occurring once every week so that it maximizes the surprise it creates?*

▷ This requires us to find a *measure of surprise*.

$\triangleright$ Let us start by posing an information theoretic counterpart of the paradox:

during a period of m days an event (such as a test given by a teacher or a surprise tax audit) occurs with probability p_i on day $i = 1, \ldots, m$.

We wish to find a probability distribution that maximizes the *average surprise* caused by the event when it occurs.

$\triangleright$ We consider a measure of surprise analogous to the one used in the celebrated definition of the *Shannon entropy* [2,4,6].

$\triangleright$ The surprise on day i is the negative of the logarithm of the *probability the event occurs on day i given that it has not occurred so far*.

$\triangleright$ As in the classical definition, $-\log p$ is used to measure the surprise associated with an event of probability p, which is also a measure of how much we learn if it occurs.

$\triangleright$ The logarithm makes the measure *additive*: the information associated with independent events should sum up when they both occur.

$\triangleright$ The use of conditional probabilities introduces some *causality*: it accounts for what is already known of the previous days.

The event *'test occurs on day i'* is simply denoted by i, and its probability is denoted by $P(i)$ or p_i. The event *'test does not occur on day i'* will be denoted by $\sim i$.

▷ Thus, we need to maximize:

$$-\sum_{i=1}^{m} P(i) \log P\Big(i \,|\, \sim 1, \ldots, \sim (i-1)\Big). \tag{1}$$

Using *Bayes' formula* for conditional probabilities, we obtain an explicit formula:

$$\begin{aligned}
&P\Big(i \,|\, \sim 1, \ldots, \sim (i-1)\Big) \\
&\quad = \frac{P\Big(\sim 1, \ldots, \sim (i-1) \,|\, i\Big)\, P(i)}{P\Big(\sim 1, \ldots, \sim (i-1)\Big)} \\
&\quad = \frac{P(i)}{1 - \Big(P(1) + \cdots + P(i-1)\Big)} \\
&\quad = \frac{P(i)}{P(i) + \cdots + P(m)}.
\end{aligned}$$

▷ We are led to the next optimization problem:

$$(\mathcal{P}_m) \quad \inf \{ S_m(\mathbf{p}) \mid \mathbf{p} \in \mathbb{R}^m,\ 1 = \langle \mathbf{u}, \mathbf{p} \rangle \} \tag{2}$$

Here, $\mathbf{u}$ is the $m-$vector of 1's and:

▷ S_m is the (m-dimensional) *surprise function*

$$S_m(\mathbf{p}) := \sum_{j=1}^m p_j \log \frac{p_j}{\frac{1}{m} \sum_{i \geq j} p_i} - \sum_{j=1}^m p_j .$$

More precisely,

$$S_m(\mathbf{p}) := \sum_{j=1}^m h\left(p_j, \frac{1}{m} \sum_{i=j}^m p_i\right), \quad \mathbf{p} \in \mathbb{R}^m,$$

where h is defined on $\mathbb{R}^2$ by

$$h(x,y) := \begin{cases} x \log \frac{x}{y} - x & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{if } x = 0 \text{ and } y \geq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

$\triangleright$ For all $\mathbf{p}$ satisfying the constraint in (2), $S_m(\mathbf{p})$ differs from the negative of the quantity in (1) only by a constant.

The factor m^{-1} makes subsequent computations more aesthetic and the limit analysis more harmonious.

$\triangleright$ Note that $S_m(\mathbf{p})$ can be viewed as the *Kullback-Leibler information measure* of $\mathbf{p}$ relative to its (normalized) *tail* $\mathbf{q}$:

$$\begin{aligned} \mathbf{q} &:= (q_1, \ldots, q_m) \qquad \text{with} \\ q_j &:= \tfrac{1}{m} \textstyle\sum_{i=j}^{m} p_i, \qquad j = 1, \ldots, m. \end{aligned} \tag{4}$$

The *Kullback-Leibler information measure* [2, 5] is an extension of Boltzmann-Shannon entropy. It is also called the *relative information measure*, *cross-entropy* or *I-divergence*.

Given two probability measures P and Q, the relative information of P with respect to Q is

$$\mathcal{K}(P||Q) := \int \left(\frac{dP}{dQ} \log \frac{dP}{dQ} - \frac{dP}{dQ}\right) dQ$$

$$= \int \left(\log \frac{dP}{dQ} - 1\right) dP$$

if P is *absolutely continuous* with respect to Q, and $\mathcal{K}(P||Q) := +\infty$ otherwise, [5].

$\triangleright$ For an extended discussion on the *Maximum Entropy Principle*, one may consult [4] and references therein.

We suppose that the event occurs at some point t in the time interval $[0, T]$, with probability density $p(t)$.

$\triangleright$ By analogy with the discrete case, we consider the following optimization problem:

$$(\mathcal{P}) \quad \inf \left\{ \mathcal{S}(p) \,\middle|\, p \in L_1\big([0, T]\big),\ 1 = \langle u \,,\, p \rangle \right\} \tag{5}$$

in which the *surprise function* $\mathcal{S}$ is the functional defined on $L_1\big([0, T]\big)$ by

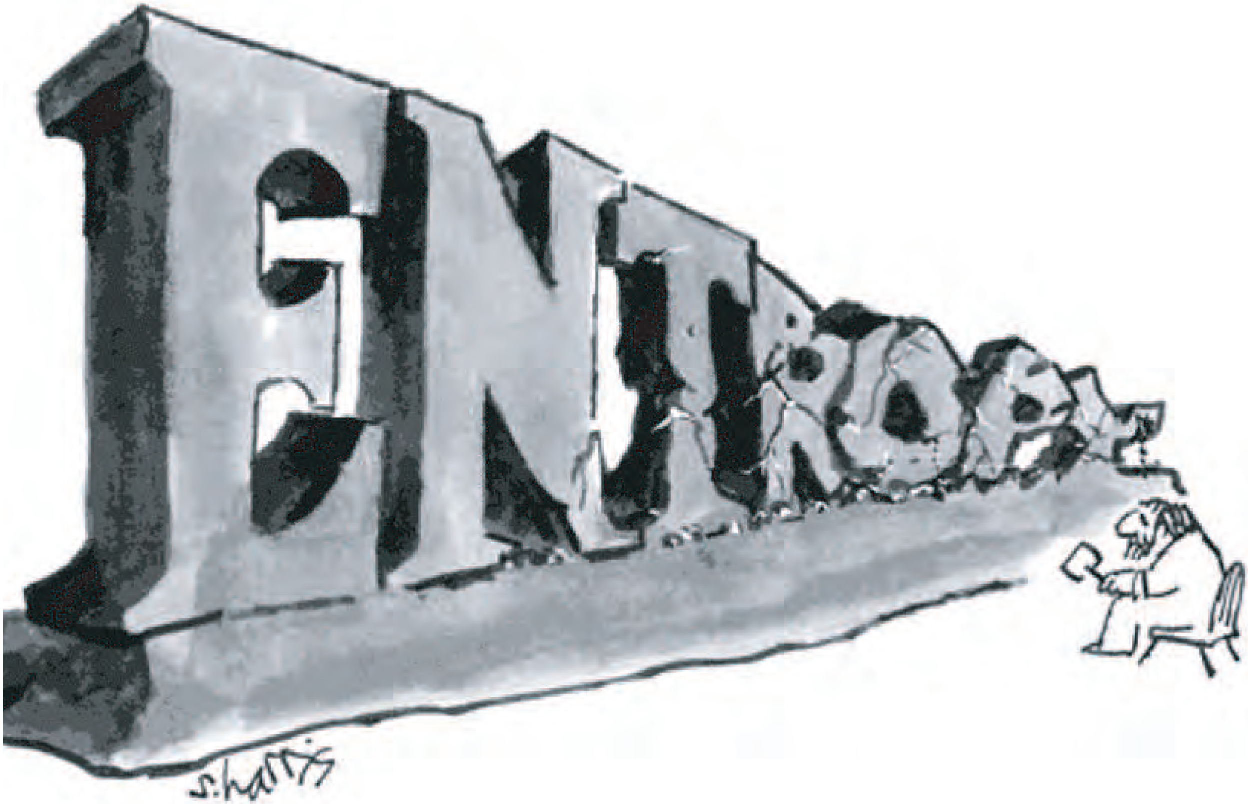
$$\mathcal{S}(p) := \int_0^T h\left(p(t), \frac{1}{T}\int_t^T p(s)\,ds\right)\,dt,$$

and $u \equiv 1\ [0, T]$.

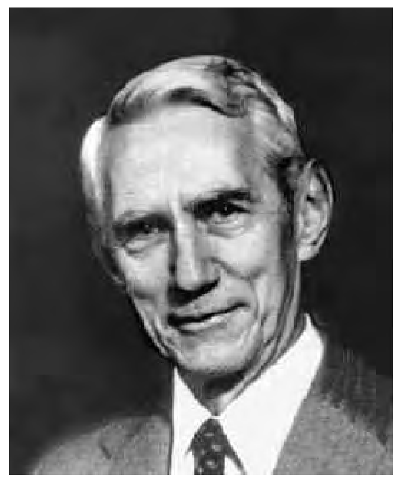
As above h is defined by

$$h(x, y) := \begin{cases} x \log \dfrac{x}{y} - x & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{if } x = 0 \text{ and } y \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

WHAT is



Boltzmann (1844-1906)



Shannon (1916-2001)

WHAT is ENTROPY?

Despite the narrative force that the concept of entropy appears to evoke in everyday writing, in scientific writing entropy remains a thermodynamic quantity and a mathematical formula that numerically quantifies disorder. When the American scientist Claude Shannon found that the mathematical formula of Boltzmann defined a useful quantity in information theory, he hesitated to name this newly discovered quantity entropy because of its philosophical baggage. The mathematician John Von Neumann encouraged Shannon to go ahead with the name entropy, however, since "no one knows what entropy is, so in a debate you will always have the advantage."

- **19C**: Boltzmann—thermodynamic disorder
- **20C**: Shannon—information uncertainty
- **21C**: JMB—potentials with superlinear growth

SURPRISINGLY, SURPRISE IS CONCAVE

$\triangleright$ We now establish the *convexity* of (the negative of) our measure of surprise. An extended real-valued function on $\mathbb{R}^n$ is *closed (convex)* if its *epigraph* (the set of points which are above or on its graph) is closed (convex) in $\mathbb{R}^{n+1}$.

$\triangleright$ The *domain* of a convex function f is the set of points where it is less than $+\infty$, denoted by dom f.

$\triangleright$ If a convex function is not identically $+\infty$ and is nowhere $-\infty$ (such functions are *proper*), then being closed is the same as being *lower semi-continuous*.

▷ Given any function f on $\mathbb{R}^n$ (convex or not), the *convex conjugate* of f is the function

$$\boxed{f^\star(\boldsymbol{\xi}) := \sup\{\langle \mathbf{x}, \boldsymbol{\xi}\rangle - f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n\}}$$

for $\boldsymbol{\xi} \in \mathbb{R}^n$.

It is easily shown that $f^\star$ is always closed and convex [2, 7]. Furthermore, if f is closed, proper, and convex, then so is $f^\star$ and the *bi-conjugate* $f^{\star\star} := (f^\star)^\star$ is f itself [2, 7].

Even without this theoretical underpinning, computation of f as a *double-conjugate* provides an accessible way of establishing both convexity and semi-continuity.

Lemma 1 *The function h defined in (3) is closed and convex.*

Proof. One may directly show that h is the convex conjugate of the *indicator function*

$$\delta\left((\xi,\eta) \mid C\right) := \begin{cases} 0 & \text{if } (\xi,\eta) \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

where C is the convex set

$$\left\{(\xi,\eta) \in \mathbb{R}^2 | \eta \leq -\exp \xi\right\}.$$

This proves that h is closed and convex. ■

Convexity of h can also be derived from the easy fact that, for any interval I, a function

$$(x,y) \mapsto y\, f(x\, y^{-1})$$

is convex on $I \times (0,\infty)$ if and only if f is convex on I. [A 'bad' way is to check the *Hessian* matrix is positive semi-definite.]

$\triangleright$ Figure 1 displays h.

Using Lemma 1, we deduce that S_m and $\mathcal{S}$ are convex. Indeed, we have

$$\begin{aligned} S_m(\mathbf{p}) &= \textstyle\sum_{i=1}^m h(p_i, [J\mathbf{p}]_i) \qquad \text{and} \\ \mathcal{S}(p) &= \textstyle\int_0^T h\Big(p(t), [\mathcal{J}p](t)\Big)\, dt, \end{aligned}$$

in which J is the $(m \times m)$-matrix whose entries are m^{-1} on and above the diagonal and 0 elsewhere, and $\mathcal{J} \colon L_1([0,T]) \to \mathcal{C}([0,T])$ is the linear mapping defined by

$$[\mathcal{J}p](t) := \frac{1}{T} \int_t^T p(s)\, ds. \tag{6}$$

In passing, we recall that the composition of a convex function with an arbitrary linear mapping is convex.

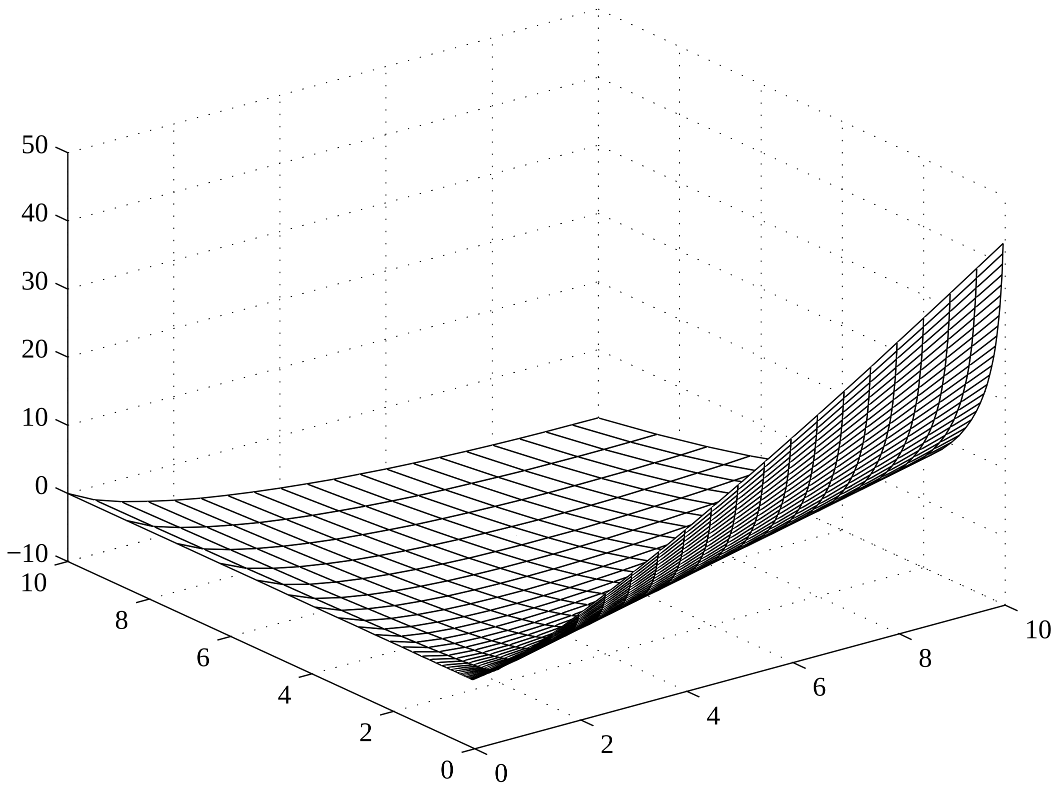


Figure 1. Graph of $(x, y) \mapsto x \log \dfrac{x}{y} - x$.

DISCRETE TIME ANALYSIS

Constrained optimization problems such as (2) are traditionally approached using concepts from *duality theory*, which flows from the theory of *Lagrange multipliers*.

> Roughly speaking, duality theory reduces constrained optimization problems to simpler or unconstrained ones.

$\triangleright$ A modern version of duality theory is posed in the language of Fenchel conjugation [2, 7].

We recall some additional basic facts. Let f be a closed proper convex function on $\mathbb{R}^n$, let A be an $(m \times n)$-matrix, and let $\mathbf{y} \in \mathbb{R}^m$.

We consider the *linearly constrained optimization problem*

$$(\mathcal{P}) \quad \inf \{ f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^n, \, \mathbf{y} - A\mathbf{x} = \mathbf{0} \} . \qquad (7)$$

▷ We denote the *optimal value* of $(\mathcal{P})$ by $V(\mathcal{P})$, the *feasible set* by $F(\mathcal{P})$ and the *solution set* by $S(\mathcal{P})$. Thus,

$$F(\mathcal{P}) := \{\mathbf{x} | \mathbf{y} - A\mathbf{x} = \mathbf{0}\}$$

and

$$S(\mathcal{P}) := \{\mathbf{x} \in F(\mathcal{P}) | f(\mathbf{x}) = V(\mathcal{P})\} .$$

▷ The *Lagrangian* of (7) is the function

$$\boxed{\mathcal{L}(\boldsymbol{\lambda}, \mathbf{x}) := f(\mathbf{x}) + \langle \boldsymbol{\lambda}, \mathbf{y} - A\mathbf{x} \rangle,}$$

for $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\quad \mathbf{x} \in \mathbb{R}^n$. For a given $\boldsymbol{\lambda}$, $\mathcal{L}(\boldsymbol{\lambda}, \mathbf{x})$ can be regarded as a "penalized" version of f.

> Each component of $\boldsymbol{\lambda}$ fixes the price (positive or negative) to be paid if the corresponding constraint is violated.

▷ Under favourable circumstances, it is possible to find a particular value $\bar{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ such that minimizers of $\mathcal{L}(\bar{\boldsymbol{\lambda}}, \cdot)$ also solve (7). Such a $\bar{\boldsymbol{\lambda}}$ is then called a *Lagrange Multiplier* or a *shadow price*.

$\triangleright$ Now minimizing $\mathcal{L}(\bar{\boldsymbol{\lambda}}, \cdot)$ is an unconstrained problem (save for any implicit constraints imposed by dom f.)

We can now state the Kuhn-Tucker Theorem which provides necessary and sufficient conditions (on $\boldsymbol{\lambda}$ and $\mathbf{x}$) for $\mathbf{x}$ to be a solution of (7), [7] or [2].

Theorem 1 (Kuhn-Tucker) *Suppose $V(\mathcal{P}) \neq -\infty$ and that*

$$\text{(CQ)} \qquad \boxed{F(\mathcal{P}) \cap \operatorname{int}\operatorname{dom} f \neq \emptyset.}$$

Then, the following are equivalent:

(i) $\mathbf{x} \in S(\mathcal{P})$;

(ii) $\sup \mathcal{L}(\cdot, \mathbf{x}) = \mathcal{L}(\bar{\boldsymbol{\lambda}}, \mathbf{x}) = \inf \mathcal{L}(\bar{\boldsymbol{\lambda}}, \cdot)$ *for some* $\bar{\boldsymbol{\lambda}}$;

(iii) $\mathbf{x} \in F(\mathcal{P})$ *and* $A^\star \bar{\boldsymbol{\lambda}} \in \partial f(\mathbf{x})$ *for some* $\bar{\boldsymbol{\lambda}}$.

$\triangleright$ In condition (iii), $A^\star$ is the matrix transpose of A and $\partial f(\mathbf{x})$ denotes the *subdifferential* of f at $\mathbf{x}$, i.e., the set of *subgradients* of f at $\mathbf{x}$.

$\triangleright$ Precisely, a vector $\boldsymbol{\xi} \in \mathbb{R}^n$ is a *subgradient* of f at $\mathbf{x}$ if the *subgradient inequality*

$$\boxed{f(\mathbf{z}) \geq g(\mathbf{z}) := f(\mathbf{x}) + \langle \boldsymbol{\xi}, \mathbf{z} - \mathbf{x} \rangle}$$

holds for all $\mathbf{z} \in \mathbb{R}^n$.

If f is convex and differentiable at $\mathbf{x}$, $\nabla f(\mathbf{x})$ is the unique subgradient of f at $\mathbf{x}$, and conversely.

• In the words of Rockafellar, the subgradient inequality says that "*the graph of the affine function g is a non-vertical supporting hyperplane to the epigraph of f at $(\mathbf{x}, f(\mathbf{x}))$.*" [7].

$\triangleright$ Points $(\bar{\boldsymbol{\lambda}}, \mathbf{x})$ satisfying condition (ii) are said to be *saddle points* of $\mathcal{L}$.

The requirements in (iii) are a form of the *Kuhn-Tucker conditions*. Notice that, in condition (ii), $\bar{\boldsymbol{\lambda}}$ appears as the maximizer of the (concave) *dual function*

$$\boxed{D(\boldsymbol{\lambda}) := \inf \mathcal{L}(\boldsymbol{\lambda}, \cdot).}$$

$\ldots$

$\triangleright$ We now return to the study of Problem (2).

The *Lagrangian* of (2) is

$$\mathcal{L}(\mathbf{p}, \lambda) := S_m(\mathbf{p}) + \lambda(1 - \langle \mathbf{u}, \mathbf{p} \rangle),$$

for $\mathbf{p} \in \mathbb{R}^m, \quad \lambda \in \mathbb{R}$.

Theorem 1 tells us that $\mathbf{p}$ is a solution for (2) if and only if:

(α) $0 = 1 - \langle \mathbf{u}, \mathbf{p} \rangle$;

(β) for some $\bar{\lambda} \in \mathbb{R}$ $\mathbf{0} \in \partial S_m(\mathbf{p}) + \bar{\lambda}\, \partial \big[1 - \langle \mathbf{u}, \cdot \rangle \big](\mathbf{p})$.

Indeed, one can check that $V(\mathcal{P}_m) \neq -\infty$ and that $(\mathcal{P}_m)$ has a feasible solution in

$$\operatorname{int} \operatorname{dom} S_m = \{ \mathbf{p} \in \mathbb{R}^m \mid \mathbf{p} > \mathbf{0} \} .$$

$\triangleright$ Furthermore, S_m is differentiable in the interior of its domain, and we have

$$\frac{\partial S_m}{\partial p_k}(\mathbf{p}) = \log m\mu_k - \sum_{i \le k} \mu_i ,$$

where

$$\mu_k := p_k / \sum_{j \ge k} p_j . \tag{8}$$

▷ Consequently, condition (β) becomes

$$0 = \log m\mu_k - \sum_{i \le k} \mu_i - \lambda, \quad k = 1, \ldots, m. \tag{9}$$

Now, by definition, $\mu_m = 1$, so setting $k = m$ in (9) gives

$$\lambda = \log m - \sum \mu_i,$$

from which we obtain the recursion

$$\mu_m = 1, \quad \mu_k = \exp\left(-\sum_{j=k+1}^m \mu_j\right), \tag{10}$$

for $k = m-1, \ldots, 1$. Also

$$\begin{aligned} \mu_{k-1} &= \exp\left(-\sum_{j=k}^m \mu_j\right) \\ &= \exp(-\mu_k)\, \exp\left(-\sum_{j=k+1}^m \mu_j\right). \end{aligned}$$

Thus, the *backward recursion* (10) can be rewritten as

$$\mu_m = 1, \quad \mu_{k-1} = \mu_k \exp\left(-\mu_k\right), \tag{11}$$

for $k = m, \ldots, 2$.

▷ Values of μ_k are shown in Figure 2, while Figure 3 shows optimal probability distributions.

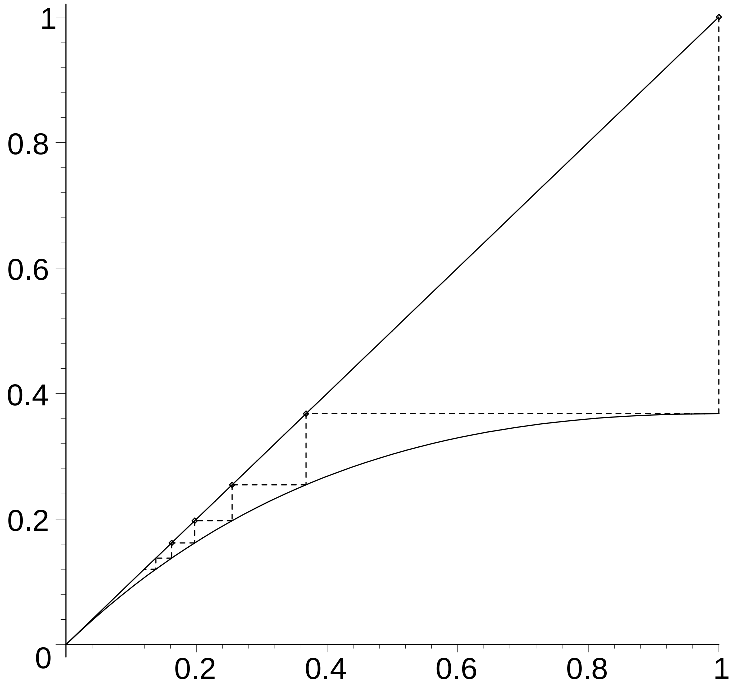


Fig. 2. **Recursion for the μ_k's.**

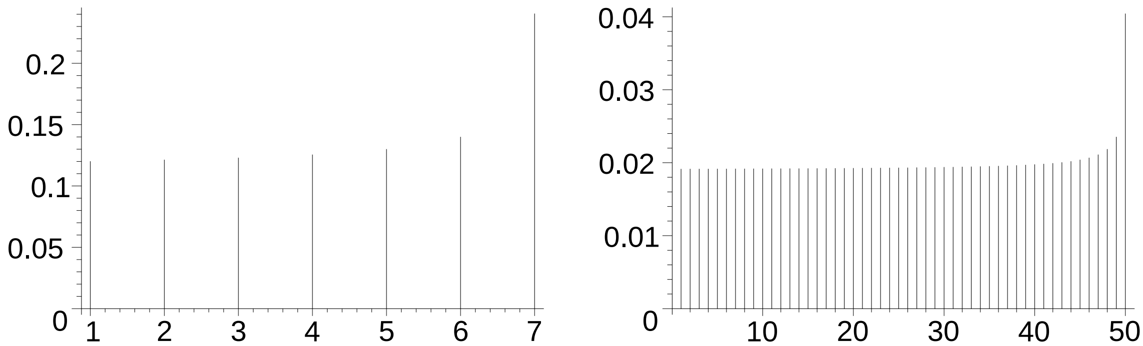


Fig. 3. **Optimal distributions**
$m = 7$ (left) and $m = 50$ (right).

▷ Finally, from condition (α) and the values of the μ_k's, we see that the components of $\mathbf{p}$ must obey the following *forward recursion*:

$$\begin{aligned} p_1 &= \mu_1, \;\; p_k = \mu_k \times \left(1 - \textstyle\sum_{j=1}^{k-1} p_j\right), \\ k &= 2, \ldots, m. \end{aligned} \tag{12}$$

The vector $\mathbf{p}$ defined in (12) satisfies conditions (α) and (β), and therefore *uniquely* solves Problem ($\mathcal{P}_m$) in (2).

Most pleasingly, the iteration is easy to handle both numerically and theoretically. For example, its components form an increasing sequence. Indeed,

$$p_k = \mu_k \,(p_k + \cdots + p_m)$$

and

$$p_{k-1} = \mu_{k-1} \,(p_{k-1} + \cdots + p_m).$$

▷ From whence we deduce, using (11), that

$$\begin{aligned}\frac{p_k}{p_{k-1}} &= \frac{\mu_k\,(1-\mu_{k-1})}{\mu_{k-1}} \\ &= \exp\mu_k \times \Big(1-\mu_k \exp(-\mu_k)\Big) \\ &= \exp\mu_k - \mu_k > 1,\end{aligned} \tag{13}$$

since $\mu_k > 0$.

▷ We recapitulate the prior discussion as:

Algorithm 1 *The unique probability distribution* $\mathbf{p}^m$ *maximizing surprise in Problem* $(\mathcal{P}_m)$*, given in (2), is strictly increasing and is determined as follows.*

a. Compute for $j = m, \ldots, 2$

$$\mu_m = 1, \qquad \mu_{j-1} = \mu_j \exp\Big(-\mu_j\Big), \tag{14}$$

and then

b. compute for $k = 2, \ldots, m$

$$p_1 = \mu_1, \;\; p_k = \mu_k \times \left(1 - \sum_{i=1}^{k-1} p_i\right). \tag{15}$$

Remark 1 As in [3, p. 50], the (optimal) conditional probability that the event occurs on the ith-to-the-last day, given that it has not occurred thus far, is *independent* of m.

$\triangleright$ This is immediate from (11) and the equality

$$P(m-i \,|\, \sim 1, \ldots, \sim (m-i-1)) \\ = p_{m-i}\left(\sum_{j=m-i}^{m} p_j\right)^{-1} = \mu_{m-i}.$$

Furthermore, as the μ_k's are defined via a backward recursion, p_{m-i}/p_{m-i-1} is also independent of m. ■

Remark 2 We may also obtain the solution to Problem ($\mathcal{P}_m$) of (2) via the optimization problem

$$\inf\left\{ S'_m(\mathbf{p},\mathbf{q}) \,\middle|\, 1 = \langle \mathbf{1}, \mathbf{p} \rangle ,\ \ \mathbf{q} = J\mathbf{p} \right\},$$

where

$$S'_m(\mathbf{p},\mathbf{q}) := \sum h(p_j, q_j).$$

▷ The needed Kuhn-Tucker conditions are

(α') $0 = 1 - \langle \mathbf{u}, \mathbf{p} \rangle$ and $\mathbf{0} = \mathbf{q} - J\mathbf{p}$;

(β') there exist $\lambda \in \mathbb{R}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ in $\mathbb{R}^m$ such that

$$\begin{aligned} \mathbf{0} \; &\in \partial S'_m(\mathbf{p}, \mathbf{q}) + \lambda \, \partial f(\mathbf{p}, \mathbf{q}) \\ &+ \lambda_1 \, \partial f_1(\mathbf{p}, \mathbf{q}) + \cdots + \lambda_m \, \partial f_m(\mathbf{p}, \mathbf{q}) \end{aligned}$$

with f and $\mathbf{f} = (f_1, \ldots, f_m)$ defined by

$$f(\mathbf{p}, \mathbf{q}) := 1 - \langle \mathbf{u}, \mathbf{p} \rangle$$

and

$$\mathbf{f}(\mathbf{p}, \mathbf{q}) := \mathbf{q} - J\mathbf{p}.$$

▷ It is then easy to check that the λ_j's derived from (α') and (β') coincide with the μ_j's of the previous discussion multiplied by m. ■

HOW THE DISTRIBUTION BEHAVES?

Striking characteristics of the optimal distribution were already shown in Remark 1. We will study asymptotic behaviour of Problem $(\mathcal{P}_m)$ as m tends to infinity.

We now establish three key properties.

$\triangleright$ First, we show that asymptotically the least probability $p_1^{(m)}$ behaves like m^{-1}.

The nub is an analysis of the rate of convergence of the *Picard-Banach iteration*,

$$t_{n+1} = g(t_n),$$

to the unique fixed point of a *contractive* self-map, g, on $[0, 1]$.

▷ But, when the fixed point, t, has $|g'(t)| = 1$, and so is not *strictly contractive*. Recall that g is contractive if

$$|g(t) - g(s)| < |t - s|$$

for all $t \neq s$ in $[0, 1]$. We use $x \mapsto x \exp(-x)$.

Proposition 1 *The quantity $mp_1^{(m)}$ tends to one as m tends to ∞.*

Proof. We define a sequence $\{t_n\}$ by setting

$$t_i := \mu_{m+1-i}^{(m)}$$

for $i = 1, \ldots, m, \quad m = 1, 2, \ldots$. Observe that t_i is independent of m, that $t_m = p_1^{(m)}$, and satisfies the recursion

$$t_1 = 1, \quad t_{k+1} = t_k \exp(-t_k),$$

for $k \geq 1$.

▷ We note that t_k tends monotonically to a limit ℓ which must necessarily be zero. Hence

$$t_{k+1}^{-1} - t_k^{-1} = t_k^{-1}(\exp t_k - 1),$$

which tends to $\exp'(0) = 1$ as k tends to infinity. Whence, since *Cesàro averaging* preserves limits,

$$\frac{1}{mt_m} = \frac{1}{m}\sum_{k=1}^{m-1}\frac{e^{t_k}-1}{t_k} + \frac{1}{mt_1}$$

also tends to 1. ■

▷ It is fun to perform a similar analysis for a general $g : [0,1] \mapsto [0,1]$.

Next, we show that the ratio between the last (biggest) and first (smallest) components converges.

Proposition 2

$$\lim_{m\to\infty} \frac{p_m^{(m)}}{p_1^{(m)}} \text{ exists and is finite.}$$

Proof. We have from (13) and the above definition of $\{t_n\}$, that

$$\begin{aligned}\lim \frac{p_m^{(m)}}{p_1^{(m)}} &= \lim_{m\to\infty} \textstyle\prod_{j=2}^m (e^{\mu_j^{(m)}} - \mu_j^{(m)}) \\ &= \lim_{m\to\infty} \textstyle\prod_{j=1}^{m-1} (e^{t_j} - t_j) \\ &\simeq 2.132979\ldots.\end{aligned}$$

The limit exists since

$$1 \le \exp t_j - t_j \le 1 + t_j^2,$$

while $\sum_j t_j^2 < \infty$ by Proposition 1.

Finally recall that $\prod_n (1 + |a_n|)$ and $\sum_n |a_n|$ converge together. ∎

Third – and more subtly - we establish that in the limit our solution value approaches that of the *uniform solution* of the next section.

Proposition 3 *The optimal value of* $(\mathcal{P}_m)$, $V(\mathcal{P}_m)$, *tends to zero as* m *tends to infinity.*

Proof. To establish this, we show that

$$\limsup V(\mathcal{P}_m) \leq 0,$$

and that

$$0 \leq \liminf V(\mathcal{P}_m).$$

a. The first inequality is easily obtained from identifying a Riemann sum:

$$\begin{aligned} V(\mathcal{P}_m) &\leq S_m\left(\frac{1}{m},\ldots,\frac{1}{m}\right) \\ &= \log m - \frac{\log m!}{m} - 1 \\ &= -\frac{1}{m}\sum_{k=1}^{m}\log\frac{k}{m} - 1 \\ &\to -\int_0^1 \log t\,dt - 1 = 0. \end{aligned}$$

b. obtain the other inequality, consider

$$\tau_m := \sum_{i=1}^{m-1}\left(p_i^{(m)}\log\frac{p_i^{(m)}}{q_{i+1}^{(m)}} - p_i^{(m)}\right)$$

and

$$\sigma_m := \sum_{i=1}^{m-1}\left(p_i^{(m)}\log\frac{p_i^{(m)}}{q_i^{(m)}} - p_i^{(m)}\right).$$

▷ We make two claims:

(i) $\tau_m - \sigma_m$ tends to 0 as m tends to infinity;

(ii) $\tau_m \geq -p_m^{(m)} \log m$.

Proof of (i). We recall from (4) and (8) that $\mu_i^{(m)} = p_i^{(m)}/(mq_i^{(m)})$ and so

$$\tau_m - \sigma_m = -\sum_{i=1}^{m-1} p_i^{(m)} \log(1 - \mu_i^{(m)}),$$

whence, as $p_i^{(m)}$ increases with i,

$$\begin{aligned} 0 &\leq \tau_m - \sigma_m = -\textstyle\sum_{i=1}^{m-1} p_{m-i}^{(m)} \log(1 - t_{i+1}) \\ &\leq -p_m^{(m)} \textstyle\sum_{i=1}^{m-1} \log(1 - t_{i+1}) \to 0, \end{aligned}$$

since $t_i \to 0$ and $mp_m^{(m)} = O(1)$.

The proof of (ii) is deferred to the next section where it is a consequence of a general integral inequality.

$\triangleright$ Now, by design,

$$V(\mathcal{P}_m) = \sigma_m + p_m^{(m)} \log m - p_m^{(m)}.$$

It follows from (ii) that

$$V(\mathcal{P}_m) \geq \sigma_m - \tau_m - p_m^{(m)}.$$

And so, since

$$p_m^{(m)} \to 0,$$

(i) shows

$$\liminf V(\mathcal{P}_m) \geq 0$$

as needed. ■

$\triangleright$ These techniques allow much more precise assertions about the asymptotics of $\mathbf{p}^m$.

CONTINUOUS TIME ANALYSIS

In the discrete case, the distribution is strictly increasing, with a sharp increase at the tip of the tail (see Figure). In measure, this is washed out in the limit.

$\triangleright$ Indeed, the optimal continuous distribution is flat, as the following theorem shows.

Theorem 2 *For all $p \in L_1([0,T])$, we have*

$$\int_0^T p(t) \log \frac{p(t)}{\dfrac{1}{T}\displaystyle\int_t^T p(s)\,ds}\,dt \geq \int_0^T p(t)\,dt$$

– equivalently $\mathcal{S}(p) \geq 0$ – with equality if and only if p is constant on $[0,T]$.

Proof. Without loss p is (a.e.) nonnegative, else $\mathcal{S}(p) = \infty$.

As in (6), set

$$q(t) := [\mathcal{J}p](t) = \frac{1}{T}\int_t^T p(s)\,ds.$$

On integrating by parts,

$$\begin{aligned}
\mathcal{S}(p) &= \int_0^T \left(p(t)\log\frac{p(t)}{q(t)} - p(t)\right)\,dt \\
&= \int_0^T \Big(p(t)\log p(t) - p(t)\Big)\,dt \\
&\qquad\qquad + T\textstyle\int_0^T q'(t)\log q(t)\,dt \\
&= \int_0^T p(t)\log p(t)\,dt - Tq(0)\log q(0),
\end{aligned}$$

▷ We shall be done once we show

$$\int_0^T p(t)\log p(t)\,dt \geq Tq(0)\log q(0). \tag{16}$$

with equality if and only if p is constant.

But, applying the integral version of *Jensen's inequality* to the strictly convex function $g := x \mapsto x \log x - x$ yields

$$\frac{1}{T}\int_0^T \left(\frac{p(t)}{q(0)}\log\frac{p(t)}{q(0)} - \frac{p(t)}{q(0)}\right)\,dt$$

$$\geq g(1) = -1,$$

from which (16) follows immediately. ∎

▷ Theorem 2 shows that the (unique) solution of Problem ($\mathcal{P}$) given in (5) is the uniform probability density on $[0, T]$.

▷ A consequence of Theorem 2, which completes the considerations of the last Section, follows:

Corollary 1 *As claimed in Section ,*

$$\tau_m \geq -p_m^{(m)} \log m.$$

Proof. Apply Theorem 2 with

$$T := 1 \quad \text{and} \quad p(t) := p_n^{(m)}$$

if

$$t \in \left(\frac{n-1}{m}, \frac{n}{m}\right] \quad (n = 1, \ldots, m).$$

For $\frac{n-1}{m} < t \leq \frac{n}{m}$ and $n \leq m - 1$,

$$\begin{aligned} q(t) \quad &\geq \quad \sum_{k=n+1}^{m} \int_{\frac{k-1}{m}}^{\frac{k}{m}} p(t)\, dt \\ &= \quad \frac{1}{m} \sum_{k=n+1}^{m} p_k^{(m)} = q_{n+1}^{(m)}, \end{aligned}$$

and, for $\frac{m-1}{m} < t \leq 1$,

$$q(t) = p_m^{(m)}(1 - t).$$

Hence τ_m majorizes

$$
\begin{aligned}
& m \sum_{n=1}^{m-1} \int_{\frac{n-1}{m}}^{\frac{n}{m}} p(t) \left\{ \log\left(\frac{p(t)}{q(t)} \right) - 1 \right\} dt \\
= \;& m \int_0^{1-\frac{1}{m}} p(t) \left\{ \log\left(\frac{p(t)}{q(t)} \right) - 1 \right\} dt \\
= \;& \int_0^1 p(t) \left\{ \log\left(\frac{p(t)}{q(t)} \right) - 1 \right\} dt \\
+ \;& m \int_{1-\frac{1}{m}}^{1} p_m^{(m)} \{ \log(1-t) + 1 \} \, dt \\
\geq \;& 0 - p_m^{(m)} \log m,
\end{aligned}
$$

on evaluating the second integral and applying Theorem 2. ■

▷ This finishes the proof that the optimal value of $(\mathcal{P}_m)$ tends to 0 ($=V(\mathcal{P})$), as claimed above.

CONCLUSION

The entropic formulation of the Surprise Examination Problem provides a beautiful case study of the application of concepts from the elementary theory of convex constrained optimization, probability and classical inequality theory. Its attractiveness comes in part from the very explicit recursive nature of the (discrete time) solution, which derives from the Kuhn-Tucker Theorem.

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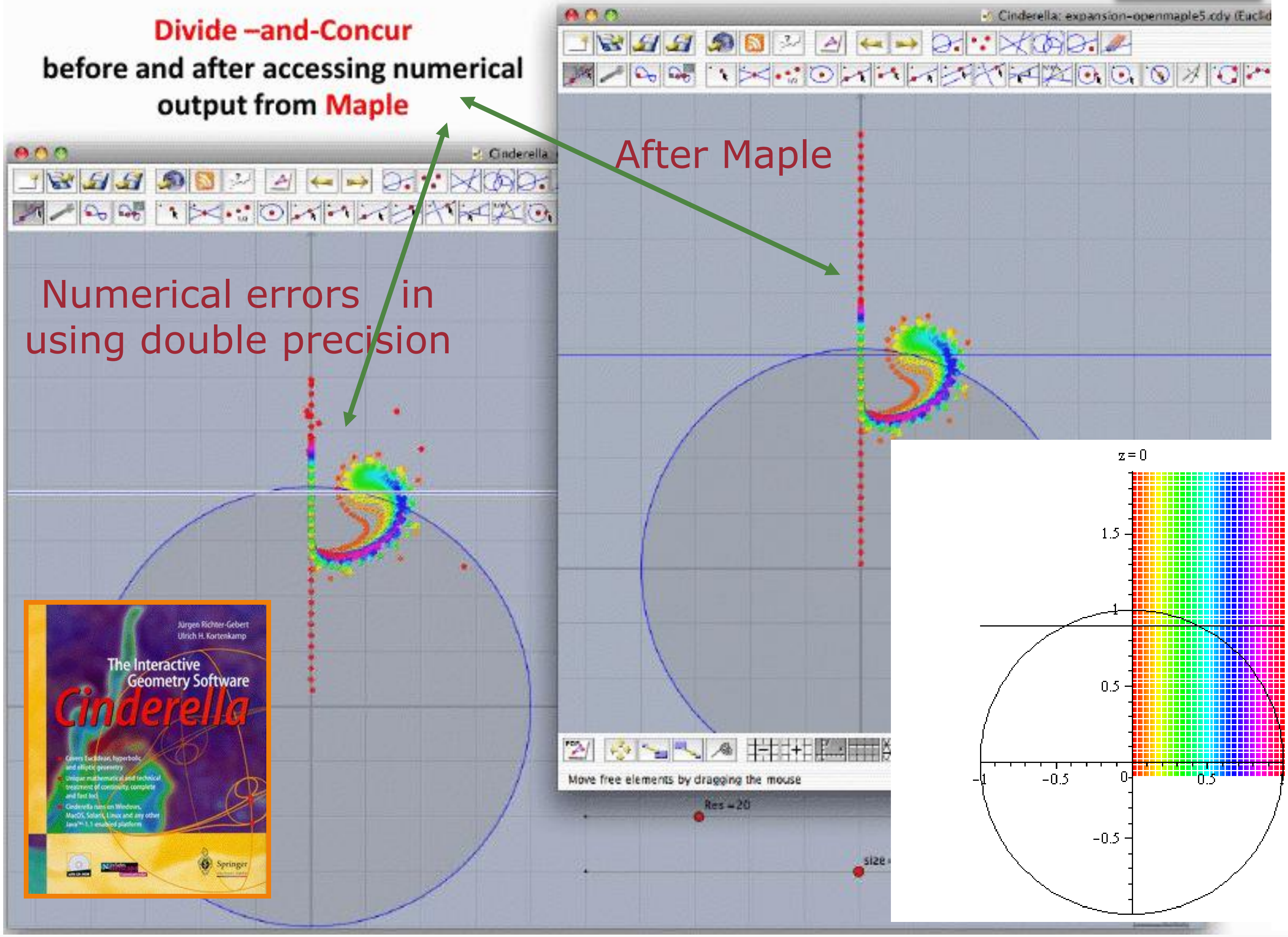
CAS+IGP: the Grief is in the GUI

Divide –and-Concur
before and after accessing numerical output from Maple



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A DUAL APPROACH TO LINEAR INVERSE PROBLEMS WITH CONVEX CONSTRAINTS*

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Abstract. A simple constraint qualification is developed and used to derive an explicit solution to a constrained optimization problem in Hilbert space. A finite parameterization is obtained for the minimum norm element in the intersection of a linear variety of finite co-dimension and a closed convex constraint set. The result extends previous duality theorems for convex cone set constraints. A fixed point iteration is presented for computing the parameters and yields a least-squares solution when the variety and constraint set have empty intersection. Proofs rely on nearest-point projections onto convex sets and the properties of monotone, firmly nonexpansive, and averaged mappings.

Key words. constrained optimization, semi-infinite convex program, constraint qualification, successive approximations, nearest-point projection, monotone operator

AMS(MOS) subject classifications. 49A, 49B, 49D

1. Introduction. The recovery of a signal from linear measurements and prior information is a central problem in signal analysis and remote sensing applications ranging from tomographic imaging and radio astronomy to well logging and respiratory physiology. Simplicity and generality are sought in characterizing and computing signals that successfully reflect available prior knowledge. To this end, the signal is abstractly represented as an element of a Hilbert space, and each known property of the signal is incorporated by restricting the reconstructed signal to lie in a specified closed convex set. In addition, the requirement that the signal be consistent with a finite number of linear measurements defines a linear variety of finite co-dimension. The intersection of this variety and the convex constraint set is termed the *feasible set* of signals. In this paper, the recovery task is formulated as the infinite-dimensional programming problem of determining the feasible signal closest to a specified nominal signal.

The desired signal is shown to admit a dual parameterization by exploiting the properties of monotone operators and nearest-point mappings onto closed convex sets. The parameter vector is seen to be a fixed point of a nonlinear, monotone, firmly nonexpansive operator in a finite-dimensional space; these properties lead both to a novel constraint qualification assuring the existence of the parameters and to iterative computational schemes. Convergence to a least-squares fit of the linear measurements is obtained when the feasible set is empty. The duality result does not require the constraint sets to have interior and allows direct derivation of the optimal L_2 solution in [8]. In addition, more recent L_p optimization results [2], [9] that likewise eschew the traditional Slater-type constraint qualification are extended, in Hilbert space, from the special case of a convex cone to general convex set constraints.

2. Problem formulation. Let $\mathcal{S}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. By the Riesz representation theorem, any N continuous, linear measurement functionals on $\mathcal{S}$ may be expressed by inner products with *measurement signals*

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$g_1, g_2, \ldots, g_N$ in $\mathcal{S}$. Accordingly, define the mapping A from S into Euclidean N-space $\Re^N$ by

$$Ax = [\langle x, g_1 \rangle, \ldots, \langle x, g_N \rangle]^t,$$

where $[\cdot]^t$ denotes vector transpose. For a given $\beta \in \Re^N$, the set of all x satisfying $Ax = \beta$ is a linear variety of co-dimension not exceeding N. The adjoint operator A^* maps a vector $\theta \in \Re^N$ with k^{th} entry θ_k to the signal $A^*\theta = \sum_{k=1}^N \theta_k g_k$. Thus, $A^*\theta$ is very simply a linear combination of the N measurement signals, and the range of A^* is the finite-dimensional subspace $\mathcal{G} \subset \mathcal{S}$ spanned by the measurement signals: range$(A^*) = \mathcal{G} = \text{span}\{g_1, \ldots, g_N\}$. Let Π be the orthogonal projection onto $\mathcal{G}$. The orthogonal complement of $\mathcal{G}$ is the null space of A, denoted $\ker(A)$; the linear variety $\{x : Ax = \beta\}$ is a translate of $\ker(A)$ and is therefore a closed convex set.

Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_M$ be closed convex sets with nonempty intersection $\mathcal{K}$. The set $\mathcal{K}$ is referred to as the *constraint set*; $\mathcal{K}$ may be infinite-dimensional and is not assumed to have interior. For a fixed measurement vector β the *feasible set* $\mathcal{F}$ is defined to be the intersection of the variety $\{x : Ax = \beta\}$ with the constraint set $\mathcal{K}$. That is, $\mathcal{F}$ is the closed and convex set $\{x \in \mathcal{K} : Ax = \beta\}$. Finally, let $\mathcal{E}$ denote the *extendible set* in $\Re^N$ defined to consist of all measurement vectors β for which the associated $\mathcal{F}$ is nonempty.

The recovery problem is to characterize and compute the signal in the feasible set $\mathcal{F}$ closest to a specified nominal signal. Without loss of generality, the nominal signal, x_{nom}, is the origin: for $x_{nom} \neq 0$, the data vector is replaced by $\beta - Ax_{nom}$, and the constraint set is translated by $-x_{nom}$. This constrained inverse problem is concisely written

$$\text{(P)} \qquad \min_{x \in K} \|x\| \text{ subject to } Ax = \beta.$$

The special case in which $\mathcal{K}$ is a convex cone is considered in [2], [8], [9], and [25], and subspace or linear variety constraints are considered in [3], [14]. Problem (P) is the linear inverse problem $Ax = \beta$ with the additional convex set constraint $x \in \mathcal{K}$.

Were the distinction between the closed convex data constraint $Ax = \beta$ and the set constraint $\mathcal{K}$ to be abandoned, the minimum norm element of the feasible set, $\mathcal{F}$, would be trivially characterized by the projection of the origin onto closed convex set $\mathcal{F}$. However, this conceptual approach is undesirable since the aim is to explicitly determine solutions. First, to combine the data constraint with the set $\mathcal{K}$ forfeits the structural advantage afforded by the finite co-dimensionality of the linear variety. Second, the nearest-point projection operator onto $\mathcal{F}$ may not be computable in a tractable manner; the set $\mathcal{K}$, on the other hand, typically arises from physically meaningful constraints that give rise to an easily implemented nearest-point projection operator onto $\mathcal{K}$. Third, the distinction between the data matching and set constraints allows for the computation of a least-squares solution when measurement noise renders the feasible set empty.

3. An optimality condition. In the absence of the constraints imposed by the convex set $\mathcal{K}$, the projection theorem, e.g., [13] simply and elegantly characterizes the minimum norm element of the variety $\{x : Ax = \beta\}$ as a linear combination $\sum_{k=1}^N \theta_k g_k$, where the parameters $\theta \in \Re^N$ are determined by the normal equations. In a similar manner, the constraints embodied by $\mathcal{K}$ are incorporated, and a particularly simple and geometrically appealing optimization result for (P) is obtained. The following theorem establishes a parsimonious parameterization of the solution $\hat{x}$.

First, two basic facts are reviewed for closed convex sets in a Hilbert space.

LEMMA 1. *Let $\mathcal{K}$ denote any closed convex subset of a Hilbert space $\mathcal{S}$. Then there exists a unique $y \in \mathcal{K}$ such that* $\inf_{z\in\mathcal{K}} \|x - z\| = \|x - y\|$.

This correspondence is denoted by $y = P_{\mathcal{K}}(x)$, where $P_{\mathcal{K}} : \mathcal{S} \mapsto \mathcal{K}$ is said to be the *nearest-point projection operator*, or simply the *projection*, of $\mathcal{S}$ onto the closed convex set $\mathcal{K}$. The operator $P_{\mathcal{K}}$ is linear if and only if $\mathcal{K}$ is a subspace.

LEMMA 2. *Let $\mathcal{K}$ be a closed convex subset of $\mathcal{S}$. Then the following are equivalent:*

$$\begin{array}{ll} (a) & P_{\mathcal{K}}(x) = y \\ (b) & \|x - y\| \leq \|x - z\| \quad \textit{for all} \quad z \in \mathcal{K} \\ (c) & \langle x - y, z - y\rangle \leq 0 \quad \textit{for all} \quad z \in \mathcal{K}. \end{array}$$

THEOREM 1. *If there exists $\theta \in \Re^N$ such that $\beta = AP_{\mathcal{K}}A^*(\theta)$, then $\hat{x} = P_{\mathcal{K}}A^*(\theta)$ is the unique solution to (P).*

Proof [18], [20]. The feasible set $\mathcal{F} := \mathcal{K}\bigcap\{x : Ax = \beta\}$ is closed and convex, and the existence of a unique minimum norm element follows from Lemma 1, provided $\mathcal{F}$ is nonempty. Let $y := A^*\theta$ where θ is the parameter vector of the hypothesis. It must be shown that

$$\inf_{x\in\mathcal{F}} \|x\| = \|P_{\mathcal{K}}(y)\|.$$

From Lemma 2, it suffices to show that $\langle P_{\mathcal{K}}(y), x - P_{\mathcal{K}}(y)\rangle \geq 0$ for all $x \in \mathcal{F}$. To this end, let x denote an arbitrary element of $\mathcal{F}$. Now write $P_{\mathcal{K}}(y)$ as $y - (y - P_{\mathcal{K}}(y))$ to yield

$$\langle P_{\mathcal{K}}(y), x - P_{\mathcal{K}}(y)\rangle = \langle y, x - P_{\mathcal{K}}(y)\rangle - \langle y - P_{\mathcal{K}}(y), x - P_{\mathcal{K}}(y)\rangle.$$

First, observe that $\langle y, x - P_{\mathcal{K}}(y)\rangle = 0$ since $y \in \mathcal{G} = \text{range}(A^*)$ and $Ax = \beta = AP_{\mathcal{K}}(y)$ implies $(x - P_{\mathcal{K}}(y)) \in \ker(A)$. Turning to the second term, observe from Lemma 2 that $\langle y - P_{\mathcal{K}}(y), x - P_{\mathcal{K}}(y)\rangle \leq 0$ for all $x \in \mathcal{K}$. In particular, $\mathcal{F}$ is a subset of $\mathcal{K}$, so the inequality holds for all x in $\mathcal{F}$. Hence, $\langle P_{\mathcal{K}}(y), x - P_{\mathcal{K}}(y)\rangle \geq 0$ for all x in $\mathcal{F}$. □

The result in Theorem 1 is a nonlinear generalization of the classical projection theorem, which follows as a simple corollary.

COROLLARY 1. *For $\mathcal{K} = \mathcal{S}$ and $\mathcal{F} \neq \emptyset$, the solution $\hat{x}$ to (P) is given by $A^*\theta$, where θ satisfies the normal equations $AA^*\theta = \beta$.*

Figure 1 provides an illustration of Theorem 1 in the Euclidean plane and, although depicting the degenerate case of $\mathcal{S} = \Re^2$, illuminates the similarities between Theorem 1 and the projection theorem. The minimum norm element x_{mn} of the variety $\{x : Ax = \beta\}$ is the orthogonal projection of the origin onto the variety. Thus, $x_{mn} = \sum_{k=1}^N \theta_k g_k$, where the coefficients θ_k are uniquely specified by the linear equations in Corollary 1. However, the minimum norm solution lies outside the constraint set $\mathcal{K}$, in general. Yet, the constrained minimum norm element $\hat{x}$ is found in an analogous manner: $\hat{x}$ is the nearest-point projection onto $\mathcal{K}$ of an element $A^*\hat{\theta} = \sum_{k=1}^N \hat{\theta}_k g_k$ in $\mathcal{G}$, where the parameter vector $\hat{\theta}$ is determined by the equations in Theorem 1. Thus, in order to constrain the minimum norm solution to lie in the constraint set $\mathcal{K}$, the linear normal equations $AA^*\theta = \beta$ are replaced by the nonlinear equations $AP_{\mathcal{K}}A^*(\hat{\theta}) = \beta$, and the solution $x_{mn} = A^*\theta$ is replaced by $\hat{x} = P_{\mathcal{K}}A^*(\hat{\theta})$.

4. A constraint qualification. The hypothesis of Theorem 1 requires the existence of a solution $\hat{\theta}$ to a nonlinear system of equations, and the optimal signal $\hat{x}$ is then parameterized by this solution via $\hat{x} = P_{\mathcal{K}}A^*(\hat{\theta})$. For a nonempty feasible set,

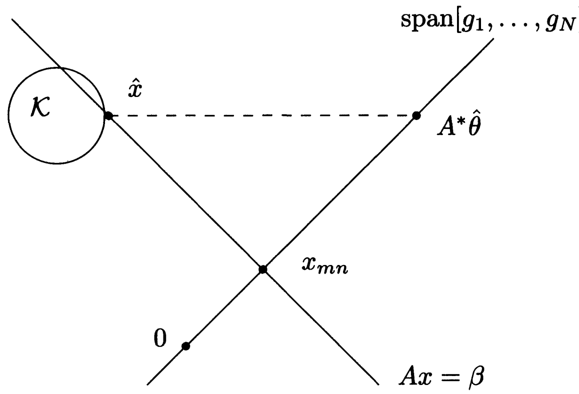

FIG. 1. *The minimum norm feasible signal is the projection onto the constraint set of an element in the span of the measurement signals.*

existence and uniqueness of a solution, $\hat{x}$, to (P) follow from Lemma 1. Therefore, the statement of the theorem immediately raises the questions: When does the representation of $\hat{x}$ by $\hat{\theta}$ exist? Is the representation unique? How may it be computed? To address the issues of existence and uniqueness requires an investigation of the ranges of the nonlinear operator $AP_{\mathcal{K}}A^* : \Re^N \mapsto \Re^N$ and its set-valued inverse. That is, there exists a solution to $AP_{\mathcal{K}}A^*(\theta) = \beta$ if and only if β is in the range of $AP_{\mathcal{K}}A^*$, and the solution is unique if and only if $(AP_{\mathcal{K}}A^*)^{-1}(\beta)$ is single valued. Pertinent properties of these ranges are derived in this section by making use of their finite dimensionality and utilizing results from the theory of monotone operators. These properties are then used both to establish a novel constraint qualification (Cor. 2), which gives a condition on the data vector β to ensure the existence of a parameterization and to characterize uniqueness. The third issue, computation, is deferred to §5, where the solution of $AP_{\mathcal{K}}A^*(\theta) = \beta$ is viewed as a nonlinear fixed point problem.

A set $\mathcal{M}$ in the Cartesian product $\Re^N \times \Re^N$ is said to be *monotone*, e.g., [27], if

$$\langle x^* - y^*, x - y\rangle \geq 0 \ \forall\ (x, x^*), (y, y^*) \in \mathcal{M}.$$

A *maximal* monotone set is one not properly contained in another monotone set. A (possibly set valued) mapping $f : \Re^N \mapsto 2^{\Re^N}$ is called a *monotone operator* if its graph $\{(x, x^*) | x^* \in f(x)\}$ is a monotone set in $\Re^N \times \Re^N$; the operator is said to be *maximal monotone* if its graph is a maximal monotone set. The operator f^{-1} is defined as the mapping which has as its graph the set $\{(x^*, x) | (x, x^*) \in \text{graph of } f\}$. Since monotonicity is invariant under transposition of the domain and range of a map, f and f^{-1} are simultaneously monotone or maximal monotone. In the sequel, set-valued mappings will be viewed as multifunctions, and the notation $f : \Re^N \mapsto \Re^N$ will be employed.

The properties of maximal monotone operators in finite-dimensional spaces and convex sets in $\Re^N$ are combined to guarantee that for any data vector β in the relative interior of the extendible set $\mathcal{E}$ there exists a parameter vector $\hat{\theta}$ providing the representation $\hat{x} = P_{\mathcal{K}}A^*(\hat{\theta})$. The requisite properties are established as two brief lemmas; the resulting theorem gives the desired constraint qualification as an immediate corollary.

LEMMA 3. *The operators $AP_{\mathcal{K}}A^* : \Re^N \mapsto \Re^N$ and $\Pi P_{\mathcal{K}} \Pi : \mathcal{S} \mapsto \mathcal{S}$ are maximal monotone operators.*

Proof. From the linearity of A and the definition of the adjoint A^*,it follows that

$$\begin{aligned}\langle AP_{\mathcal{K}}A^*(x) - AP_{\mathcal{K}}A^*(y), x-y\rangle &= \langle P_{\mathcal{K}}A^*(x) - P_{\mathcal{K}}A^*(y), A^*(x-y)\rangle \\ &= \langle P_{\mathcal{K}}(x^*) - P_{\mathcal{K}}(y^*), x^* - y^*\rangle,\end{aligned}$$

where $x^* = A^*x$ and $y^* = A^*y$. Immediately, this inner product is nonnegative since the projection $P_{\mathcal{K}}$ is monotone [7]. Next, since $AP_{\mathcal{K}}A^*$ is continuous and defined for all x in $\Re^N$, it is maximal monotone [15]. The proof for $\Pi P_{\mathcal{K}}\Pi$ is identical. □

LEMMA 4 ([17]). *The closure of the range of a maximal monotone operator is a convex set.*

As defined above, the set of all data vectors β that can be generated by measuring some signal x from the constraint set $\mathcal{K}$ is termed the *extendible set.* This set, $\mathcal{E} := \{\beta \in \Re^N : \beta = Ax, x \in \mathcal{K}\}$, is convex (immediately from the linearity of A) but not necessarily closed. Lemmas 3 and 4 are used to establish that the extendible set and the range of $AP_{\mathcal{K}}A^*$ are the same to within closure.

THEOREM 2. *The closure of the extendible set is equal to the closure of the set of all measurement vectors obtainable from parameterized signals of the form* $x = P_{\mathcal{K}}A^*(\theta)$, $\theta \in \Re^N$; *i.e.,* $cl(\mathcal{E}) := cl\{Ax : x \in \mathcal{K}\} = cl(range(AP_{\mathcal{K}}A^*))$.

Proof. [19] It must be shown that

$$\inf_{\theta\in\Re^N} \|Aq - AP_{\mathcal{K}}A^*(\theta)\|^2 = 0 \quad \forall q \in \mathcal{K}.$$

To this end, observe that $\ker(A)$ is orthogonal to range$(A^*) = \mathcal{G}$ and recall Π is the orthogonal projection onto $\mathcal{G}$. Thus, it must be shown that

$$\inf_{p\in\mathcal{G}} \|\Pi(q - P_{\mathcal{K}}(p))\|^2 = 0 \quad \forall q \in \mathcal{K}.$$

If $\mathcal{G} = \text{range}(A^*) = 0$, then $\Pi = 0$ and the claim is proven; so attention is restricted to the case of $\mathcal{G}$ nontrivial.

Proceeding by contradiction, assume there exists some $q \in \mathcal{K}$ for which the infimum is $\epsilon > 0$. The closure of range$(\Pi P_{\mathcal{K}}\Pi)$ is convex from Lemma 4. Let z denote the nearest point in cl(range$(\Pi P_{\mathcal{K}}\Pi)$) to Πq, with $\|\Pi q - z\|^2 = \epsilon$. Then, there exists a hyperplane $\mathcal{H}$ in the finite dimensional subspace $\mathcal{G}$ containing $h = \frac{1}{2}(\Pi q + z)$ and normal to $(\Pi q - z)$. The hyperplane $\mathcal{H}$ separates Πq from range$(\Pi P_{\mathcal{K}}\Pi)$.

Next, a point p_t is constructed to provide a contradiction. Let $p_t = \Pi q + t(\Pi q - z)$, $t > 0$. For t sufficiently large, p_t is closer to q than to $\mathcal{H}$. In particular, let $Q = \|(I - \Pi)q\|^2$ and observe

$$\|p_t - q\|^2 = \|\Pi q + t(\Pi q - z) - (\Pi q + (I - \Pi)q)\|^2 = t^2\epsilon + Q$$

On the other hand, the projection of p_t onto $\mathcal{H}$ is h for all $t > 0$. Thus,

$$\inf_{y\in\mathcal{H}} \|p_t - y\|^2 = \|\Pi q + t(\Pi q - z) - \frac{1}{2}(\Pi q + z)\|^2 = \left(t + \frac{1}{2}\right)^2 \epsilon.$$

Hence, for $(t + \frac{1}{4})\epsilon > Q$, $d(p_t,\, q) < d(p_t,\, \mathcal{H})$, where $d(\cdot,\, \cdot)$ is adopted as a distance notation. Now, let $\mathcal{J} = \mathcal{H} \oplus \mathcal{G}^\perp$ and let $\mathcal{J}^+$ denote the halfspace in $\mathcal{S}$ containing range$(\Pi P_{\mathcal{K}}\Pi)$. Then,

$$d(p_t,\, \mathcal{K}) \leq d(p_t,\, q) < d(p_t,\, \mathcal{H}) = d(p_t,\, \mathcal{J}) \leq d(p_t,\, \mathcal{J}^+ \cap \mathcal{K})$$

implying $P_{\mathcal{K}}(p_t) \in \mathcal{J}^-$, whence $\Pi P_{\mathcal{K}}(p_t) \in \mathcal{J}^-$ and $\Pi P_{\mathcal{K}}(p_t) \notin \mathrm{range}(\Pi P_{\mathcal{K}} \Pi)$. But $\Pi p_t = p_t$, providing a contradiction. Therefore, it must be the case that the infimum is indeed zero. □

The *relative interior* of a convex set $\mathcal{C} \subset \Re^N$, denoted $\mathrm{ri}(\mathcal{C})$, is defined as the interior that results when $\mathcal{C}$ is regarded as a subset of the intersection of all closed linear varieties containing $\mathcal{C}$. Given convex sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\Re^N$, $\mathrm{cl}(\mathcal{C}_1) = \mathrm{cl}(\mathcal{C}_2)$ if and only if $\mathrm{ri}(\mathcal{C}_1) = \mathrm{ri}(\mathcal{C}_2)$ [23]. The desired existence result now follows immediately from the theorem. This result can also be developed from convex duality theory [4].

COROLLARY 2 (CONSTRAINT QUALIFICATION). *If $\beta \in ri(\mathcal{E})$, then there exists θ such that $AP_{\mathcal{K}}A^*(\theta) = \beta$, i.e., $\beta \in range(AP_{\mathcal{K}}A^*)$.*

Proof. From Theorem 2, $\mathrm{cl}(\mathcal{E}) = \mathrm{cl}(\mathrm{range}(AP_{\mathcal{K}}A^*))$. Hence, equivalence of the relative interiors follows: $\mathrm{ri}(\mathcal{E}) = \mathrm{ri}(\mathrm{range}(AP_{\mathcal{K}}A^*))$. □

In an infinite-dimensional Hilbert space there exist closed convex sets $\mathcal{K}$ without interior for which support points are only dense in the boundary and form only a set of the first category, the complementary set being dense as well [10]. Yet, a simple consequence of Corollary 2 is that for $\beta \in \mathrm{ri}(\mathcal{E})$, the solution $\hat{x}$ to (P) is, in fact, a support point of $\mathcal{K}$ and, moreover, some normal to $\mathcal{K}$ at $\hat{x}$ intersects the subspace $\mathcal{G}$.

Two commonly employed but more restrictive constraint qualifications found in the literature follow as corollaries to the result in Theorem 2.

COROLLARY 3 (SLATER CONSTRAINT). *If $\mathcal{K}$ has interior and the feasible set $\mathcal{F} := \mathcal{K} \bigcap \{x : Ax = \beta\}$ contains points interior to $\mathcal{K}$, then there exists $\hat{\theta}$ such that $AP_{\mathcal{K}}A^*(\hat{\theta}) = \beta$.*

COROLLARY 4 ([2], [8]). *Let $\mathcal{S} = L_2$ and let $\mathcal{K}$ be the closed convex cone of nonnegative functions in L_2. If $\beta \in int(\mathcal{E})$, then there exists $\hat{\theta}$ such that $AP_{\mathcal{K}}A^*(\hat{\theta}) = \beta$.*

Theorem 2 answers the question of existence of the parameterization $\hat{x} = P_{\mathcal{K}}A^*(\theta)$. The second issue, uniqueness of a parameter vector, is equivalent to the single-valuedness of the operator $f = (AP_{\mathcal{K}}A^*)^{-1}$.

PROPOSITION 1. *If $\mathcal{E}$ has nonempty interior, then a parameter vector θ satisfying $AP_{\mathcal{K}}A^*(\theta) = \beta$ is unique for almost every $\beta \in \mathcal{E}$.*

Proof. The mapping $(AP_{\mathcal{K}}A^*)^{-1}$ is a monotone operator by Lemma 3. From [27, Thm. 1], the set of points where a monotone operator on a finite-dimensional Hilbert space is not single valued has zero Lebesgue measure. □

Furthermore, the set of points in $\mathcal{E}$ for which the representation is unique is a subset of the relative interior of $\mathrm{range}(AP_{\mathcal{K}}A^*)$ [22, Cor. 1.1]. For linearly dependent measurement signals $\{g_1, \ldots, g_N\}$ the extendible set $\mathcal{E} \subset \Re^N$ is contained in a subspace of dimension less than N and $\mathrm{int}(\mathcal{E}) = \emptyset$.

5. Iterative computation. From Theorems 1 and 2, the solution $\hat{x}$ to (P) is parameterized by $\hat{x} = P_{\mathcal{K}}A^*(\hat{\theta})$, where the vector $\hat{\theta}$ solves the nonlinear system $AP_{\mathcal{K}}A^*(\theta) = \beta$. Equivalently, the parameter vector $\hat{\theta}$ is a fixed point of the operator $T : \Re^N \mapsto \Re^N$ defined by $T(\theta) = \theta + \beta - AP_{\mathcal{K}}A^*(\theta)$. The operator T is not a contraction, nor does it have a compact domain; therefore, the well-known Banach and Brouwer fixed point results are not applicable. Nonetheless, the properties of firmly nonexpansive and averaged mappings are exploited to show that the sequence of Picard iterations

$$\theta^{(n+1)} = \theta^{(n)} + \lambda[\beta - AP_{\mathcal{K}}A^*(\theta^{(n)})], \qquad \lambda \in (0, 2) \tag{1}$$

converges to a fixed point of T. Additionally, the sequence is shown to characterize a least-squares solution to (P) when there exists no fixed point.

Case 1. First, the sequence $\{\theta^{(n)}\}$ is considered for the case in which T has a fixed point. Convergence is established by relying on three simple lemmas in Euclidean N-space. A mapping $f : \Re^N \mapsto \Re^N$ is said to be *nonexpansive* if $\|f(x) - f(y)\| \leq \|x - y\|$ for all x, y in $\Re^N$. Further, f is *firmly nonexpansive* if and only if $2f - I$ is nonexpansive [7].

LEMMA 5. *As defined above, let T be the operator given by $T(\theta) = \theta + \beta - AP_{\mathcal{K}}A^*(\theta)$. If the measurement signals $\{g_1, \ldots, g_N\}$ satisfy $\sum_{k=1}^N \|g_k\|^2 \leq 1$, then T is firmly nonexpansive.*

Proof. To show that T is firmly nonexpansive, $2T - I$ is shown to be nonexpansive. To this end, direct computation using the definitions of T and of the adjoint A^* yields

$$\|(2T-I)x-(2T-I)y\|^2 \leq \|x-y\|^2 \Leftrightarrow \langle P_{\mathcal{K}}(x')-P_{\mathcal{K}}(y'), x'-y'\rangle \geq \|A(P_{\mathcal{K}}(x')-P_{\mathcal{K}}(y'))\|^2,$$

where $x' := A^*x$ and $y' := A^*y$. From Lemma 2, $\langle P_{\mathcal{K}}(x') - P_{\mathcal{K}}(y'), x' - y'\rangle \geq \|P_{\mathcal{K}}(x') - P_{\mathcal{K}}(y')\|^2$. Furthermore, by hypothesis on the measurement signals and application of the Cauchy-Bunyakovskiĭ-Schwarz inequality, A is nonexpansive:

$$\|Aw\|^2 = \sum_{k=1}^{N}(\langle g_k, w\rangle)^2 \leq \sum_{k=1}^{N}\|g_k\|^2\|w\|^2 \leq \|w\|^2 \; \forall w \in \mathcal{S}.$$

Hence,

$$\|P_{\mathcal{K}}(x') - P_{\mathcal{K}}(y')\|^2 \geq \|A(P_{\mathcal{K}}(x') - P_{\mathcal{K}}(y'))\|^2,$$

and T is firmly nonexpansive. □

LEMMA 6 ([7]). *Let $f : \Re^N \mapsto \Re^N$ be a nonexpansive operator with a fixed point. Then, $\{f^n(x)\}$ converges to a fixed point of f if and only if f is* asymptotically regular, *i.e.,*

$$\lim_n f^n(x) - f^{n+1}(x) = 0 \textit{ for all } x \in \Re^N.$$

As an example of a nonexpansive operator $\Re$ to $\Re$ with a fixed point and not asymptotically regular, consider $f(x) = -x - 1$. Although a nonexpansive operator f may not be asymptotically regular, the *averaged mapping* $f_\lambda := \lambda f + (1 - \lambda)I$, where $0 < \lambda < 1$, shares the same fixed point set and has desirable asymptotic properties.

LEMMA 7 ([5]). *Let f be a nonexpansive operator in $\Re^N$. Although the operator f itself may not be asymptotically regular, if f has a fixed point, then the averaged mapping f_λ is asymptotically regular.*

The result now follows directly.

THEOREM 3. *Assume T has a fixed point. For $\lambda \in (0, 2)$ let $f : \Re^N \mapsto \Re^N$ be defined by*

$$f(\theta) = \theta + \lambda[\beta - AP_{\mathcal{K}}A^*(\theta)]$$

If the measurement signals $\{g_1, \ldots, g_N\}$ satisfy $\sum_{k=1}^N \|g_k\|^2 \leq 1$, then the sequence of Picard iterates $\{f^n(\theta)\}$ converges to a fixed point of T for any $\theta \in \Re^N$.

Proof. From Lemma 5, T is firmly nonexpansive, so $2T - I$ is nonexpansive and has the same fixed point set as T. Simply note that for $0 < \delta < 1$, f is the averaged mapping $f = \delta(2T - I) + (1 - \delta)I = 2\delta T + (1 - 2\delta)I$. By Lemma 7, f is asymptotically regular. Application of Lemma 6 then yields Picard iterates $\{f^n(\theta)\}$ converging to a fixed point of $2T - I$. Thus, the limit $\hat{\theta}$ is a fixed point of T and, therefore, satisfies $AP_{\mathcal{K}}A^*(\hat{\theta}) = \beta$. □

For all β in the relative interior of the extendible set $\mathcal{E}$, T has a fixed point by Corollary 2, and the Picard iteration $\{f^n(\theta)\}$ yields the solution to (P). The hypothesis that the measurement signals have square sum not exceeding one can always be satisfied by simple scaling.

Case 2. Next, the behavior of the sequence $\{f^n(\theta)\}$ is considered for the general case in which T may be fixed point free. First, T is trivially fixed point free when the measurement vector β is not extendible, i.e., when there exists no signal x in the constraint set $\mathcal{K}$ for which $Ax = \beta$, and hence, no solution to (P). In application, such a nonextendible vector β may result from either measurement noise or from failure of the constraint set $\mathcal{K}$ to reflect physical reality. In addition, T may have no fixed point for β in the relative boundary of $\mathcal{E}$.

The objective of determining a feasible signal $x \in \mathcal{S}$ satisfying both $x \in \mathcal{K}$ and $Ax = \beta$ is unobtainable when β fails to lie in the extendible set. A well-motivated and popular recourse is to find a signal in the constraint set $\mathcal{K}$ that best matches the measurement vector β in the least-squares sense: $\inf_{x \in \mathcal{K}} \|Ax - \beta\|$. (This choice implicitly supposes greater confidence in the knowledge expressed by the constraint set $\mathcal{K}$ than in the noisy measurement β.) If more than one signal achieves this infimum, then the unique infimizer of minimum norm is termed the *minimum norm least-squares* solution and solves

$$(\mathrm{P}') \qquad \min_{x \in \mathcal{K}} \|x\| \text{ subject to } \|Ax - \beta\| = \inf_{y \in \mathcal{K}} \|Ay - \beta\|.$$

A weighted least-squares formulation is easily adopted with corresponding change in the definition of the adjoint, A^*. For a closed convex constraint set $\mathcal{K}$ and a linear measurement operator A, the extendible set $\mathcal{E} = A(\mathcal{K}) \subset \Re^N$, though not necessarily closed, is convex. Hence, for a measurement vector $\beta \notin \mathcal{E}$, there exists a unique closest vector in the closure of $\mathcal{E}$, namely, the projection of β onto $cl(\mathcal{E})$, $P_{\bar{\mathcal{E}}}(\beta)$.

PROPOSITION 2. *The infimum in (P$'$) is achieved if and only if $P_{\bar{\mathcal{E}}}(\beta)$ is in $\mathcal{E}$.*

Proof. With $P_{\bar{\mathcal{E}}}$ as above, $\inf_{y \in \mathcal{K}} \|Ay - \beta\| = \|P_{\bar{\mathcal{E}}}(\beta) - \beta\|$. □

Therefore, for $P_{\bar{\mathcal{E}}}(\beta) \in \mathcal{E}$, problem (P$'$) is equivalent to (P) with measurement vector $P_{\bar{\mathcal{E}}}(\beta)$.

The asymptotic behavior of averaged mappings provides the solution to (P$'$), as readily demonstrated using the following asymptotic property of nonexpansive maps.

LEMMA 8 ([1]). *Let $h : \Re^N \mapsto \Re^N$ be nonexpansive and define the averaged mapping $h_\lambda = \lambda h + (1 - \lambda)I$, $\lambda \in (0, 1)$. Then for all θ in $\Re^N$*

$$\text{(a)} \quad \lim_{n\to\infty} \frac{h_\lambda^n(\theta)}{n} = -\nu$$

$$\text{(b)} \quad \lim_{n\to\infty} [h_\lambda^n(\theta) - h_\lambda^{n+1}(\theta)] = \nu,$$

where ν is the unique point of least norm in cl(range($I - h_\lambda$)). Additionally, h has no fixed point if and only if $\lim_{n\to\infty} \|h_\lambda^n(\theta)\| = \infty$ for all θ in $\Re^N$.

In relation to the asymptotic regularity condition of Lemma 6, observe that $h^n(\theta)$ is a Cauchy sequence if and only if h has a fixed point and ν is the zero vector.

THEOREM 4. *Let $\beta \in \Re^N$ be an observed measurement vector, and let $f : \Re^N \mapsto \Re^N$ be defined by $f(\theta) = \theta + \lambda[\beta - AP_{\mathcal{K}}A^*(\theta)]$ for $\lambda \in (0, 2)$. Also, assume $\sum_{k=1}^N \|g_k\|^2 \leq 1$. Let $\{\theta^{(n)}\}$ denote the sequence of Picard iterates $\theta^{(n)} = f^n(\theta^{(0)})$*

with initial iterate $\theta^{(0)}$. *Then, for any* $\theta^{(0)} \in \Re^N$, *the sequence* $\{AP_{\mathcal{K}}A^*(\theta^{(n)})\}$ *converges to* $P_{\bar{\mathcal{E}}}(\beta)$, *the projection of* β *onto the closure of the extendible set.*

Proof. [19] By Theorem 2, $\mathrm{cl}(\mathrm{range}(AP_{\mathcal{K}}A^*)) = \mathrm{cl}(\mathcal{E})$. Therefore, the closure of the range of $(I - f)$ is simply a scaled translate of the closure of the extendible set:

$$\mathrm{cl}(\mathrm{range}(I - f)) = \mathrm{cl}\{\gamma : \gamma = \lambda(AP_{\mathcal{K}}A^*(\theta) - \beta), \theta \in \Re^N\} = \lambda\{\mathrm{cl}(\mathcal{E}) - \beta\}.$$

Then, the minimum norm element of $\mathrm{cl}(\mathrm{range}(I - f))$ is $\lambda\nu$, where ν is the minimum norm element of $\mathrm{cl}(\mathcal{E}) - \beta$. Hence, the projection of β onto $\mathrm{cl}(\mathcal{E})$ is given by the sum $P_{\bar{\mathcal{E}}}(\beta) = \beta + \nu$. By Lemma 8, given $\epsilon > 0$, there exists some integer M_ϵ such that for all n exceeding M_ϵ

$$\begin{aligned} \epsilon &> \|\lambda\nu - (\theta^{(n)} - \theta^{(n+1)})\| \\ &= \|\lambda\nu - \theta^{(n)} + \theta^{(n)} + \lambda[\beta - AP_{\mathcal{K}}A^*(\theta^{(n)})]\| \\ &= \lambda\|P_{\bar{\mathcal{E}}}(\beta) - AP_{\mathcal{K}}A^*(\theta^{(n)})\|. \end{aligned}$$

Hence, $AP_{\mathcal{K}}A^*(\theta^{(n)}) \to P_{\bar{\mathcal{E}}}(\beta)$ as $n \to \infty$. □

COROLLARY 5. *If, in addition to the hypotheses of Theorem 4,* $P_{\bar{\mathcal{E}}}(\beta)$ *is contained in the extendible set* $\mathcal{E}$, *then the sequence of approximate reconstructions* $\{x^{(n)}\}$, $x^{(n)} := P_{\mathcal{K}}A^*(\theta^{(n)})$, *is bounded, and there exists a subsequence* $\{x^{(n_j)}\}$ *that converges weakly to* $\hat{x}$, *the solution to* (P′).

Proof. From Proposition 2, there exists a solution, $\hat{x}$, to (P′). By Theorem 4, $\epsilon_n := \|\Pi(\hat{x} - x^{(n)})\| \to 0$. Then, employ Lemma 2, a direct sum decomposition with $\mathcal{G}$, and the Cauchy-Bunyakovskiĭ-Schwarz inequality to learn

$$\begin{aligned} 0 &\le \|\hat{x} - A^*\theta^{(n)}\| - \|P_{\mathcal{K}}A^*\theta^{(n)} - A^*\theta^{(n)}\| \\ &\le \|(I - \Pi)(\hat{x})\| - \|(I - \Pi)(P_{\mathcal{K}}A^*\theta^{(n)})\| + \\ &\quad \Big| \|\Pi(\hat{x} - A^*\theta^{(n)})\| - \|\Pi(P_{\mathcal{K}}A^*\theta^{(n)} - A^*\theta^{(n)})\| \Big| \\ &\le \|(I - \Pi)(\hat{x})\| - \|(I - \Pi)(P_{\mathcal{K}}A^*\theta^{(n)})\| + \epsilon_n. \end{aligned}$$

Therefore, $x^{(n)} = P_{\mathcal{K}}A^*\theta^{(n)}$ is a bounded sequence in $\mathcal{K}$, and consequently there exists some subsequence $\{x^{(n_j)}\}$ that converges weakly. Let y be the weak limit. Now, $y \in \mathcal{K}$, $\Pi y = \Pi\hat{x}$, and $Ay = P_{\bar{\mathcal{E}}}(\beta)$ by Theorem 4. Thus, $\|\hat{x}\| \le \|y\|$ by definition of $\hat{x}$. Conversely, $\|(I - \Pi)(y)\| \le \|(I - \Pi)(\hat{x})\|$ from above, whence $\|y\| = \|\hat{x}\|$. Since, from Lemma 1, $\hat{x}$ is the unique element of minimum norm in $\mathcal{K}$ for which $A\hat{x} = P_{\bar{\mathcal{E}}}(\beta)$, it follows that $y = \hat{x}$. □

A practical criterion for convergence in computer implementation of Theorem 4 is to test the sequence of differences in successive iterations for convergence to $\lambda\nu$ within a given tolerance. However, the vector ν is not known a priori and is zero if and only if β is an observation vector in the closure of the extendible set. Nonetheless, $\{\theta^{(n)} - \theta^{(n+1)}\}$ is indeed a Cauchy sequence in $\Re^N$ by Lemma 8. Therefore, observing that

$$\theta^{(n)} - \theta^{(n+1)} = \theta^{(n)} - f(\theta^{(n)}) = \lambda[P_{\mathcal{K}}A^*(\theta^{(n)}) - \beta]$$

is simply the residual error scaled by λ, the iterations may be terminated when the change in the residual error from iterate n to $n+1$ is less than some prescribed value. Moreover, this sequence of residual errors is monotonically nonincreasing in norm due to the nonexpansiveness of f. Although $\{\theta^{(n)}\}$ is divergent, it grows only linearly as $n\lambda\nu$. Therefore, the divergence presents no practical computational overflow problems, even for a large number of iterations, since $\|\nu\|^2$ is bounded by the noise power in the measurement β.

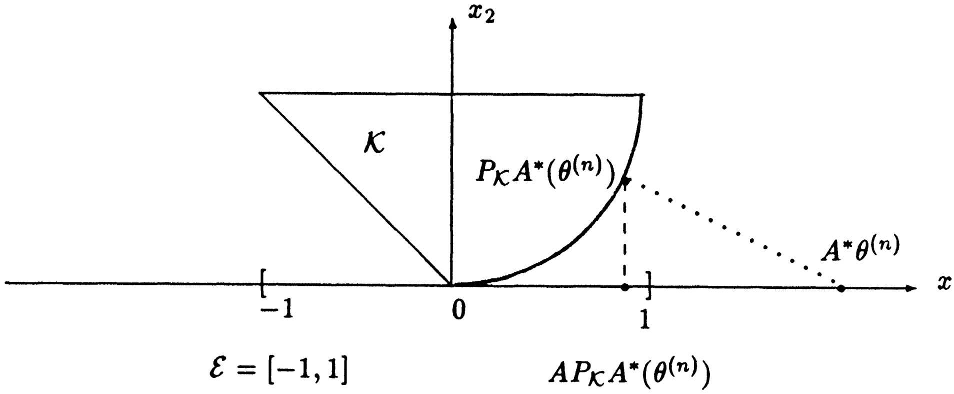

FIG. 2. *An example in the plane.*

6. Example. The results of §§3, 4, and 5 are illustrated by a simple example in the Euclidean plane. Although analytical nuances are lost from the infinite-dimensional case, the relationships among the relative interior of the extendible set, fixed points, and Picard iterations are clearly illuminated. (An example application to an infinite-dimensional problem is found in [21].) In the Hilbert space $\mathcal{S} = \Re^2$ let the constraint set $\mathcal{K}$ be the closed convex set depicted in Fig. 2. Let the measurement β be simply the first coordinate of a vector in $\Re^2$. Accordingly, the single measurement signal is $g_1 = [1\ 0]^t$, yielding $A : \Re^2 \mapsto \Re$ given by $[1\ 0]$ and $A^* = g_1$. The extendible set $\mathcal{E} := A(\mathcal{K})$ is the closed interval $[-1, 1]$. That $\mathcal{E}$ is closed is implied by the boundedness of $\mathcal{K}$. For a given β, the feasible set $\mathcal{F}$ is the intersection of $\mathcal{K}$ with the line $x_1 = \beta$.

By Corollary 2, if the measurement is in the open interval $\mathrm{ri}(\mathcal{E}) = (-1, 1)$, then there exists some scalar θ such that $\hat{x} = P_{\mathcal{K}}A^*(\theta)$ is the solution to (P). For the measurement $\beta = 1$ on the boundary of $\mathcal{E}$, no finite θ provides a parameterization; the measurement $\beta = -1$ is likewise on the boundary of $\mathcal{E}$, yet $\theta = -2$ provides the solution to (P). The dense uniqueness in Proposition 1 is illustrated by the infinitely many parameterizations, $\theta \in (\infty, -2]$, for $\beta = -1$. (A translation of $\mathcal{K}$ by $[0\ 1]^t$ provides an example of nonunique parameterization for β in the interior of $\mathcal{E}$.)

The iterative procedure of Theorems 3 and 4 is given by

$$f^{n+1}(\theta^{(0)}) = \theta^{(n+1)} = \theta^{(n)} + \lambda[\beta - AP_{\mathcal{K}}A^*(\theta^{(n)})], \quad \lambda \in (0, 2).$$

The action of $AP_{\mathcal{K}}A^* : \Re \mapsto \Re$ is depicted in Fig. 2 and is given by

$$AP_{\mathcal{K}}A^*(\theta) = \begin{cases} -1, & \theta \le -2 \\ \frac{1}{2}\theta, & -2 < \theta < 0 \\ \theta(\theta^2 + 1)^{-\frac{1}{2}}, & \theta \ge 0. \end{cases}$$

The existence of a parameterization θ is equivalent to the existence of a fixed point for f. For $\beta \in [-1, 1)$, there exists a fixed point for f, and by Theorem 3, the sequence $\{\theta^{(n)}\}$ converges to a parameter yielding $\hat{x}$, the minimum norm element of the feasible set. For $\beta \ge 1$, f has no fixed point and $\{|\theta^{(n)}|\}$ diverges as $\lambda n \nu$, where $\nu = \beta - 1$ is the distance of β from the extendible set. Yet, by Theorem 4, the sequence $\{AP_{\mathcal{K}}A^*(\theta^{(n)})\}$ converges to $P_{\bar{\mathcal{E}}}$, and by Corollary 5, a subsequence of the approximate reconstructions $\{P_{\mathcal{K}}A^*(\theta^{(n)})\}$ converges to the minimum norm, least-squares solution to (P′), $\hat{x} = [1\ 1]^t$. Finally, the parameter vector is unique for every β in $\mathcal{E}$ except $\beta = -1$, where the solution to (P) is not a regular point of $\mathcal{K}$.

7. Discussion. The method of successive projections is an alternative scheme for computing an element in the feasible set [26]. Treating the variety $Ax = \beta$ as an additional closed convex constraint set $\mathcal{K}_{M+1}$, the iteration

$$x^{(n+1)} = (P_{\mathcal{K}_{M+1}} P_{\mathcal{K}_M} P_{\mathcal{K}_{M-1}} \cdots P_{\mathcal{K}_1}) x^{(n)} \tag{2}$$

converges weakly in $\mathcal{S}$ to an element of the feasible set, provided one exists. The method is attractive in that any number of convex constraint sets may be incorporated without requiring synthesis of the projection onto the intersection $\mathcal{K} = \bigcap_{j=1}^{M} \mathcal{K}_j$. However, the technique does not allow the preferential selection of one feasible signal over others, as provided by the optimality criterion in (P′). In general, the limit point of Eq. (2) depends on both the initial estimate $x^{(0)}$ and the ordering of the composition of projection operators. Moreover, the iterations are performed in the (perhaps infinite-dimensional) signal space $\mathcal{S}$ rather than in $\Re^N$ and typically suffer from slow convergence rates and high computational cost per iteration [11], [24]. In addition, successive projections do not in general provide a least-squares solution when no feasible signal exists.

In contrast, the signal recovery algorithm established in Theorems 1–4 provides a finite-dimensional parameterization for a signal reconstruction. The iterative algorithm is performed in the parameter space to preferentially produce the unique least-squares solution consistent with the constraints and closest to a specified nominal signal. Moreover, Newton-Raphson iterations may typically be applied in $\Re^N$ for quadratically convergent iterative computation; the requisite derivatives are guaranteed to exist almost everywhere since $AP_{\mathcal{K}}A^*$ is Lipschitz. A potential difficulty in implementing the iterative scheme in Eq. (1) is the need to construct $AP_{\mathcal{K}}A^*$, which may require numerical approximation of the projection onto $\mathcal{K}$, the intersection of constraint sets. Yet, in application, $\mathcal{K}$ is physically motivated and, as such, typically gives rise to an intuitive and tractable projection operator. Furthermore, sensitivity of the solution $\hat{x}$ to errors in the parameters $\hat{\theta}$ is low since $P_{\mathcal{K}}A^*$ is nonexpansive. Although the constraint set must be convex and the signal space is Hilbertian, the formulation admits a large and relevant class of sets for incorporating prior information.

Many well-known linear reconstruction results follow immediately from Theorem 1 for the special case of constraint sets $\mathcal{K}$ that are subspaces, e.g., [3], [6], [12], [16]. Likewise, Theorems 1 and 2 extend, in Hilbert space, the optimization results in [2], [8], and [9] from closed convex cones to arbitrary closed convex constraint sets. For example, the minimum energy correlation extension presented in [8] and [25] may be directly obtained with $\mathcal{S} = L_2$ and $\mathcal{K}$ the convex cone of nonnegative spectral estimates. Kuhn-Tucker, Lagrange multiplier, and Fenchel duality theorems, e.g., [13] are similar dual optimization results that have been applied to signal recovery problems and, in addition, admit cost functions more general than the weighted norm. However, the hypotheses of these classical results require nonempty interior and regularity conditions that are absent in Theorems 1–4. These seemingly technical restrictions are, in fact, of great importance to application in many practical reconstruction tasks since, for example, the set of nonnegative signals in L_2 is without interior.

8. Conclusion. Motivated by practical reconstruction, estimation, and interpolation problems, an explicit solution to a constrained minimization problem has been derived. The finite parameterization led to a simple and computationally attractive iterative algorithm. The constraint qualification for the infinite-dimensional program with linear equality constraints and a convex set constraint extended previous results for a convex cone set constraint.

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5.1.4 Continuity of Multifunctions

The basic definition is given below.

Definition 5.1.15 (Continuity of Multifunction) *Let X and Y be two Hausdorff topological spaces and let $F\colon X \to 2^Y$ be a multifunction. We say that F is upper (lower) semicontinuous at $\bar{x} \in X$ provided that for any open set U in Y with $F(\bar{x}) \subset U$, $(F(\bar{x}) \cap U \neq \emptyset)$,*

$$\{x \in X \mid F(x) \subset U\} \quad (\{x \in X \mid F(x) \cap U \neq \emptyset\})$$

is an open set in X. We say that F is continuous at $\bar{x}$ if it is both upper and lower semicontinuous at $\bar{x}$. We say that F is upper (lower) continuous on $S \subset X$ if it is upper (lower) continuous at every $x \in S$. We omit S when it coincides with the domain of F.

We will also need a sequential approach to limits and continuity of multifunctions. This is mainly for applications in the subdifferential theory because the corresponding topological approach often yields objects that are too big.

Definition 5.1.16 (Sequential Lower and Upper Limits) *Let X and Y be two Hausdorff topological spaces and let $F\colon X \to 2^Y$ be a multifunction. We define the sequential lower and upper limit of F at $\bar{x} \in X$ by*

$$\text{s-}\liminf_{x \to \bar{x}} F(x) := \bigcap \{\liminf_{i \to \infty} F(x_i) \mid x_i \to \bar{x}\}$$

and

$$\text{s-}\limsup_{x \to \bar{x}} F(x) := \bigcup \{\limsup_{i \to \infty} F(x_i) \mid x_i \to \bar{x}\}.$$

When

$$\text{s-}\liminf_{x \to \bar{x}} F(x) = \text{s-}\limsup_{x \to \bar{x}} F(x)$$

we call the common set the sequential limit of F at $\bar{x}$ and denote it by $\text{s-}\lim_{x \to \bar{x}} F(x)$.

Definition 5.1.17 (Semicontinuity and Continuity) *Let X and Y be two Hausdorff topological spaces and let $F\colon X \to 2^Y$ be a multifunction. We say that F is sequentially lower (upper) semicontinuous at $\bar{x} \in X$ provided that*

$$F(\bar{x}) \subset \text{s-}\liminf_{x \to \bar{x}} F(x) \ (\text{s-}\limsup_{x \to \bar{x}} F(x) \subset F(\bar{x})).$$

When F is both upper and lower semicontinuous at $\bar{x}$ we say it is continuous at $\bar{x}$. In the notation introduced above,

$$F(\bar{x}) = \text{s-}\lim_{x \to \bar{x}} F(x).$$

Clearly, when Y is a metric space the sequential and the topological (semi) continuity coincide.

The following example illustrates how the semicontinuity and continuity of multifunctions relate to that of functions.

Example 5.1.18 (Profile Mappings) Let X be a Banach space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a function. Then the epigraphic profile of f, E_f is upper (lower) semicontinuous at $\bar{x}$ if and only if f is lower (upper) semicontinuous at $\bar{x}$. Consequently, E_f is continuous at $\bar{x}$ if and only if f is continuous.

Example 5.1.19 (Sublevel Set Mappings) Let X be a Banach space and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then the sublevel set mapping $S(a) = f^{-1}((-\infty, a])$ is upper semicontinuous.

When X and Y are metric spaces we have the following characterizations of the sequential lower and upper limit.

Theorem 5.1.20 (Continuity and Distance Functions) *Let X and Y be two metric spaces and let $F\colon X \to 2^Y$ be a multifunction. Then F is sequentially lower (upper) semicontinuous at $\bar{x} \in X$ if and only if for every $y \in Y$, the distance function $x \to d(F(x); y)$ is upper (lower) semicontinuous. Consequently, F is continuous at $\bar{x}$ if and only if for every $y \in Y$, the distance function $x \to d(F(x); y)$ is continuous.*

Proof. This follows from Lemma 5.1.11. Details are left as Exercise 5.1.15. ●

5.1.5 Uscos and Cuscos

The acronym *usco* (*cusco*) stands for a (convex) upper semicontinuous nonempty valued compact multifunction. Such multifunctions are interesting because they describe common features of the maximal monotone operators, of the convex subdifferential and of the Clarke generalized gradient.

Definition 5.1.21 *Let X be a Banach space and let Y be a Hausdorff topological vector space. We say $F\colon X \to 2^Y$ is an* usco *(*cusco*) provided that F is a nonempty (convex) compact valued upper semicontinuous multifunction. An usco (cusco) is* minimal *if it does not properly contain any other usco (cusco).*

A particularly useful case is when $Y = X^*$ with its weak-star topology. In this case we use the terminology weak*-usco (-cusco).

Closed multifunctions and uscos have an intimate relationship.

Proposition 5.1.22 *Let X and Y be two Hausdorff topological spaces and let $F\colon X \to 2^Y$ be a multifunction. Suppose that F is an usco. Then it is closed. If in addition,* range F *is compact, then F is an usco if and only if F is closed.*

Proof. It is easy to check that if $F\colon X \to 2^Y$ is an usco, then its graph is closed (Exercise 5.1.16). Now suppose F is closed and range F is compact. Then clearly F is compact valued. We show it is upper semicontinuous. Suppose on the contrary that F is not upper semicontinuous at $\bar{x} \in X$. Then there exists an open set $U \subset Y$ containing $F(\bar{x})$ and a net $x_\alpha \to \bar{x}$ and $y_\alpha \in F(x_\alpha)\backslash U$ for each α. Since range F is compact, we can take subnet (x_β, y_β) of (x_α, y_α) such that $x_\beta \to \bar{x}$ and $y_\beta \to \bar{y} \notin U$. On the other hand it follows from F is closed that $\bar{y} \in F(\bar{x}) \subset U$, a contradiction. ●

An important feature of an usco (cusco) is that it always contains a minimal one.

Proposition 5.1.23 (Existence of Minimal usco) *Let X and Y be two Hausdorff topological spaces and let $F\colon X \to 2^Y$ be an usco (cusco). Then there exists a minimal usco (cusco) contained in F.*

Proof. By virtue of of Zorn's lemma we need only show that any decreasing chain (F_α) of usco (cusco) maps contained in F in terms of set inclusion has a minimal element. For $x \in X$ define $F_0(x) = \bigcap F_\alpha(x)$. Since $F_\alpha(x)$ are compact, $F_0(x)$ is nonempty, (convex) and compact. It remains to show that F_0 is upper semicontinuous. Suppose that $x \in X$, U is open in Y and $F_0(x) \subset U$. Then $F_\alpha(x) \subset U$ for some α. Indeed, if each $F_\alpha(x)\backslash U$ were nonempty then the intersection of these compact nested sets would be a nonempty subset of $F_0(x)\backslash U$, a contradiction. By upper semicontinuity of F_α, there exists an open set V containing x such that $F_0(V) \subset F_\alpha(V) \subset U$. ●

When $Y = \mathbb{R}$ the proposition below provides a procedure of constructing a minimal usco contained in a given usco.

Proposition 5.1.24 *Let X be a Hausdorff topological space and $F\colon X \to 2^{\mathbb{R}}$ an usco. For each $x \in X$, put $f(x) := \min\{r \mid r \in F(x)\}$. Let $G\colon X \to 2^{\mathbb{R}}$ be the closure of f (i.e., the set-valued mapping whose graph is the closure of the graph of f). Now put $g(x) := \max\{r \mid r \in G(x)\}$ for each $x \in X$. Finally let $H\colon X \to 2^{\mathbb{R}}$ be the closure of g. Then H is a minimal usco contained in F.*

Proof. Since the graph of F is closed, G is contained in F, and G is an usco as G is closed and F is an usco. For the same reason H is an usco contained in G.

To show that H is minimal, consider open sets $U \subset X$ and $W \subset \mathbb{R}$, such that there is some $w \in H(U) \cap W$. It is sufficient to find a nonempty open subset of U, whose image under H is entirely contained in W.

Fix some $\varepsilon < d(\mathbb{R} \setminus W; w)$. Since $w \in H(U)$, there is some $x \in U$ such that $g(x) \in (w - \varepsilon; w + \varepsilon)$. This means that $G(x) \subset (-\infty; w + \varepsilon)$ and by upper semi-continuity of G there is an open $V \subset U$, $V \ni x$, such that $G(V) \subset (-\infty\,;\, w + \varepsilon)$.

As $g(x) \in (w-\varepsilon, w+\varepsilon)$, there is some $x' \in V$ with $f(x') \in (w-\varepsilon, w+\varepsilon)$. This means that $F(x') \subset (w-\varepsilon, +\infty)$ and by upper semi-continuity of F there is an open $V' \subset V$, $V' \ni x'$, such that $F(V') \subset (w-\varepsilon, +\infty)$.

Now $H(V') \subset F(V') \cap G(V) \subset (w-\varepsilon, w+\varepsilon) \subset W$. Thus H is a minimal usco. ●

Maximal monotone operators, in particular, subdifferentials of convex functions provide interesting examples of w*-cuscos. We leave the verification of the following example as a guided exercise (Exercise 5.1.17).

Example 5.1.25 Let X be a Banach space, let $F\colon X \to 2^{X^*}$ be a maximal monotone multifunction and let S be an open subset of $\operatorname{dom} F$. Then the restriction of F to S is a w*-cusco.

To further explore the relationship of maximal monotone multifunctions and cuscos we need to extend the notion of maximal monotone multifunctions to arbitrary set.

Definition 5.1.26 (Maximal Monotone on a Set) *Let X be a Banach space, let $F\colon X \to 2^{X^*}$ be a monotone multifunction and let S be a subset of X. We say that F is maximal monotone in S provided the monotone set*

$$\operatorname{graph} F \cap (S \times X^*) := \{(x, x^*) \in S \times X^* \mid x \in S \text{ and } x^* \in F(x)\}$$

is maximal under the set inclusion in the family of all monotone sets contained in $S \times X^*$.

It turns out that a monotone cusco on an open set is maximal.

Lemma 5.1.27 *Let X be a Banach space, let $F\colon X \to 2^{X^*}$ be a monotone multifunction and let S be an open subset of X. Suppose that $S \subset \operatorname{dom} F$ and F is a w*-cusco on S. Then F is maximal monotone in S.*

Proof. We need only show that if $(y, y^*) \in S \times X^*$ satisfies

$$\langle y^* - x^*, y - x\rangle \geq 0 \text{ for all } x \in S, x^* \in F(x), \tag{5.1.15}$$

then $y^* \in F(y)$. If not, by the separation theorem there exists $z \in X\backslash\{0\}$ such that $F(y) \subset \{z^* \in X^* \mid \langle z^*, z\rangle < \langle y^*, z\rangle\} = W$. Since W is weak* open and F is w*-upper semicontinuous on S, there exists an $h > 0$ with $B_h(y) \subset S$ such that $F(B_h(y)) \subset W$. Now, for $t \in (0, h/\|z\|)$, we have $y + tz \in B_h(y)$, and therefore $F(y + tz) \subset W$. Applying (5.1.15) to any $u^* \in F(y + tz)$ we get

$$0 \leq \langle y^* - u^*, y - (y + tz)\rangle = -t\langle y^* - u^*, z\rangle,$$

which implies $\langle u^*, z\rangle \geq \langle y^*, z\rangle$, that is $u^* \notin W$, a contradiction. ●

As a corollary we have

Corollary 5.1.28 *Let X be a Banach space, let $F\colon X \to 2^{X^*}$ be a maximal monotone multifunction and let S be an open subset of X. Suppose that $S \subset \operatorname{dom} F$. Then F is maximal monotone in S.*

Proof. By Example 5.1.25 the maximal monotonicity of F implies that F is a w*-cusco on S, so the result follows from Lemma 5.1.27. ●

Now we can prove the interesting relation that a maximal monotone multifunction on an open set is a minimal cusco.

Theorem 5.1.29 (Maximal Monotonicity and Minimal cusco) *Let X be a Banach space, let S be an open subset of X and let F be a maximal monotone multifunction in S. Then F is a minimal w*-cusco.*

Proof. We know by Example 5.1.25 that F is a w*-cusco. Suppose that $G\colon S \to 2^{X^*}$ is a w*-cusco and $\operatorname{graph} G \subset \operatorname{graph} F$. By Lemma 5.1.27, G is maximal monotone, and therefore $G = F$. ●

Note that a maximal monotone multifunction need not be a minimal usco. The following example clarifies the difference whose easy proof is left as Exercise 5.1.18.

Example 5.1.30 Define monotone multifunctions F_0, F_1 and F_2 from $\mathbb{R} \to 2^{\mathbb{R}}$ by

$$F_0(x) = F_1(x) = F_2(x) = \operatorname{sgn} x \text{ if } x \neq 0,$$

while

$$F_0(0) = \{-1\}, F_1(0) = \{-1, 1\} \text{ and } F_2(0) = [-1, 1].$$

Then $\operatorname{graph} F_0 \subset \operatorname{graph} F_1 \subset \operatorname{graph} F_2$, and they are all distinct. The multifunction F_2 is maximal monotone and minimal cusco, F_1 is minimal usco and F_0 does not have a closed graph.

5.1.6 Monotone Operators and the Fitzpatrick Function

Throughout this subsection, $(X, \|\cdot\|)$ is a reflexive Banach space with dual X^* and $T\colon X \to 2^{X^*}$ is maximal monotone. The *Fitzpatrick function* F_T, associated with T, is the proper closed convex function defined on $X \times X^*$ by

$$\begin{aligned} F_T(x, x^*) &:= \sup_{y^* \in Ty} [\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle] \\ &= \langle x^*, x\rangle + \sup_{y^* \in Ty} \langle x^* - y^*, y - x\rangle. \end{aligned}$$

Since T is maximal monotone

$$\sup_{y^* \in Ty} \langle x^* - y^*, y - x\rangle \geq 0$$

and the equality holds if and only if $x^* \in Tx$, it follows that

$$F_T(x, x^*) \geq \langle x^*, x\rangle \tag{5.1.16}$$

with equality holding if and only if $x^* \in Tx$. Thus, we capture much of a maximal monotone multifunction via an associated convex function.

Using only the Fitzpatrick function and the decoupling lemma we can prove the following fundamental result remarkably easily.

Theorem 5.1.31 (Rockafellar) *Let X be a reflexive Banach space and let $T\colon X \to 2^{X^*}$ be a maximal monotone operator. Then* range$(T + J) = X^*$. *Here J is the* duality map *defined by $J(x) := \partial\|x\|^2/2$.*

Proof. The Cauchy inequality and (5.1.16) implies that for all x, x^*,

$$F_T(x, x^*) + \frac{\|x\|^2 + \|x^*\|^2}{2} \geq 0. \tag{5.1.17}$$

Applying the decoupling result of Lemma 4.3.1 to (5.1.17) we conclude that there exists a point $(w^*, w) \in X^* \times X$ such that

$$\begin{aligned} 0 \leq\ & F_T(x, x^*) - \langle w^*, x\rangle - \langle x^*, w\rangle \\ &+ \frac{\|y\|^2 + \|y^*\|^2}{2} + \langle w^*, y\rangle + \langle y^*, w\rangle \end{aligned} \tag{5.1.18}$$

Choosing $y \in -Jw^*$ and $y^* \in -Jw$ in inequality (5.1.18) we have

$$F_T(x, x^*) - \langle w^*, x\rangle - \langle x^*, w\rangle \geq \frac{\|w\|^2 + \|w^*\|^2}{2}. \tag{5.1.19}$$

For any $x^* \in Tx$, adding $\langle w^*, w\rangle$ to both sides of the above inequality and noticing $F_T(x, x^*) = \langle x^*, x\rangle$ we obtain

$$\langle x^* - w^*, x - w\rangle \geq \frac{\|w\|^2 + \|w^*\|^2}{2} + \langle w^*, w\rangle \geq 0. \tag{5.1.20}$$

Since (5.1.20) holds for all $x^* \in Tx$ and T is maximal we must have $w^* \in Tw$. Now setting $x^* = w^*$ and $x = w$ in (5.1.20) yields

$$\frac{\|w\|^2 + \|w^*\|^2}{2} + \langle w^*, w\rangle = 0,$$

which implies $-w^* \in Jw$. Thus, $0 \in (T + J)w$. Since the argument applies equally well to all translations of T, we have range$(T + J) = X^*$ as required. ●

There is a tight relationship between nonexpansive mappings and monotone operators in Hilbert spaces, as stated in the next lemma.

Lemma 5.1.32 *Let H be a Hilbert space. Suppose that P and T are two multifunctions from subsets of H to 2^H whose graphs are related by the condition $(x, y) \in$ graph P if and only if $(v, w) \in$ graph T where $x = w + v$ and $y = w - v$. Then*

(i) *P is nonexpansive (and single-valued) if and only if T is monotone.*
(ii) $\operatorname{dom} P = \operatorname{range}(T + I)$.

Proof. Exercise 5.1.29. ●

This very easily leads to the Kirszbraun–Valentine theorem [161, 254] on the existence of nonexpansive extensions to all of Hilbert space of nonexpansive mappings on subsets of Hilbert space. The proof is left as a guided exercise.

Theorem 5.1.33 (Kirszbraun–Valentine) *Let H be a Hilbert space and let D be a non-empty subset of H. Suppose that $P\colon D \to H$ is a nonexpansive mapping. Then there exists a nonexpansive mapping $\widehat{P}\colon H \to H$ defined on all of H such that $\widehat{P}|_D = P$.*

Proof. Exercise 5.1.30. ●

Alternatively [226], one may directly associate a convex Fitzpatrick function F_P with a non-expansive mapping P, and thereby derive the Kirszbraun–Valentine theorem, see Exercise 5.1.31.

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ON PROJECTION ALGORITHMS FOR SOLVING CONVEX FEASIBILITY PROBLEMS*

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Abstract. Due to their extraordinary utility and broad applicability in many areas of classical mathematics and modern physical sciences (most notably, computerized tomography), algorithms for solving convex feasibility problems continue to receive great attention. To unify, generalize, and review some of these algorithms, a very broad and flexible framework is investigated. Several crucial new concepts which allow a systematic discussion of questions on behaviour in general Hilbert spaces and on the quality of convergence are brought out. Numerous examples are given.

Key words. angle between two subspaces, averaged mapping, Cimmino's method, computerized tomography, convex feasibility problem, convex function, convex inequalities, convex programming, convex set, Fejér monotone sequence, firmly nonexpansive mapping, Hilbert space, image recovery, iterative method, Kaczmarz's method, linear convergence, linear feasibility problem, linear inequalities, nonexpansive mapping, orthogonal projection, projection algorithm, projection method, Slater point, subdifferential, subgradient, subgradient algorithm, successive projections

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1. Introduction, preliminaries, and notation.

A very common problem in diverse areas of mathematics and physical sciences consists of trying to find a point in the intersection of convex sets. This problem is referred to as the *convex feasibility problem*; its precise mathematical formulation is as follows.

Suppose X is a Hilbert space and $C_1, \ldots, C_N$ are closed convex subsets with *nonempty* intersection C:

$$C = C_1 \cap \cdots \cap C_N \neq \emptyset.$$

Convex feasibility problem: Find some point x in C.

We distinguish two major types.

1. The set C_i is "simple" in the sense that the projection (i.e., the nearest point mapping) onto C_i can be calculated explicitly; C_i might be a hyperplane or a halfspace.

2. It is not possible to obtain the projection onto C_i; however, it is at least possible to describe the projection *onto some approximating superset of* C_i. (There is always a trivial approximating superset of C_i, namely, X.) Typically, C_i is a lower level set of some convex function.

One frequently employed approach in solving the convex feasibility problem is algorithmic. The idea is to involve the projections onto each set C_i (resp., onto a superset of C_i) to generate a sequence of points that is supposed to converge to a solution of the convex feasibility problem. This is the approach we will investigate. We are aware of four distinct (although intertwining) branches, which we classify by their applications.

I. Best approximation theory.

Properties: Each set C_i is a closed subspace. The algorithmic scheme is simple ("cyclic" control).

Basic results: von Neumann [103, Thm. 13.7], Halperin [61].

Comments: The generated sequence converges in norm to the point in C that is *closest* to the starting point. Quality of convergence is well understood.

References: Deutsch [44].

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Areas of application: Diverse. Statistics (linear prediction theory), partial differential equations (Dirichlet problem), and complex analysis (Bergman kernels, conformal mappings), to name only a few.

II. Image reconstruction: Discrete models.

Properties: Each set C_i is a halfspace or a hyperplane. X is a Euclidean space (i.e., a finite-dimensional Hilbert space). Very flexible algorithmic schemes.

Basic results: Kaczmarz [71], Cimmino [29], Agmon [1], Motzkin and Schoenberg [83].

Comments: Behaviour in general Hilbert space and quality of convergence only partially understood.

References: Censor [21, 23, 24], Censor and Herman [27], Viergever [102], Sezan [91].

Areas of application: Medical imaging and radiation therapy treatment planning (computerized tomography), electron microscopy.

III. Image reconstruction: Continuous models.

Properties: X is usually an infinite-dimensional Hilbert space. Fairly simple algorithmic schemes.

Basic results: Gubin, Polyak, and Raik [60].

Comments: Quality of convergence is fairly well understood.

References: Herman [63], Youla and Webb [108], Stark [95].

Areas of application: Computerized tomography, signal processing.

IV. Subgradient algorithms.

Properties: Some sets C_i are of type 2. Fairly simple algorithmic schemes ("cyclic" or "weighted" control).

Basic results: Eremin [52], Polyak [86], Censor and Lent [28].

Comments: Quality of convergence is fairly well understood.

References: Censor [22], Shor [92].

Areas of application: Solution of convex inequalities, minimization of convex nonsmooth functions.

To improve, unify, and review algorithms for these branches, we must study a *flexible algorithmic scheme* in *general Hilbert space* and be able to draw conclusions on the *quality of convergence*. This is our objective in this paper.

We will analyze algorithms in a very broad and adaptive framework that is essentially due to Flåm and Zowe [53]. (Related frameworks with somewhat different ambitions were investigated by Browder [17] and Schott [89].) The algorithmic scheme is as follows.

Given the current iterate $x^{(n)}$, the next iterate $x^{(n+1)}$ is obtained by

$$(*) \qquad x^{(n+1)} := A^{(n)}x^{(n)} := \left(\sum_{i=1}^{N} \lambda_i^{(n)} \left[(1-\alpha_i^{(n)})\mathrm{Id} + \alpha_i^{(n)} P_i^{(n)}\right]\right) x^{(n)},$$

where every $P_i^{(n)}$ is the projection onto some approximating superset $C_i^{(n)}$ of C_i, every $\alpha_i^{(n)}$ is a relaxation parameter between 0 and 2, and the $\lambda_i^{(n)}$'s are nonnegative weights summing up to 1. In short, $x^{(n+1)}$ *is a weighted average of relaxed projections of* $x^{(n)}$.

Censor and Herman [27] expressly suggested the study of a (slightly) restricted version of $(*)$ in the context of computerized tomography. It is worthwhile to point out that the scheme $(*)$ can be thought of as a combination of the schemes investigated by Aharoni, Berman, and Censor [2] and Aharoni and Censor [3]. In Euclidean spaces, norm convergence results were obtained by Flåm and Zowe for $(*)$ and by Aharoni and Censor [3] for the restricted version. However, neither behaviour in general Hilbert spaces nor quality of convergence has been much discussed so far. To do this comprehensively and clearly, it is important to bring out some underlying recurring concepts. We feel these concepts lie at the heart of many algorithms and will be useful for other researchers as well.

The paper is organized as follows.

In §2, the two important concepts of *attracting mappings* and *Fejér monotone sequences* are investigated. The former concept captures essential properties of the *operator* $A^{(n)}$, whereas the latter deals with inherent qualities of the *sequence* $(x^{(n)})$.

The idea of a *focusing algorithm* is introduced in §3. The very broad class of focusing algorithms admits results on convergence. In addition, the well-known ideas of cyclic and weighted control are subsumed under the notion of *intermittent control*. Weak topology results on intermittent focusing algorithms are given. We actually study a more general form of the iteration $(*)$ without extra work; as a by-product, we obtain a recent result by Tseng [100] and make connections with work by Browder [17] and Baillon [7].

At the start of §4, we exclusively consider algorithms such as $(*)$, which we name *projection algorithms*. Prototypes of focusing and *linearly focusing* (a stronger, more quantitative version) projection algorithms are presented. When specialized to Euclidean spaces, our analysis yields basic results by Flåm and Zowe [53] and Aharoni and Censor [3].

The fifth section discusses norm and particularly linear convergence. Many known sufficient sometimes ostensibly different looking conditions for linear convergence can be thought of as special instances of a single new geometric concept—*regularity*. Here the N-tuple $(C_1, \ldots, C_N)$ is called regular if "closeness to all sets C_i implies closeness to their intersection C." Four quantitative versions of (bounded) (linear) regularity are described. Having gotten all the crucial concepts together, we deduce our main results, one of which states in short that

$$\left.\begin{array}{c}\text{linearly focusing projection algorithm}\\ +\\ \text{intermittent control}\\ +\\ \text{"nice" relaxation parameters and weights}\\ +\\ (C_1, \ldots, C_N)\text{ boundedly linearly regular}\end{array}\right\}\text{ imply linear convergence.}$$

This section ends with results on (bounded) (linear) regularity, including a characterization of regular N-tuples of closed subspaces.

Section 6 contains a multitude of examples of algorithms from branches I, II, and III.

The final section examines the subgradient algorithms of branch IV, to which our previous results also apply. Thus, a well-known Slater point condition emerges as a sufficient condition for a subgradient algorithm to be linearly focusing, thus yielding a conceptionally simple proof of an important result by De Pierro and Iusem [40]. It is very satisfactory that analogous results are obtained for algorithms suggested by Dos Santos [47] and Yang and Murty [105].

For the reader's convenience, an index is included.

We conclude this section with a collection of frequently-used facts, definitions, and notation.

The "stage" throughout this paper is a real Hilbert space X; its unit ball $\{x \in X : \|x\| \leq 1\}$ is denoted B_X.

FACTS 1.1.

(i) (parallelogram law) If $x, y \in X$, then
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

(ii) (strict convexity) If $x, y \in X$, then
$$\|x+y\| = \|x\| + \|y\| \quad \text{implies} \quad \|y\| \cdot x = \|x\| \cdot y.$$

(iii) Every bounded sequence in X possesses a weakly convergent subsequence.

Proof. (i) is easy to verify and implies (ii). (iii) follows from the Eberlein–Šmulian theorem (see, for example, Holmes [67, §18]). □

All "actors" turn out to be members of the distinguished class of nonexpansive mappings. A mapping $T : D \longrightarrow X$, where the domain D is a closed convex nonempty subset of X, is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in D.$$

If $\|Tx - Ty\| = \|x - y\|$, for all $x, y \in D$, then we say T is an *isometry*. In contrast, if $\|Tx - Ty\| < \|x - y\|$, for all distinct $x, y \in D$, then we speak of a *strictly nonexpansive mapping*. If T is a nonexpansive mapping, then the set of all *fixed points* Fix T, which is defined by

$$\operatorname{Fix} T = \{x \in D : x = Tx\},$$

is always closed and convex [58, Lem. 3.4].

FACT 1.2 (demiclosedness principle). If D is a closed convex subset of X, $T : D \longrightarrow X$ is nonexpansive, (x_n) is a sequence in D, and $x \in D$, then

$$\left.\begin{array}{r} x_n \rightharpoonup x \\ x_n - Tx_n \to 0 \end{array}\right\} \quad \text{implies} \quad x \in \operatorname{Fix} T,$$

where, by convention, "$\to$" (resp., "$\rightharpoonup$") stands for norm (resp., weak) convergence.

Proof. This is a special case of Opial's [84, Lem. 2]. □

It is obvious that the identity Id is nonexpansive and easy to see that convex combinations of nonexpansive mappings are also nonexpansive. In particular, if N is a nonexpansive mapping, then so is

$$(1 - \alpha)\mathrm{Id} + \alpha N \quad \text{for all } \alpha \in [0, 1[.$$

These mappings are called *averaged mappings*. A *firmly nonexpansive mapping* is a nonexpansive mapping that can be written as

$$\tfrac{1}{2}\mathrm{Id} + \tfrac{1}{2}N \quad \text{for some nonexpansive mapping } N.$$

FACT 1.3. If D is a closed convex subset of X and $T : D \longrightarrow X$ is a mapping, then the following conditions are equivalent.

(i) T is firmly nonexpansive.

(ii) $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in D$.

(iii) $2T - \mathrm{Id}$ is nonexpansive.

Proof. See, for example, Zarantonello's [109, §1] or Goebel and Kirk's [56, Thm. 12.1]. □

A mapping is called *relaxed firmly nonexpansive* if it can be expressed as

$$(1 - \alpha)\mathrm{Id} + \alpha F \quad \text{for some firmly nonexpansive mapping } F.$$

COROLLARY 1.4. *Suppose D is a closed convex subset of X and $T : D \longrightarrow X$ is a mapping. Then T is averaged if and only if it is relaxed firmly nonexpansive.*

The "principal actor" is the projection operator. Given a closed convex nonempty subset C of X, the mapping that sends every point to its nearest point in C (in the norm induced by the inner product of X) is called the *projection onto* C and denoted P_C.

FACTS 1.5. Suppose C is a closed convex nonempty subset of X with projection P_C. Then
(i) P_C is firmly nonexpansive.
(ii) If $x \in X$, then $P_C x$ is characterized by

$$P_C x \in C \quad \text{and} \quad \langle C - P_C x, x - P_C x\rangle \leq 0.$$

Proof. See, for example, [109, Lem. 1.2] for (i) and [109, Lem. 1.1] for (ii). □

Therefore,

$$\begin{gathered}\text{projection}\\ \Downarrow\\ \text{firmly nonexpansive}\\ \Downarrow\\ \text{relaxed firmly nonexpansive} = \text{averaged}\\ \Downarrow\\ \text{isometry} \Rightarrow \text{nonexpansive} \Leftarrow \text{strictly nonexpansive.}\end{gathered}$$

The associated function $d(\cdot, C) : X \longrightarrow \mathbb{R} : x \longmapsto \inf_{c\in C} \|x - c\| = \|x - P_C x\|$ is called the *distance function to* C; it is easy to see that $d(\cdot, C)$ is convex and continuous (hence weakly lower semicontinuous).

A good reference on nonexpansive mappings is Goebel and Kirk's recent book [58]. Many results on projections are in Zarantonello's [109].

The algorithms' quality of convergence will be discussed in terms of linear convergence: a sequence (x_n) in X is said to *converge linearly to its limit x (with rate β)* if $\beta \in [0, 1[$ and there is some $\alpha \geq 0$ such that (s.t.)

$$\|x_n - x\| \leq \alpha\beta^n \qquad \text{for all } n.$$

PROPOSITION 1.6. *Suppose (x_n) is a sequence in X, p is some positive integer, and x is a point in X. If $(x_{pn})_n$ converges linearly to x and $(\|x_n - x\|)_n$ is decreasing, then the entire sequence $(x_n)_n$ converges linearly to x.*

Proof. There is some $\alpha > 0$ and $\beta \in [0, 1[$ s.t.

$$\|x_{pn} - x\| \leq \alpha\beta^n \qquad \text{for all } n.$$

Now fix an arbitrary positive integer m and divide by p with remainder; i.e., write

$$m = p \cdot n + r, \qquad \text{where } r \in \{0, 1, \ldots, p-1\}.$$

We estimate

$$\begin{aligned}\|x_m - x\| &\leq \|x_{pn} - x\| \leq \alpha\left(\beta^{\frac{1}{p}}\right)^{np}\\ &= \frac{\alpha\left(\beta^{\frac{1}{p}}\right)^{np+r}}{\left(\beta^{\frac{1}{p}}\right)^{r}} \leq \left(\frac{\alpha}{\left(\beta^{\frac{1}{p}}\right)^{p-1}}\right)\left(\beta^{\frac{1}{p}}\right)^{m},\end{aligned}$$

and the result follows. □

Finally, we recall the meaning of the following.

If S and Y are any subsets of X, then span S, $\overline{\text{conv}}S$, $\overline{S}$, $\text{int}_Y S$, $\text{icr}S$, and $\text{int}S$ denote, respectively, the *span of S*, the *closed convex hull of S*, the *closure of S*, the *interior of S with* *respect to (w.r.t.)* Y, the *intrinsic core of* S ($= \operatorname{int}_{\overline{\operatorname{aff}} S} S$, where $\overline{\operatorname{aff}} S$ is the *closed affine span* of S), and the *interior of* S ($= \operatorname{int}_X S$).

S is called a *cone* if it is nonempty, convex, and closed under nonnegative scalar multiplication. If S is the intersection of finitely many halfspaces, then S is a *polyhedron*.

If r is a real number, then $r^+ := \max\{r, 0\}$ is called the *positive part of* r. In the context of sequences of real numbers, $\overline{\lim}$ (resp., $\underline{\lim}$) stands for *limes superior* (resp., *limes inferior*). Occasionally, we will use the quantifiers $\forall$ (for all) and $\exists$ (there exists) to avoid monstrous sentences.

2. Two useful tools: Attracting mappings and Fejér monotone sequences. We focus on two important concepts. The first generalizes the idea of averaged (resp., strictly) nonexpansive mappings.

DEFINITION 2.1. Suppose D is a closed convex nonempty set, $T : D \longrightarrow D$ is nonexpansive, and F is a closed convex nonempty subset of D. We say that T is *attracting w.r.t.* F if for every $x \in D \setminus F$, $f \in F$,

$$\|Tx - f\| < \|x - f\|.$$

In other words, every point in F attracts every point outside F. A more quantitative and stronger version is the following.

We say that T is *strongly attracting w.r.t.* F if there is some $\kappa > 0$ s.t. for every $x \in D$, $f \in F$,

$$\kappa \|x - Tx\|^2 \leq \|x - f\|^2 - \|Tx - f\|^2.$$

Alternatively, if we want to emphasize κ explicitly, we say that T is κ*-attracting w.r.t.* F. In several instances, F is $\operatorname{Fix} T$; in this case, we simply speak of *attracting, strongly attracting, or* κ*-attracting mappings*.

REMARKS 2.2. Some authors do not require nonexpansivity in the definition of attracting mappings; see, for example, Bruck's "strictly quasi-nonexpansive mappings" [18], Elsner, Koltracht, and Neumann's "paracontracting mappings" [51], Eremin's "F-weakly Fejér maps" [52], and Istratescu's "T-mappings" [69, Chap. 6]. For our purposes, however, the above definitions are already general enough. As we will see, the class of strongly attracting mappings contains *properly* all averaged nonexpansive mappings and thus all relaxed projections—the mappings we are primarily interested in.

We would like to mention (but will not use) the fact that the class of attracting mappings *properly* contains all three of the following classes: the class of strictly nonexpansive mappings, Bruck and Reich's strongly nonexpansive mappings [19], and a very nice class of nonexpansive mappings introduced by De Pierro and Iusem [41, Def. 1]. The mapping $x \longmapsto 1 - \ln(1 + e^x)$ is a first example of a mapping that is strictly nonexpansive but not averaged; hence the class of attracting mappings is genuinely bigger than the class of strongly attracting mappings. Finally, neither class contains isometries with fixed points.

The asserted proper containment statements are demonstrated by the following example.

EXAMPLE 2.3. Suppose D is a closed convex symmetric interval in $\mathbb{R}$ that contains the interval $[-1, +1]$ strictly. Let

$$T : D \longrightarrow D : x \longmapsto \begin{cases} \frac{1}{2}|x|^2 & \text{if } |x| \leq 1, \\ |x| - \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then

- T is nonexpansive and $\operatorname{Fix} T = \{0\}$.
- T is *not* strictly nonexpansive.

- T is *not* strongly nonexpansive (in the sense of Bruck and Reich [19]); in particular, T is *not* averaged [19, Prop. 1.3].
- T is *not* nonexpansive in the sense of De Pierro and Iusem [41, Def. 1].
- T is attracting.
- T is strongly attracting if and only if D is compact.

LEMMA 2.4 (prototype of a strongly attracting mapping). *Suppose D is a closed convex nonempty set, $T : D \longrightarrow D$ is firmly nonexpansive with fixed points, and $\alpha \in]0, 2[$. Let $R := (1-\alpha)\mathrm{Id} + \alpha T$ and fix $x \in D$, $f \in \operatorname{Fix} T$. Then*

(i) $\operatorname{Fix} R = \operatorname{Fix} T$.
(ii) $\langle x - f, x - Tx\rangle \geq \|x - Tx\|^2$ *and* $\langle x - Tx, Tx - f\rangle \geq 0$.
(iii) $\|x - f\|^2 - \|Rx - f\|^2 = 2\alpha\langle x - f, x - Tx\rangle - \alpha^2\|x - Tx\|^2$.
(iv) *R is $(2-\alpha)/\alpha$-attracting:*

$$\|x - f\|^2 - \|Rx - f\|^2 \geq (2-\alpha)/\alpha\|x - Rx\|^2 = (2-\alpha)\alpha\|x - Tx\|^2.$$

Proof. (i) is immediate.
(ii): Since T is firmly nonexpansive, we obtain

$$\begin{aligned}
&\|Tx - f\|^2 \leq \langle Tx - f, x - f\rangle \\
\iff &\|Tx - x\|^2 + \|x - f\|^2 + 2\langle Tx - x, x - f\rangle \leq \langle Tx - f, x - f\rangle \\
\iff &\|Tx - x\|^2 \leq \langle x - Tx, x - f\rangle = \langle x - Tx, (x - Tx) + (Tx - f)\rangle \\
\iff &0 \leq \langle x - Tx, Tx - f\rangle.
\end{aligned}$$

(iii) is a direct calculation:

$$\begin{aligned}
&\|x - f\|^2 - \|Rx - f\|^2 \\
&= \|x - f\|^2 - \|(1-\alpha)(x - f) + \alpha(Tx - f)\|^2 \\
&= \|x - f\|^2 - [(1-\alpha)^2\|x - f\|^2 + \alpha^2\|Tx - f\|^2 + 2\alpha(1-\alpha)\langle x - f, Tx - f\rangle] \\
&= 2\alpha\|x - f\|^2 - \alpha^2\|x - f\|^2 - \alpha^2\|Tx - f\|^2 \\
&\quad + 2\alpha^2\langle x - f, Tx - f\rangle - 2\alpha\langle x - f, Tx - f\rangle \\
&= 2\alpha\langle x - f, (x - f) - (Tx - f)\rangle \\
&\quad - \alpha^2[\|x - f\|^2 + \|Tx - f\|^2 - 2\langle x - f, Tx - f\rangle] \\
&= 2\alpha\langle x - f, x - Tx\rangle - \alpha^2\|x - Tx\|^2.
\end{aligned}$$

(iv): By (ii), (iii), and the definition of R, we get

$$\begin{aligned}
\|x - f\|^2 - \|Rx - f\|^2 &= 2\alpha\langle x - f, x - Tx\rangle - \alpha^2\|x - Tx\|^2 \\
&\geq 2\alpha\|x - Tx\|^2 - \alpha^2\|x - Tx\|^2 \\
&= \alpha(2-\alpha)\|x - Tx\|^2 \\
&= (2-\alpha)/\alpha\|x - Rx\|^2. \qquad \square
\end{aligned}$$

Note that (i) and (iii) are actually true for an arbitrary nonexpansive mapping T; this will, however, not be needed in what follows. Since projections are firmly nonexpansive (Facts 1.5.(i)), we immediately obtain the following result which slightly improves Flåm and Zowe's [53, Lem. 1].

COROLLARY 2.5. *If P is the projection onto some closed convex nonempty set S and $\alpha \in]0, 2[$, then $R := (1-\alpha)\mathrm{Id} + \alpha P$ is $(2-\alpha)/\alpha$-attracting w.r.t. S and for $x \in X$, $s \in S$,*

$$\|x - s\|^2 - \|Rx - s\|^2 \geq (2-\alpha)\alpha d^2(x, S).$$

DEFINITION 2.6. Suppose (x_n) is a sequence in X. We say that (x_n) is *asymptotically regular* if

$$x_n - x_{n+1} \longrightarrow 0.$$

EXAMPLE 2.7. Suppose D is a closed convex nonempty set, F is a closed convex nonempty subset of D, and $(T_n)_{n\geq 0}$ is a sequence of nonexpansive self mappings of D, where each T_n is κ_n-attracting w.r.t. F and $\underline{\lim}_n \kappa_n > 0$. Suppose further the sequence (x_n) is defined by

$$x_0 \in D \text{ arbitrary}, \qquad x_{n+1} := T_n x_n \qquad \text{for all } n \geq 0.$$

Then (x_n) is asymptotically regular.

Proof. Fix $f \in F$ and choose $0 < \kappa < \underline{\lim}_n \kappa_n$. Then for all large n,

$$\kappa \|x_{n-1} - x_n\|^2 \leq \|x_{n-1} - f\|^2 - \|x_n - f\|^2.$$

Summing these inequalities shows that $\sum_n \|x_{n-1} - x_n\|^2$ is finite; the result follows. □

COROLLARY 2.8. *Suppose D is a closed convex nonempty set and $T : D \longrightarrow D$ is strongly attracting with fixed points. Then the sequence of iterates $(T^n x_0)_{n\geq 0}$ is asymptotically regular for every $x_0 \in D$.*

REMARK 2.9. The corollary is well known for firmly nonexpansive and, more generally, strongly nonexpansive mappings (see [19, Prop. 1.3 and Cor. 1.1]). In the literature, the conclusion of the corollary is often "T is asymptotically regular at x_0." We hope the reader accepts this as an a posteriori justification for introducing the notion of an asymptotically regular sequence.

The next propositions show that (strongly) attracting mappings respect compositions and convex combinations.

PROPOSITION 2.10. *Suppose D is a closed convex nonempty set, $T_1, \ldots, T_N : D \longrightarrow D$ are attracting, and $\bigcap_{i=1}^N \operatorname{Fix} T_i$ is nonempty. Then*

(i) $\operatorname{Fix}(T_N T_{N-1} \cdots T_1) = \bigcap_{i=1}^N \operatorname{Fix} T_i$ *and $T_N T_{N-1} \cdots T_1$ is attracting.*

(ii) *If every T_i is κ_i-attracting, then $T_N T_{N-1} \cdots T_1$ is $\min\{\kappa_1, \ldots, \kappa_N\}/2^{N-1}$-attracting.*

Proof. It is enough to prove the proposition for $N = 2$; the general case follows inductively.

(i): Clearly, $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 \subseteq \operatorname{Fix}(T_2 T_1)$. To prove the other inclusion, pick $f \in \operatorname{Fix}(T_2 T_1)$. It is enough to show that $f \in \operatorname{Fix} T_1$. If this were false, then $T_1 f \notin \operatorname{Fix} T_2$. Now fix $\bar{f} \in \operatorname{Fix} T_1 \cap \operatorname{Fix} T_2$. Then, since T_2 is attracting,

$$\|f - \bar{f}\| = \|T_2(T_1 f) - \bar{f}\| < \|T_1 f - \bar{f}\| \leq \|f - \bar{f}\|,$$

which is absurd. Thus $\operatorname{Fix} T_1 \cap \operatorname{Fix} T_2 = \operatorname{Fix}(T_2 T_1)$. It remains to show that $T_2 T_1$ is attracting. Fix $x \in D \setminus \operatorname{Fix}(T_2 T_1)$, $f \in \operatorname{Fix}(T_2 T_1)$. If $x = T_1 x$, then $T_2 x \neq x$ and hence $\|T_2 T_1 x - f\| = \|T_2 x - f\| < \|x - f\|$. Otherwise $x \neq T_1 x$; then $\|T_2 T_1 x - f\| \leq \|T_1 x - f\| < \|x - f\|$. In either case, $T_2 T_1$ is attracting.

(ii): Given $x \in D$, $f \in \operatorname{Fix}(T_2 T_1)$, we estimate

$$\begin{aligned}
\|x - T_2 T_1 x\|^2 &\leq (\|x - T_1 x\| + \|T_1 x - T_2 T_1 x\|)^2 \\
&\leq 2(\|x - T_1 x\|^2 + \|T_1 x - T_2 T_1 x\|^2) \\
&\leq \frac{2}{\kappa_1}(\|x - f\|^2 - \|T_1 x - f\|^2) \\
&\quad + \frac{2}{\kappa_2}(\|T_1 x - f\|^2 - \|T_2 T_1 x - f\|^2) \\
&\leq \frac{2}{\min\{\kappa_1, \kappa_2\}}(\|x - f\|^2 - \|T_2 T_1 x - f\|^2). \qquad \square
\end{aligned}$$

REMARK 2.11. Let $X \supsetneqq \{0\}$, $D := X$, $N := 2$, and $T_1 := T_2 := -\mathrm{Id}$. Then

$$\mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 = \{0\} \subsetneqq X = \mathrm{Fix}\,(T_2 T_1);$$

hence the formula on the fixed point sets given in (i) of the last proposition does not hold in general for nonexpansive mappings.

PROPOSITION 2.12. *Suppose D is a closed convex nonempty set, $T_1, \ldots, T_N : D \longrightarrow D$ are attracting, and $\bigcap_{i=1}^N \mathrm{Fix}\, T_i$ is nonempty. Suppose further $\lambda_1, \ldots, \lambda_N > 0$ with $\sum_{=1}^N \lambda_i = 1$. Then*

(i) *$\mathrm{Fix}\,(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N \mathrm{Fix}\, T_i$ and $\sum_{i=1}^N \lambda_i T_i$ is attracting.*

(ii) *If every T_i is κ_i-attracting, then $\sum_{i=1}^N \lambda_i T_i$ is $\min\{\kappa_1, \ldots, \kappa_N\}$-attracting.*

Proof. Again we have only to consider the case when $N = 2$.

(i): Once more, $\mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 \subseteq \mathrm{Fix}\,(\lambda_1 T_1 + \lambda_2 T_2)$. Conversely, pick $f \in \mathrm{Fix}\,(\lambda_1 T_1 + \lambda_2 T_2)$, $\bar{f} \in \mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2$. Then

$$\begin{aligned}
\|f - \bar{f}\| &= \|\lambda_1 T_1 f + \lambda_2 T_2 f - \lambda_1 \bar{f} - \lambda_2 \bar{f}\| \\
&\leq \lambda_1 \|T_1 f - \bar{f}\| + \lambda_2 \|T_2 f - \bar{f}\| \\
&\leq \lambda_1 \|f - \bar{f}\| + \lambda_2 \|f - \bar{f}\| = \|f - \bar{f}\|.
\end{aligned}$$

Hence the above chain of inequalities is actually one of equalities. This, together with the strict convexity of X, implies $f = T_1 f = T_2 f$. Next, we show that $\lambda_1 T_1 + \lambda_2 T_2$ is attracting. Suppose $x \neq \lambda_1 T_1 x + \lambda_2 T_2 x$ and $f \in \mathrm{Fix}\,(\lambda_1 T_1 + \lambda_2 T_2)$. Then $x \notin \mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2$ and thus

$$\begin{aligned}
\|\lambda_1 T_1 x + \lambda_2 T_2 x - f\| &\leq \lambda_1 \|T_1 x - f\| + \lambda_2 \|T_2 x - f\| \\
&< \lambda_1 \|x - f\| + \lambda_2 \|x - f\| = \|x - f\|.
\end{aligned}$$

(ii): If $\kappa := \min\{\kappa_1, \kappa_2\}$, $x \in D$, and $f \in \mathrm{Fix}\,(\lambda_1 T_1 + \lambda_2 T_2)$, then

$$\begin{aligned}
\kappa \|x - (\lambda_1 T_1 x + \lambda_2 T_2 x)\|^2 &\leq \kappa(\lambda_1 \|x - T_1 x\| + \lambda_2 \|x - T_2 x\|)^2 \\
&\leq \kappa(\lambda_1 \|x - T_1 x\|^2 + \lambda_2 \|x - T_2 x\|^2) \\
&\leq \lambda_1 \kappa_1 \|x - T_1 x\|^2 + \lambda_2 \kappa_2 \|x - T_2 x\|^2 \\
&\leq \lambda_1 (\|x - f\|^2 - \|T_1 x - f\|^2) \\
&\quad + \lambda_2 (\|x - f\|^2 - \|T_2 x - f\|^2) \\
&\leq \|x - f\|^2 - \|(\lambda_1 T_1 x + \lambda_2 T_2 x) - f\|^2. \qquad \square
\end{aligned}$$

REMARK 2.13. In contrast to the last remark, the above proof shows that the formula $\mathrm{Fix}\,(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N \mathrm{Fix}\, T_i$ holds even if the T_i's are not attracting.

EXAMPLE 2.14. Suppose $S_1, \ldots, S_N$ are closed convex nonempty sets with projections $P_1, \ldots, P_N$ and with nonempty intersection. Then

$$T := \frac{P_1 + P_2 P_1 + \cdots + P_N P_{N-1} \cdots P_1}{N}$$

is strongly attracting, $\mathrm{Fix}\, T = \bigcap_{i=1}^N S_i$, and the sequence of iterates $(T^n x_0)$ is asymptotically regular for every x_0.

The second concept captures essential properties of iterates of nonexpansive mappings.

DEFINITION 2.15. Suppose C is a closed convex nonempty set and (x_n) is a sequence in X. We say that $(x_n)_{n \geq 0}$ is *Fejér monotone w.r.t. C* if

$$\|x_{n+1} - c\| \leq \|x_n - c\| \quad \text{for all } c \in C \text{ and every } n \geq 0.$$

THEOREM 2.16 (basic properties of Fejér monotone sequences). *Suppose the sequence* $(x_n)_{n\geq 0}$ *is Fejér monotone w.r.t. C. Then*

(i) (x_n) *is bounded and* $d(x_{n+1}, C) \leq d(x_n, C)$.

(ii) (x_n) *has* at most *one weak cluster point in C. Consequently,* (x_n) *converges weakly to some point in C if and only if* all *weak cluster points of* (x_n) *lie in C.*

(iii) *If the interior of C is nonempty, then* (x_n) *converges in norm.*

(iv) *The sequence* $(P_C x_n)$ *converges in norm.*

(v) *The following are equivalent:*

1. (x_n) *converges in norm to some point in C.*
2. (x_n) *has norm cluster points, all lying in C.*
3. (x_n) *has norm cluster points, one lying in C.*
4. $d(x_n, C) \longrightarrow 0$.
5. $x_n - P_C x_n \longrightarrow 0$.

Moreover, if (x_n) *converges to some* $x \in C$, *then* $\|x_n - x\| \leq 2d(x_n, C)$ *for all* $n \geq 0$.

(vi) *If there is some constant* $\alpha > 0$ *s.t.* $\alpha d^2(x_n, C) \leq d^2(x_n, C) - d^2(x_{n+1}, C)$ *for every n, then* (x_n) *converges linearly to some point x in C. More precisely,*

$$\|x_n - x\| \leq 2(1-\alpha)^{n/2} d(x_0, C) \quad \textit{for every } n \geq 0.$$

Proof. (i) is obvious.

(ii): For any $c \in C$, the sequence $(\|x_n\|^2 - 2\langle x_n, c\rangle)$ converges. Hence if we suppose c_1, c_2 are two weak cluster points of (x_n) in C, then we conclude that the sequence $(\langle x_n, c_1 - c_2\rangle)$ converges and that $\langle c_1, c_1 - c_2\rangle = \langle c_2, c_1 - c_2\rangle$. Thus $c_1 = c_2$.

(iii): Fix $c_0 \in \operatorname{int} C$ and get $\epsilon > 0$ s.t. $c_0 + \epsilon B_X \subseteq C$.

Claim: $$2\epsilon\|x_n - x_{n+1}\| \leq \|x_n - c_0\|^2 - \|x_{n+1} - c_0\|^2 \quad \text{for all } n \geq 0.$$

We can assume $x_n \neq x_{n+1}$. Then $c_0 + \epsilon(x_n - x_{n+1})/\|x_n - x_{n+1}\| \in C$ and hence, by Fejér monotonicity,

$$\left\|\left(c_0 + \epsilon\frac{x_n - x_{n+1}}{\|x_n - x_{n+1}\|}\right) - x_{n+1}\right\| \leq \left\|\left(c_0 + \epsilon\frac{x_n - x_{n+1}}{\|x_n - x_{n+1}\|}\right) - x_n\right\|.$$

Squaring and expanding yields the claim.

The claim implies $2\epsilon\|x_n - x_{n+k}\| \leq \|x_n - c_0\|^2 - \|x_{n+k} - c_0\|^2$ for all $n, k \geq 0$. Because the sequence $(\|x_n - c_0\|^2)$ converges, we recognize (x_n) as a Cauchy sequence.

(iv): Applying the parallelogram law $\|a - b\|^2 = 2\|a\|^2 + 2\|b\|^2 - \|a + b\|^2$ to $a := P_C x_{n+k} - x_{n+k}$ and $b := P_C x_n - x_{n+k}$, we obtain for all $n, k \geq 0$,

$$\begin{aligned}
\|P_C x_{n+k} - P_C x_n\|^2 &= 2\|P_C x_{n+k} - x_{n+k}\|^2 + 2\|P_C x_n - x_{n+k}\|^2 \\
&\quad - 4\|(P_C x_{n+k} + P_C x_n)/2 - x_{n+k}\|^2 \\
&\leq 2\|P_C x_{n+k} - x_{n+k}\|^2 + 2\|P_C x_n - x_{n+k}\|^2 \\
&\quad - 4\|P_C x_{n+k} - x_{n+k}\|^2 \\
&\leq 2\|P_C x_n - x_{n+k}\|^2 - 2\|P_C x_{n+k} - x_{n+k}\|^2 \\
&\leq 2(\|P_C x_n - x_n\|^2 - \|P_C x_{n+k} - x_{n+k}\|^2).
\end{aligned}$$

We identify $(P_C x_n)$ as a Cauchy sequence because $(\|x_n - P_C x_n\|)$ converges by (i).

(v): The equivalences follow easily from (i), (iv), and the definition of Fejér monotonicity. The estimate follows from letting k tend to infinity in

$$\begin{aligned}\|x_{n+k} - x_n\| &\le \|x_{n+k} - P_C x_n\| + \|P_C x_n - x_n\| \\ &\le \|x_n - P_C x_n\| + \|P_C x_n - x_n\| = 2d(x_n, C).\end{aligned}$$

(vi): Summing the given inequalities shows that $d^2(x_n, C)$ tends to 0; therefore, (x_n) converges to some point x in C by (v). The estimate on the rate of convergence of (x_n) follows easily from the estimate given in (v). □

REMARKS 2.17. As far as we know, the notion of Fejér monotonicity was coined by Motzkin and Schoenberg [83] in 1954. Moreau [81] inspired (iii); see also [69, Thm. 6.5.3]. (iv) rests on an idea by Baillon and Brezis [8, Lemme 3] and partially extends [46, Thm. 3.4.(c)]. Finally, (v) and (vi) appeared implicitly in Gubin, Polyak, and Raik's [60, Proof of Lem. 6].

EXAMPLE 2.18 (Krasnoselski/Mann iteration). Suppose C is a closed convex nonempty set, $T : C \longrightarrow C$ is nonexpansive with fixed points, and the sequence (x_n) is given by

$$x_0 \in C, \qquad x_{n+1} := (1 - t_n)x_n + t_n T x_n$$

for all $n \ge 0$ and some sequence $(t_n)_{n\ge 0}$ in $[0, 1]$. Then (x_n) is Fejér monotone w.r.t. Fix T.

REMARKS 2.19. In the early days, the Krasnoselski/Mann iteration was studied in Hilbert space. Some authors then implicitly used properties of Fejér monotone sequences; see [79, Proof of Thm. 1] and [90, Proof of Thm. 2]. However, tremendous progress has been made and today the iteration is studied in normed or even more general spaces (see [15, 57] for further information).

EXAMPLE 2.20 (Example 2.14 continued). The sequence $(T^n x_0)$ converges weakly to some fixed point of T for every x_0.

Proof. $(T^n x_0)$ is asymptotically regular (Example 2.14) and Fejér monotone w.r.t. Fix T (Example 2.18). By the demiclosedness principle, every weak limit point of $(T^n x_0)$ lies in Fix T. The result now follows from Theorem 2.16.(ii). □

REMARK 2.21. Alternatively, one can use Baillon, Bruck, and Reich's results on averaged mappings [9, Thms. 1.2 and 2.1] to understand the last example. In fact, using a suitable modification of [9, Thm. 1.1], one can show that $(T^n x_0)$ converges *in norm* whenever $S_1, \ldots, S_N$ are closed affine subspaces.

REMARK 2.22. We conclude this section by mentioning a method due to Halpern [62] which generates a sequence that converges *in norm* to the fixed point of T that is closest to the starting point. For extensions of Halpern's result, the interested reader is referred to Lions's [77], Wittmann's [104], and the first author's [11].

3. The algorithm: Basic properties and convergence results.

Setting. Suppose D is a closed convex nonempty set and $C_1, \ldots, C_N$ are finitely many closed convex subsets of D with nonempty intersection:

$$C := \bigcap_{i=1}^N C_i \neq \emptyset.$$

For every $i \in \{1, \ldots, N\}$ (we will often refer to i as an *index*) and all $n \ge 0$, suppose that $T_i^{(n)} : D \longrightarrow D$ is firmly nonexpansive with

$$\text{Fix } T_i^{(n)} \supseteq C_i,$$

that $\alpha_i^{(n)} \in [0, 2]$ is a *relaxation parameter* and

$$R_i^{(n)} := (1 - \alpha_i^{(n)})\text{Id} + \alpha_i^{(n)} T_i^{(n)}$$

is the corresponding relaxation of $T_i^{(n)}$ (*underrelaxed* if $\alpha_i^{(n)} \in [0, 1]$, *overrelaxed* if $\alpha_i^{(n)} \in [1, 2]$), that $(\lambda_i^{(n)})_{i=1}^N$ in $[0, 1]$ is a *weight*, i.e., $\sum_{i=1}^N \lambda_i^{(n)} = 1$, and finally that

$$A^{(n)} := \sum_{i=1}^N \lambda_i^{(n)} R_i^{(n)}$$

is the corresponding weighted average of the relaxations.

With these abbreviations, we define the *algorithm* by the *sequence*

$$x^{(0)} \in D \text{ arbitrary}, \qquad x^{(n+1)} := A^{(n)} x^{(n)} \quad \text{for all } n \geq 0,$$

with the implicit assumption that the sequence $(x^{(n)})$ lies in D. We also define the set of *active indices*

$$I^{(n)} := \{i \in \{1, \ldots, N\} : \lambda_i^{(n)} > 0\},$$

and we say *i is active at n* or *n is active for i* if $\lambda_i^{(n)} > 0$; i.e., $i \in I^{(n)}$. We always assume that *every index is picked infinitely often*; i.e., i is active at infinitely many n (this is sometimes referred to as *repetitive control*; see [22]). To facilitate the presentation, we abbreviate

$$\mu_i^{(n)} := \lambda_i^{(n)} \alpha_i^{(n)} [2 - \textstyle\sum_{j=1}^N \lambda_j^{(n)} \alpha_j^{(n)}] \quad \text{for every index } i \text{ and } n \geq 0.$$

For convenience, we introduce some more notions. We say the algorithm is *asymptotically regular* if every sequence generated by the algorithm is. We say the algorithm is *unrelaxed* if $\alpha_i^{(n)} = 1$ for all n and every index i active at any n; note that in this case the algorithm reduces to a product of firmly nonexpansive mappings. We say the algorithm is *singular* if $I^{(n)}$ is a singleton for every n. Singular algorithms are also called *row–action methods* (see, for example, [21]). Finally, we say the algorithm is *weighted* if $I^{(n)} = \{1, \ldots, N\}$ for all n; the reader may also find the words "parallel" or "simultaneous" in the literature.

REMARK 3.1. The algorithm is a direct generalization of Flåm and Zowe's algorithm [53]. In fact, one just chooses X finite dimensional and $T_i^{(n)}$ as the projection onto a hyperplane containing C_i. We will examine their algorithm in detail in §4.

LEMMA 3.2 (basic properties of the algorithm).

(i) *If $x \in D$ and $n \geq 0$, then*

$$\begin{aligned}\|x^{(n)} - x\|^2 - \|x^{(n+1)} - x\|^2 = &\sum_{i<j} \lambda_i^{(n)} \lambda_j^{(n)} \alpha_i^{(n)} \alpha_j^{(n)} \|T_i^{(n)} x^{(n)} - T_j^{(n)} x^{(n)}\|^2 \\ &+ 2\sum_i \lambda_i^{(n)} \alpha_i^{(n)} \langle x^{(n)} - T_i^{(n)} x^{(n)}, T_i^{(n)} x^{(n)} - x \rangle \\ &+ \sum_i \lambda_i^{(n)} \alpha_i^{(n)} [2 - \textstyle\sum_j \lambda_j^{(n)} \alpha_j^{(n)}] \|x^{(n)} - T_i^{(n)} x^{(n)}\|^2.\end{aligned}$$

(ii) *If $x \in \bigcap_{i \in I^{(n)}} C_i$ and $n \geq 0$, then*

$$\|x^{(n)} - x\|^2 - \|x^{(n+1)} - x\|^2 \geq \sum_i \mu_i^{(n)} \|x^{(n)} - T_i^{(n)} x^{(n)}\|^2.$$

(iii) *If $x \in \bigcap_{l=n}^{m-1} \bigcap_{i \in I^{(l)}} C_i$ and $m \geq n \geq 0$, then*

$$\|x^{(n)} - x\|^2 - \|x^{(m)} - x\|^2 \geq \sum_{l=n}^{m-1} \sum_i \mu_i^{(l)} \|x^{(l)} - T_i^{(l)} x^{(l)}\|^2.$$

In particular, this estimate holds whenever $x \in C$.

(iv) *The sequence* $(x^{(n)})$ *is Fejér monotone w.r.t.* C *and hence is bounded. Also,*

$$+\infty > \sum_{l=0}^{+\infty} \sum_i \mu_i^{(l)} \|x^{(l)} - T_i^{(l)} x^{(l)}\|^2.$$

(v) *If* $n \geq 0$, *then*

$$\|x^{(n+1)} - x^{(n)}\| \leq \sum_i \lambda_i^{(n)} \alpha_i^{(n)} \|x^{(n)} - T_i^{(n)} x^{(n)}\|.$$

Proof. (i): We omit the somewhat tedious and lengthy but elementary calculation. It is easy to see that "(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv)" (use Lemma 2.4.(ii)). (v) is immediate by the definition of the algorithm. □

Using the tools from the previous section, we obtain the following corollary.

COROLLARY 3.3 (sufficient conditions for norm convergence).

(i) *If the interior of* C *is nonempty, then the sequence* $(x^{(n)})$ *converges in norm to some point in* D.

(ii) *If the sequence* $(x^{(n)})$ *has a subsequence* $(x^{(n')})$ *with* $d(x^{(n')}, C) \longrightarrow 0$, *then the entire sequence* $(x^{(n)})$ *converges in norm to some point in* C.

Proof. This follows from Lemma 3.2.(iv) and Theorem 2.16. □

REMARK 3.4. If the interior of C is empty, then the convergence might only be weak: Genel and Lindenstrauss [56] present an example of a firmly nonexpansive self mapping T of some closed convex nonempty set in ℓ_2 such that the sequence of iterates $(T^n x_0)$ converges weakly *but not in norm* for some starting point x_0. (A *norm convergent* method is mentioned in Remark 2.22.) Since $(T^n x_0)$ is Fejér monotone w.r.t. Fix T (Example 2.18), we conclude that Fix T has empty interior and norm convergence of the algorithm needs *some* hypothesis.

COROLLARY 3.5 (asymptotically regular algorithms). *The algorithm is asymptotically regular whenever*

(i) $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index* i *or*

(ii) $\overline{\lim}_{n:n \text{ active for } i}\, \alpha_i^{(n)} < 2$ *for every index* i.

Proof. (i): There exists an $\epsilon > 0$ s.t. for all n sufficiently large, $\mu_i^{(n)} \geq \epsilon$ for every index i active at n. Lemma 3.2.(iv) implies that $\sum_n \sum_{i:i \text{ active at } n} \|x^{(n)} - T_i x^{(n)}\|^2$ is finite. Consequently,

$$\textstyle\sum_{i:i \text{ active at } n} \|x^{(n)} - T_i x^{(n)}\| \longrightarrow 0.$$

On the other hand, by Lemma 3.2.(v),

$$\|x^{(n+1)} - x^{(n)}\| \leq \textstyle\sum_{i:i \text{ active at } n} \lambda_i^{(n)} \alpha_i^{(n)} \|x^{(n)} - T_i x^{(n)}\|.$$

Hence $(x^{(n)})$ is asymptotically regular.

(ii): By Lemma 2.4.(iv) and Proposition 2.12, every $A^{(n)}$ is κ_n-attracting w.r.t. C, where $\kappa_n := \min\{(2 - \alpha_i^{(n)})/\alpha_i^{(n)} : i \text{ active at } n\}$. The hypothesis guarantees $\underline{\lim}_n \kappa_n > 0$, so the conclusion follows from Example 2.7. □

The following simple example shows that the algorithm is not necessarily asymptotically regular.

EXAMPLE 3.6. Suppose $X := \mathbb{R}$, $N := 1$, $T_1^{(n)} :\equiv P_{\{0\}} = 0$, $\alpha_1^{(n)} :\equiv 2$. Then $x^{(n)} = (-1)^n x^{(0)}$; hence the sequence $(x^{(n)})$ is not asymptotically regular for $x^{(0)} \neq 0$.

The algorithm should converge at least weakly to some point; however, as the last example shows, further assumptions are necessary.

DEFINITION 3.7. We say the algorithm is *focusing* if for every index i and every subsequence $(x^{(n_k)})_k$ of the algorithm,

$$\left.\begin{array}{c} x^{(n_k)} \rightharpoonup x \\ x^{(n_k)} - T_i^{(n_k)} x^{(n_k)} \to 0 \\ i \text{ active at } n_k \end{array}\right\} \text{ implies } x \in C_i.$$

Thanks to the demiclosedness principle, we immediately obtain a first example.

EXAMPLE 3.8. Suppose $N := 1$, $T_1^{(n)} :\equiv T$, and $C_1 := \operatorname{Fix} T$. Then the algorithm is focusing.

REMARK 3.9. As almost all upcoming results show, the concept of a focusing algorithm is crucial. It can be viewed as a generalization of the demiclosedness principle for firmly nonexpansive mappings. The concept itself is investigated in Proposition 3.16 (cf. Theorem 4.3, Corollary 4.9, Theorem 7.7, and Theorem 7.12).

THEOREM 3.10 (dichotomy I). *Suppose the algorithm is focusing. If* $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index* i, *then the sequence* $(x^{(n)})$ *either converges in norm to some point in* C *or has no norm cluster points at all.*

Proof. In view of the Fejér monotonicity of $(x^{(n)})$ and Theorem 2.16.(v), it suffices to show that any norm cluster point of $(x^{(n)})$ lies in C. Suppose to the contrary that the theorem were false. Then there is some subsequence $(x^{(n_k)})_k$ converging to some point $x \notin C$. Let us define

$$I_{\text{in}} := \{i \in \{1, \ldots, N\} : x \in C_i\} \quad \text{and} \quad I_{\text{out}} := \{i \in \{1, \ldots, N\} : x \notin C_i\};$$

then I_{out} is nonempty. We assume, after passing to a subsequence if necessary, that

$$I^{(n_k)} \cup I^{(n_k+1)} \cup \cdots \cup I^{(n_{k+1}-1)} = \{1, \ldots, N\}.$$

Now get $m_k \in \{n_k, \ldots, n_{k+1} - 1\}$ minimal s.t. $I^{(m_k)} \cap I_{\text{out}} \neq \emptyset$. Thus for $n_k \leq m < m_k$, we have $I^{(m)} \subseteq I_{\text{in}}$. Since $x \in \bigcap_{i \in I_{\text{in}}} C_i$, Lemma 3.2.(iii) yields $\|x^{(n_k)} - x\| \geq \|x^{(m_k)} - x\|$, which implies

$$x^{(m_k)} \longrightarrow x. \tag{1}$$

After passing to another subsequence if necessary, we can assume that there is some index i s.t.

$$i \in I^{(m_k)} \cap I_{\text{out}} \qquad \text{for all } k. \tag{2}$$

By Lemma 3.2.(iv), $+\infty > \sum_k \mu_i^{(m_k)} \|x^{(m_k)} - T_i^{(m_k)} x^{(m_k)}\|^2$. By (2) and the hypothesis on $(\mu_i^{(n)})$, we conclude that

$$x^{(m_k)} - T_i^{(m_k)} x^{(m_k)} \longrightarrow 0. \tag{3}$$

Because the algorithm is focusing, (1), (2), and (3) imply $x \in C_i$, which is a contradiction to $i \in I_{\text{out}}$. Therefore, the proof is complete. □

REMARKS 3.11. The finite-dimensional version of the last theorem is relatively recent and was discovered (in some form or another) independently by Flåm and Zowe [53], Tseng [100], and Elsner, Koltracht, and Neumann [51]. Unfortunately, since firmly nonexpansive mappings are not weakly continuous in general (see, for example, Zarantonello's [109, Example on p. 245]), *the proof does not work in the weak topology.*

COROLLARY 3.12. *Suppose X is finite dimensional and the algorithm is focusing. If* $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index i, then the sequence $(x^{(n)})$ converges in norm to some point in C.*

Proof. $(x^{(n)})$ is bounded (Lemma 3.2.(iv)) and hence possesses a norm cluster point. Now apply Theorem 3.10. □

REMARK 3.13 (guaranteeing the "liminf" condition). A simple way to guarantee $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for some index i is to assume the existence of some $\epsilon > 0$ s.t.

$$\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon \quad \text{and} \quad \epsilon \leq \lambda_i^{(n)} \quad \text{for all large } n \text{ active at } i,$$

because then $\mu_i^{(n)} \geq \epsilon^3$. Moreover, this assumption is *equivalent* to (cf. to Corollary 3.5)

$$\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0 \quad \text{and} \quad \overline{\lim}_{n:n \text{ active for } i}\, \alpha_i^{(n)} < 2.$$

Flåm and Zowe [53] used this assumption with great success; see also Example 4.18.

EXAMPLE 3.14 (Tseng's framework [100, Thm. 1]). Suppose X is finite dimensional and the algorithm is singular. Suppose further $T_i^{(n)} :\equiv T_i$, $C_i :=$ Fix T_i, and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2-\epsilon$ for all n and every index i. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. The demiclosedness principle readily implies that the algorithm is focusing. By the last remark, $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for every index i. The result follows from the last corollary. □

DEFINITION 3.15. Given an algorithm, we say that $(T_i^{(n)})$ *converges actively pointwise* to T_i for some index i if

$$\lim_{n:n \text{ active for } i} T_i^{(n)} d = T_i d \quad \text{for every } d \in D.$$

PROPOSITION 3.16 (prototype of a focusing algorithm). *Suppose $T_1, \ldots, T_N : D \longrightarrow D$ are firmly nonexpansive and let $C_i :=$ Fix T_i for every index i. If $(T_i^{(n)})$ converges actively pointwise to T_i for every index i, then the algorithm is focusing.*

Proof. Fix an index i and a subsequence $(x^{(n_k)})$ of $(x^{(n)})$ with $x^{(n_k)} \rightharpoonup x \in D$, $x^{(n_k)} - T_i^{(n_k)} x^{(n_k)} \to 0$, and i active at every n_k. We must show that $x \in C_i$. Fix any $u \in X$. Because $T_i^{(n_k)} P_D$ is nonexpansive,

$$\langle (x^{(n_k)} - u) - (T_i^{(n_k)} x^{(n_k)} - T_i^{(n_k)} P_D u), x^{(n_k)} - u \rangle \geq 0 \quad \text{for all } k.$$

Let k tend to infinity; then, by hypothesis on $(T_i^{(n)})$ and $(x^{(n_k)})$, we conclude that

$$\langle T_i P_D u - u, x - u \rangle \geq 0.$$

Since u has been chosen arbitrarily, we might as well choose $u = x + tv$, where v is an arbitrary vector and $t > 0$. Then

$$\langle T_i P_D(x + tv) - (x + tv), v \rangle \leq 0;$$

hence, by letting t tend to 0, we get $\langle T_i P_D x - x, v \rangle \leq 0$. For $v = x - T_i P_D x$, we obtain

$$x = T_i P_D x.$$

But $x \in D$, so $P_D x = x$ and therefore $x \in$ Fix $T_i = C_i$. □

REMARKS 3.17.

- The last proof is a special case of an argument of Baillon [7, Chapitre 6, Démonstration du Théorème I.3].
- Note that the last proposition gives another explanation of the fact that the algorithms of Examples 3.8 and 3.14 are focusing.

DEFINITION 3.18 (control). We say the algorithm is *cyclic* if

$$I^{(n-1)} = \{n \bmod N\} \quad \text{for } n \geq 1,$$

where we use $\{1, \ldots, N\}$ as remainders. If there is a positive integer p s.t.

$$i \in I^{(n)} \cup I^{(n+1)} \cup \cdots I^{(n+p-1)} \quad \text{for every index } i \text{ and all } n \geq 0,$$

then we speak of an *intermittent* or *p-intermittent* algorithm or of *intermittent control*. Following Censor [21], we call an algorithm *almost cyclic* if it is intermittent and singular. We say the algorithm *considers only blocks* and speak of *block control* (cf. [25]) and a *block algorithm* if the following two conditions hold.

1. There is a decomposition $J_1 \cap \cdots \cap J_M = \{1, \ldots, N\}$ with $J_m \neq \emptyset$ and $J_m \cap J_{m'} = \emptyset$ for all $m, m' \in \{1, \ldots, M\}$ and $m \neq m'$.

2. There is a positive integer p s.t. for all $n \geq 0$ and every $m \in \{1, \ldots, M\}$, $I^{(n')} = J_m$ for some $n' \in \{n, n+1, \ldots, n+p-1\}$.

Finally, if we want to emphasize that the active indices do not necessarily follow some form of control, then we use the phrases *random control* and *random algorithm*. Clearly,

$$\begin{tikzcd}
\text{cyclic} \arrow[r] & \text{almost cyclic} \arrow[dr] & & \\
& \text{considers only blocks} \arrow[r] & \text{intermittent} \arrow[r] & \text{random.} \\
& \text{weighted} \arrow[ur] & &
\end{tikzcd}$$

REMARKS 3.19.

- Recently, block algorithms have received much attention in radiation therapy treatment planning; see [25] and the subsection on polyhedra in §6.
- Equivalent to the phrase "almost cyclic" is Amemiya and Ando's "quasi-periodic" [5] or Browder's "admissible (for finitely many sets)" [17].

THEOREM 3.20 (weak topology results).

(i) *Suppose the algorithm is focusing and intermittent. If* $\underline{\lim}_{n: n \text{ active for } i} \mu_i^{(n)} > 0$ *for every index* i, *then the sequence* $(x^{(n)})$ *is asymptotically regular and converges weakly to some point in* C.

(ii) *Suppose the algorithm is focusing and* p*-intermittent for some positive integer* p. *Let*

$$\nu_n := \min\{\mu_i^{(l)} : np \leq l \leq (n+1)p - 1 \text{ and } i \text{ active at } l\} \quad \textit{for all } n \geq 0.$$

If $\sum_n \nu_n = +\infty$, *then the sequence* $(x^{(n)})$ *has a unique weak cluster point in* C. *More precisely, there is a subsequence* $(x^{(n_k p)})$ *converging weakly to this unique weak cluster point of* $(x^{(n)})$ *in* C *s.t.*

$$(*) \qquad \sum_{l=n_k p}^{(n_k+1)p-1} \sum_{i \in I^{(l)}} \|x^{(l)} - T_i^{(l)} x^{(l)}\| \longrightarrow 0, \quad \textit{which implies}$$

$$(**) \qquad x^{(n_k p + r_k)} - x^{(n_k p + s_k)} \longrightarrow 0$$

for all sequences $(r_k), (s_k)$ *in* $\{0, \ldots, p-1\}$. *In particular, this happens whenever* $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index* i.

(iii) *Suppose the algorithm is focusing and the sequence* $(x^{(n)})$ *converges weakly to some point* x. *If* $\sum_n \mu_i^{(n)} = +\infty$ *for some index* i, *then* $x \in C_i$. *Consequently, if* $\sum_n \mu_i^{(n)} = +\infty$ *for every index* i, *then* $x \in C$.

Proof. (i): $(x^{(n)})$ is asymptotically regular (Corollary 3.5.(i)). Suppose to the contrary that $(x^{(n)})$ does not converge weakly to some point in C. Then, by the Fejér monotonicity of $(x^{(n)})$ and Theorem 2.16.(ii), there exists an index i and a subsequence $(x^{(n_k)})_k$ converging weakly to some point $x \notin C_i$. Because the algorithm is intermittent, we obtain m_k with

$$n_k \leq m_k \leq n_k + p - 1 \quad \text{and} \quad i \in I^{(m_k)} \quad \text{for all } k \geq 0.$$

Since the algorithm is asymptotically regular, we have $x^{(n_k)} - x^{(m_k)} \longrightarrow 0$ and hence

$$(x^{(m_k)})_k \ \text{ converges weakly to } \ x.$$

Since the algorithm is focusing, we conclude that

$$\underline{\lim}_k \, \|x^{(m_k)} - T_i^{(m_k)} x^{(m_k)}\| > 0.$$

On the other hand, by Lemma 3.2.(iv), $+\infty > \sum_k \mu_i^{(m_k)} \|x^{(m_k)} - T_i^{(m_k)} x^{(m_k)}\|^2$. This contradicts the hypothesis on $(\mu_i^{(n)})$; thus (i) holds.

(ii): Fix momentarily $c \in C$. Then, by Lemma 3.2.(iii) and the definition of ν_n,

$$\|x^{(np)} - c\|^2 - \|x^{((n+1)p)} - c\|^2 \geq \nu_n \sum_{l=np}^{(n+1)p-1} \sum_{i \in I^{(l)}} \|x^{(l)} - T_i^{(l)} x^{(l)}\|^2$$

for all $n \geq 0$. Summing over n and considering the hypothesis on (ν_n), we obtain a subsequence $(x^{(n_k p)})_k$ s.t.

$$(*) \qquad \sum_{l=n_k p}^{(n_k+1)p-1} \sum_{i \in I^{(l)}} \|x^{(l)} - T_i^{(l)} x^{(l)}\|^2 \longrightarrow 0.$$

By Lemma 3.2.(v), we also have

$$(**) \qquad x^{(n_k p + r_k)} - x^{(n_k p + s_k)} \longrightarrow 0$$

for all sequences (r_k), (s_k) in $\{0, 1, \ldots, p-1\}$. After passing to a subsequence if necessary, we may assume that $(x^{(n_k p)})_k$ converges weakly to some $x \in D$.

Claim: $\qquad x \in C.$

Fix any index i. Since the algorithm is intermittent, there is some sequence (r_k) in $\{0, 1, \ldots, p-1\}$ s.t.

$$(1) \qquad x^{(n_k p + r_k)} \rightharpoonup x$$

(this follows from $(**)$ with $s_k \equiv 0$) and

$$(2) \qquad i \in I^{(n_k p + r_k)} \quad \text{for all } k.$$

By $(*)$,

$$x^{(n_k p+r_k)} - T_i^{(n_k p+r_k)} x^{(n_k p+r_k)} \longrightarrow 0. \tag{3}$$

Since the algorithm is focusing, (1), (2), and (3) imply $x \in C_i$. The claim follows.

By Theorem 2.16.(ii), x is the unique weak cluster point of $(x^{(n)})$ in C. The proof of (ii) is complete.

(iii): By Lemma 3.2.(iv), $+\infty > \sum_n \mu_i^{(n)} \|x^{(n)} - T_i^{(n)} x^{(n)}\|^2$. Since we assume $\sum_n \mu_i^{(n)} = +\infty$, the limes inferior of the sequence

$$(\|x^{(n)} - T_i^{(n)} x^{(n)}\|)_{n:n \text{ active for } i}$$

must equal 0. Since the algorithm is focusing, we readily see that x is in C_i. The entire theorem is proven. □

REMARKS 3.21. (i) is our basic weak convergence result. (ii) is a generalization of an idea due to Trummer [97, Thm. 1]. Tseng's [100, Thm. 2] is also a result on the existence of a unique weak cluster point of $(x^{(n)})$ in C; his hypothesis, however, is somewhat contrasting: he considers less general relaxation parameters and weights but more general control. (iii) can be viewed as a generalization of Flåm and Zowe's [53, Thm. 2] (see also Corollary 3.24 and Example 4.18.(ii)) and Aharoni and Censor's [3, Thm. 1] (see also Corollary 4.17 and Example 4.19).

COROLLARY 3.22. *Suppose $T_1, \ldots, T_N : D \longrightarrow D$ are firmly nonexpansive, $C_i :=$ Fix T_i, and $(T_i^{(n)})$ converges actively pointwise to T_i. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all $n \geq 0$ and every index i active at n. If the algorithm is intermittent, then the sequence $(x^{(n)})$ converges weakly to some point in C.*

Proof. The algorithm is focusing (Proposition 3.16) and $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for every index i (Remark 3.13). The result now follows from Theorem 3.20.(i). □

REMARKS 3.23.

(i) (a special case of a theorem of Browder) If the algorithm is almost cyclic and $T_i^{(n)} \equiv T_i$, then the last corollary gives [17, Thm. 5 for finitely many sets].

(ii) (a remark of Baillon) If the algorithm is almost cyclic and unrelaxed, then the last corollary gives Baillon's [7, Chapitre 6, Remarque II.2].

COROLLARY 3.24. *Suppose the algorithm is focusing and the interior of C is nonempty. If $\sum_n \mu_i^{(n)} = +\infty$ for every index i, then the sequence $(x^{(n)})$ converges in norm to some point in C.*

Proof. It is immediate from Corollary 3.3.(i) and Theorem 3.20.(iii). □

COROLLARY 3.25. *Suppose X is finite dimensional and the algorithm is focusing and p-intermittent. If $\sum_n \nu_n = +\infty$ (where ν_n is defined as in Theorem* 3.20.(ii)), *then the sequence $(x^{(n)})$ converges in norm to some point in C.*

Proof. By Theorem 3.20.(ii), $(x^{(n)})$ has a weak cluster point $x \in C$. Since X is finite dimensional, the point x is a norm cluster point of $(x^{(n)})$. Now apply Corollary 3.3.(ii). □

REMARK 3.26 (guaranteeing the "divergent sum" condition). One way to guarantee $\sum_n \mu_i^{(n)} = +\infty$ for some index i is to assume that there exists some $\epsilon > 0$ s.t.

$$\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon \ \text{ for all } n \quad \text{and} \quad \textstyle\sum_n \lambda_i^{(n)} = +\infty.$$

This corresponds (in the case when the T_i's are projections) to Flåm and Zowe's [53, Thm. 2] (see also Example 4.18.(ii)). Another way is to assume that

$$\text{the algorithm is singular} \quad \text{and} \quad \textstyle\sum_{n:n \text{ active for } i} \alpha_i^{(n)}(2 - \alpha_i^{(n)}) = +\infty,$$

because then the preceding sum equals $\sum_n \mu_i^{(n)}$.

4. Harvest time I: Projection algorithms. *From now on, we consider exclusively the following situation.*

Setting. We maintain the hypotheses of the last section, where we defined the algorithm. In addition, we assume that $T_i^{(n)}$ is the projection onto some closed convex nonempty set $C_i^{(n)}$ containing C_i:

$$T_i^{(n)} := P_i^{(n)} := P_{C_i^{(n)}} \text{ and } C_i^{(n)} \supseteq C_i \quad \text{for every index } i \text{ and all } n \geq 0.$$

We also assume that $D := X$; that is possible since projections are defined everywhere. We abbreviate

$$P_i := P_{C_i} \quad \text{for every index } i \in \{1, \ldots, N\}$$

and refer to the algorithm in this setting as the *projection algorithm*. We say the projection algorithm *has constant sets* if $C_i^{(n)} \equiv C_i$ for all $n \geq 0$ and every index i.

REMARK 4.1. The projection algorithm is formally a slight generalization of Flåm and Zowe's algorithm [53] (cf. Remark 3.1). Nevertheless, since we allow infinite-dimensional Hilbert spaces and assume less restrictive hypotheses, we will obtain a number of genuinely more general results.

Of course, all the results of the previous section may be applied to the projection algorithm. However, before we can do so, we first must understand the meaning of a *focusing* projection algorithm. A first prototype is formulated in terms of *set convergence in the sense of Mosco* [82] (see [10] for a good survey article on set convergence). It is essentially a reformulation of Tsukada's [101] characterization of Mosco convergence.

LEMMA 4.2. *Suppose (S_n) is a sequence of closed convex sets and there is some closed convex nonempty set S with $S \subseteq S_n$ for all n. Then the following conditions are equivalent.*

(i) *$P_{S_n} \longrightarrow P_S$ pointwise in norm.*

(ii) *$S_n \longrightarrow S$* in the sense of Mosco; *i.e., the following two conditions are satisfied.*

(a) *For every $s \in S$, there exists a sequence (s_n) converging in norm to s with $s_n \in S_n$ for all n.*

(b) *If $(s_{n_k})_k$ is a weakly convergent sequence with $s_{n_k} \in S_{n_k}$ for all k, then its weak limit lies in S.*

(iii) *If $(x_{n_k})_k$ is a weakly convergent sequence with $x_{n_k} - P_{S_{n_k}} x_{n_k} \longrightarrow 0$, then its weak limit lies in S.*

Moreover, if one (and hence each) of the above conditions is satisfied, then

$$S = \bigcap_n S_n.$$

Proof. "(i)$\Longleftrightarrow$(ii)": This is the Hilbert space case of Tsukada's [101, Thm. 3.2]. The proof of "(ii)$\Longleftrightarrow$(iii)" and the "Moreover" part is easy and is thus omitted. □

THEOREM 4.3 (first prototype of a focusing projection algorithm). *If $(P_i^{(n)})$ converges actively pointwise to P_i for every index i, then the projection algorithm is focusing and $C_i = \bigcap_{n: n \text{ active for } i} C_i^{(n)}$ for every index i.*

Proof. Apply Lemma 4.2 to $(C_i^{(n)})_{n: n \text{ active for } i}$ for every index i. □

EXAMPLE 4.4. Suppose $C_i = \bigcap_n C_i^{(n)}$ and $(C_i^{(n)})_n$ is *decreasing*; i.e.,

$$C_i^{(1)} \supseteq C_i^{(2)} \supseteq \cdots \supseteq C_i^{(n)} \supseteq C_i^{(n+1)} \supseteq \cdots \qquad \text{for all } n \geq 0 \text{ and every index } i.$$

Then the projection algorithm is focusing. If, furthermore, the projection algorithm is intermittent and $\underline{\lim}_{n: n \text{ active for } i}\, \mu_i^{(n)} > 0$ for every index i, then the sequence $(x^{(n)})$ is asymptotically regular and converges weakly to some point in C.

Proof. Mosco proved that a decreasing sequence of closed convex sets converges to its intersection in his sense [82, Lem. 1.3]. The last theorem and the last lemma imply that the projection algorithm is focusing. The result now follows from Theorem 3.20.(i). □

REMARK 4.5. Baillon obtained the last example when the algorithm is in addition almost cyclic and unrelaxed [7, Chapitre 6, Remarque II.6].

EXAMPLE 4.6 (random projections). Suppose the projection algorithm is singular, unrelaxed, and has constant sets. If for some index j the set C_j is boundedly compact, then the sequence $(x^{(n)})$ converges in norm to some point in C. In particular, this holds whenever X is finite dimensional.

Proof. The last example shows that the algorithm is focusing. Also, $\mu_i^{(n)} = 1$ for all $n \geq 0$ and every index i active at n. The sequence $(x^{(n)})_{n:n \text{ active for } j}$ lies in C_j and thus must have a norm cluster point; therefore, by Theorem 3.10, the entire sequence $(x^{(n)})$ converges in norm to some point in C. □

REMARKS 4.7. The finite-dimensional version of the last example also follows from Aharoni and Censor's [3, Thm. 1], Flåm and Zowe's [53, Thm. 1], Tseng's [100, Thm. 1], and Elsner, Koltracht, and Neumann's [51, Thm. 1]. We discuss generalizations of Example 4.6 in §6. Not too much is known when the compactness assumption is dropped. It is known that *weak convergence* is obtained whenever

(i) $N = 2$ or
(ii) each C_i is a subspace,

but no example is known where the convergence is not actually in norm.

Case (i) is also known as *von Neumann's alternating projection algorithm.* Since projections are idempotent, one can view the sequence generated by the random projection algorithm as an alternating projection algorithm. In [13], we discussed this algorithm in some detail and provided sufficient conditions for norm (or even linear) convergence.

In 1965, Amemiya and Ando [5] proved weak convergence for Case (ii)—this is still one of the best results. Recently, the first author [12] obtained norm convergence for Case (ii) whenever a certain condition (which holds, for example, if all subspaces have finite co-dimension) is satisfied.

In order to formulate the second prototype of a focusing projection algorithm (as well as the norm and linear convergence results in the following sections), we require some more definitions.

DEFINITION 4.8. We say a projection algorithm is *linearly focusing* if there is some $\beta > 0$ s.t.

$$\beta d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)}) \quad \text{for all large } n \text{ and every index } i \text{ active at } n.$$

We speak of a *strongly focusing* projection algorithm if

$$\left.\begin{array}{c} x^{(n_k)} \rightharpoonup x \\ d(x^{(n_k)}, C_i^{(n_k)}) \to 0 \\ i \text{ active at } n_k \end{array}\right\} \text{ implies } d(x^{(n_k)}, C_i) \to 0$$

for every index i and every subsequence $(x^{(n_k)})_k$ of $(x^{(n)})$.

By Definition 3.7 and the weak lower semicontinuity of $d(\cdot, C_i)$, we obtain the following:

$$\text{linearly focusing} \implies \text{strongly focusing} \implies \text{focusing}.$$

COROLLARY 4.9 (second prototype of a focusing projection algorithm). *Every linearly focusing projection algorithm is focusing.*

REMARK 4.10. Flåm and Zowe [53] used linearly focusing projection algorithms in Euclidean spaces with great success (see also Example 4.18).

COROLLARY 4.11 (prototype of a linearly focusing projection algorithm). *If the projection algorithm has constant sets, then it is linearly focusing.*

COROLLARY 4.12 (prototype of a strongly focusing projection algorithm). *Suppose the projection algorithm is focusing. If the terms of the sequence $(x^{(n)})$ form a relatively compact set, then the projection algorithm is strongly focusing. In particular, this happens whenever X is finite dimensional or the interior of C is nonempty.*

Proof. Suppose not. Then we get $\epsilon > 0$, $x \in X$, an index i, and a subsequence $(x^{(n_k)})_k$ with $x^{(n_k)} \rightharpoonup x$, $x^{(n_k)} - P_i^{(n_k)} x^{(n_k)} \to 0$, i active at n_k, but $\| x^{(n_k)} - P_i x^{(n_k)} \| \geq \epsilon$ for all k. Since the algorithm is focusing, $x \in C_i$. After passing to a subsequence if necessary, we may assume (by the compactness assumption) that $x^{(n_k)} \to x$. But then $x^{(n_k)} - P_i x^{(n_k)} \to x - P_i x = 0$, which is absurd. Therefore, the projection algorithm is strongly focusing. If X is finite dimensional, then the terms of $(x^{(n)})$ form a relatively compact set because $(x^{(n)})$ is bounded (Lemma 3.2.(iv)). Finally, if int $C \neq \emptyset$, then $(x^{(n)})$ converges in norm (Corollary 3.3.(i)). The proof is complete. □

The two prototypes of a focusing projection algorithm (cf. Theorem 4.3 and Corollary 4.9) are unrelated, as the following examples demonstrate.

EXAMPLE 4.13. Suppose $X := \mathbb{R}$, $N := 1$, $C := C_1 := \{0\}$, $C_1^{(n)} := [0, 1/(n+1)]$, and $x^{(0)} := 2$. Then the projection algorithm is strongly focusing and the sequence $(C_1^{(n)})$ of compact convex decreasing sets *converges to C_1 in the sense of Mosco* (Example 4.4 and Corollary 4.12). However, the projection algorithm *is not linearly focusing*. Indeed, for $n \geq 1$,

$$x^{(n)} = \frac{1}{n} \quad \text{and} \quad \frac{d(x^{(n)}, C_1^{(n)})}{d(x^{(n)}, C_1)} = \frac{1}{n+1} \longrightarrow 0.$$

EXAMPLE 4.14. Suppose $X := \mathbb{R}$, $N := 1$, $C := C_1 := \{0\}$, $C_1^{(n)} := (-1)^n[0, 1]$, and $x^{(0)} \in X$ arbitrary. Then the projection algorithm *is linearly focusing* since $x^{(n)} \equiv 0 \in C$ for $n \geq 2$. However, the sequence $(C_1^{(n)})$ of compact convex sets *does not converge to C in the sense of Mosco.*

Having gotten a feeling for the concept of a linearly focusing projection algorithm, we document its usefulness through a dichotomy result inspired by Aharoni and Censor's [3, Proof of Thm. 1].

THEOREM 4.15 (dichotomy II). *Suppose the projection algorithm is linearly focusing and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ either converges in norm or has no norm cluster points at all.*

Proof. Assume to the contrary that $(x^{(n)})$ has at least two distinct norm cluster points, say y and z. Get $\beta > 0$ s.t. $\beta d(x^{(l)}, C_i) \leq d(x^{(l)}, C_i^{(l)})$ for all large l and every index i active at l. Fix $c \in C$. Since $y \notin C$ (otherwise, the sequence $(x^{(n)})$ would converge in norm by Corollary 3.3.(ii)), the set of indices $I := \{i \in \{1, \ldots, N\} : y \notin C_i\}$ is nonempty. Define $B := y + rB_X$, where $r := (1/2)\min(\{\|y - z\|\} \cup \{d(y, C_i) : i \in I\})$.

Claim 1: $$\underset{\gamma_1 > 0}{\exists} \left.\begin{array}{c} x^{(l)} \in B \\ l \text{ large} \end{array}\right\} \text{ implies } \|x^{(l)} - c\| - \|x^{(l+1)} - c\| \geq \gamma_1 \sum_{i \in I} \lambda_i^{(l)}.$$

On the one hand, by Lemma 3.2.(ii), the definition of β, and $\|y - x^{(l)}\| \geq d(y, C_i) - d(x^{(l)}, C_i)$,

$$\begin{aligned} \|x^{(l)} - c\|^2 - \|x^{(l+1)} - c\|^2 &\geq \sum_{i \in I} \mu_i^{(l)} d^2(x^{(l)}, C_i^{(l)}) \\ &\geq \sum_{i \in I} \lambda_i^{(l)} \epsilon^2 \beta^2 d^2(x^{(l)}, C_i) \\ &\geq \epsilon^2 \beta^2 r^2 \sum_{i \in I} \lambda_i^{(l)}. \end{aligned}$$

On the other hand,

$$\|x^{(l)} - c\|^2 - \|x^{(l+1)} - c\|^2 = (\|x^{(l)} - c\| - \|x^{(l+1)} - c\|) \times (\|x^{(l)} - c\| + \|x^{(l+1)} - c\|),$$

and the norm of the latter factor is at most $2(r + \|y - c\|)$. Altogether, $\gamma_1 = \epsilon^2\beta^2r^2/(2(r + \|y - c\|))$ does the job and Claim 1 is verified.

Claim 2: $$\underset{\gamma_2>0}{\exists} \left.\begin{array}{c} x^{(l)} \in B \\ l \text{ large} \end{array}\right\} \text{ implies } \|x^{(l+1)} - y\| - \|x^{(l)} - y\| \leq \gamma_2 \sum_{i\in I} \lambda_i^{(l)}.$$

For every $i \in \{1, \ldots, N\} \setminus I$, the point y is fixed under the nonexpansive mapping $R_i^{(l)}$ (cf. Facts 1.5.(i), Fact 1.3.(iii), and Corollary 1.4); thus we estimate

$$\begin{aligned}
\|x^{(l+1)} - y\| &= \left\| \sum_{i\in\{1,\ldots,N\}\setminus I} \lambda_i^{(l)}(R_i^{(l)}x^{(l)} - y) + \sum_{i\in I} \lambda_i^{(l)}(R_i^{(l)}x^{(l)} - y) \right\| \\
&\leq \sum_{i\in\{1,\ldots,N\}\setminus I} \lambda_i^{(l)}\|x^{(l)} - y\| + \sum_{i\in I} \lambda_i^{(l)}\|R_i^{(l)}x^{(l)} - y\| \\
&\leq \|x^{(l)} - y\| + \sum_{i\in I} \lambda_i^{(l)}\{\|R_i^{(l)}x^{(l)} - x^{(l)}\| + \|x^{(l)} - y\|\} \\
&\leq \|x^{(l)} - y\| + \sum_{i\in I} \lambda_i^{(l)}\{\alpha_i^{(l)}\|x^{(l)} - P_i^{(l)}x^{(l)}\| + \|x^{(l)} - y\|\} \\
&\leq \|x^{(l)} - y\| + \sum_{i\in I} \lambda_i^{(l)}\{2d(x^{(l)}, C_i) + r\} \\
&\leq \|x^{(l)} - y\| + \sum_{i\in I} \lambda_i^{(l)}\{2(d(y, C_i) + \|x^{(l)} - y\|) + r\}.
\end{aligned}$$

Therefore, $\gamma_2 = 2\max\{d(y, C_i) : i \in I\} + 3r$ does the job and Claim 2 is also verified.

The rest is done quickly. Set

$$\delta := r\frac{\gamma_1}{\gamma_1 + \gamma_2} \quad (< r)$$

and find n large s.t. $\|x^{(n)} - y\| < \delta$; thus $x^{(n)} \in B$. Now z is another norm cluster point of $(x^{(n)})$ and has positive distance to B, so there is a *minimal* $m > n$ with $x^{(m)} \notin B$. By the Fejér monotonicity of $(x^{(n)})$ and Claim 1,

$$\begin{aligned}
\|y - c\| \leq \|x^{(m)} - c\| &\leq \|x^{(n)} - c\| - \gamma_1 \sum_{l=n}^{m-1} \sum_{i\in I} \lambda_i^{(l)} \\
&< \delta + \|y - c\| - \gamma_1 \sum_{l=n}^{m-1} \sum_{i\in I} \lambda_i^{(l)};
\end{aligned}$$

thus

$$\sum_{l=n}^{m-1} \sum_{i\in I} \lambda_i^{(l)} < \frac{\delta}{\gamma_1}.$$

By Claim 2, however,

$$\|x^{(m)} - y\| \leq \|x^{(n)} - y\| + \gamma_2 \sum_{l=n}^{m-1} \sum_{i\in I} \lambda_i^{(l)} < \delta + \frac{\gamma_2}{\gamma_1}\delta = r,$$

which contradicts $y \notin B$. Therefore, the sequence $(x^{(n)})$ has at most one norm cluster point. □

REMARK 4.16. As Example 3.6 demonstrates, some assumption is necessary to guarantee at most one norm cluster point.

COROLLARY 4.17. *Suppose the projection algorithm is linearly focusing and there is some $\epsilon > 0$ s.t. $\epsilon \le \alpha_i^{(n)} \le 2 - \epsilon$ for all large n and every index i active at n. Suppose further that X is finite dimensional or the interior of C is nonempty. Then the sequence $(x^{(n)})$ converges in norm to some point x. If $\sum_n \mu_i^{(n)} = +\infty$ for some index i, then $x \in C_i$. Consequently, if $\sum_n \mu_i^{(n)} = +\infty$ for every index i, then $x \in C$.*

Proof. If $\operatorname{int} C \neq \emptyset$, then $(x^{(n)})$ converges in norm by Corollary 3.3.(i). If X is finite dimensional, then $(x^{(n)})$ has a norm cluster point; thus, by the last theorem, $(x^{(n)})$ is also norm convergent. The result now follows from Theorem 3.20.(iii). □

The next two examples follow immediately.

EXAMPLE 4.18 (Flåm and Zowe's framework [53, Thms. 1 and 2]). Suppose X is finite dimensional, the projection algorithm is linearly focusing, and there is some $\epsilon > 0$ s.t. $\epsilon \le \alpha_i^{(n)} \le 2 - \epsilon$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ converges in norm to some point x.

(i) If $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for every index i, then $x \in C$.

(ii) If $\operatorname{int} C \neq \emptyset$ and $\sum_n \mu_i^{(n)} = +\infty$ for every index i, then $x \in C$.

EXAMPLE 4.19 (Aharoni and Censor's framework [3, Thm. 1]). Suppose X is finite dimensional, the projection algorithm has constant sets (and is therefore linearly focusing by Corollary 4.11), and there is some $\epsilon > 0$ s.t. $\epsilon \le \alpha_i^{(n)} \equiv \alpha^{(n)} \le 2 - \epsilon$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ is norm convergent and its limit lies in $\bigcap_{i \in I} C_i$, where $I := \{i \in \{1, \ldots, N\} : \sum_n \mu_i^{(n)} = +\infty\}$.

REMARKS 4.20.

- Under the assumption on the relaxation parameters in the preceding examples, the condition $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ is equivalent to $\underline{\lim}_{n:n \text{ active for } i}\, \lambda_i^{(n)} > 0$ (cf. to Remark 3.13) and the condition $\sum_n \mu_i^{(n)} = +\infty$ is equivalent to $\sum_n \lambda_i^{(n)} = +\infty$ (cf. to Remark 3.26) for every index i.
- Example 4.18.(i) follows not only from Corollary 4.17 but also from Theorem 3.10.

The next example shows that if one drops the assumption $\sum_n \mu_i^{(n)} = +\infty$ in Example 4.19, then one cannot expect the limit of $(x^{(n)})$ to lie in C_i.

EXAMPLE 4.21. Let $X := \mathbb{R}$, $N := 2$, $C_1 := C_1^{(n)} :\equiv\,]-\infty, 0]$, and $C_2 := C_2^{(n)} :\equiv [0, +\infty[$. Suppose $x^{(0)} > 0$, $\alpha_1^{(n)} :\equiv \alpha_2^{(n)} :\equiv 3/2$, and $\lambda_1^{(n)} < 2/3$ for all n. Then

$$x^{(n)} = \left(1 - \frac{3}{2}\lambda_1^{(n-1)}\right) \cdots \left(1 - \frac{3}{2}\lambda_1^{(0)}\right) x^{(0)},$$

and therefore

$$\lim_n x^{(n)} \in C_1 \iff \lim_n x^{(n)} = 0 \iff \sum_n \mu_1^{(n)} = +\infty.$$

THEOREM 4.22. *Given a projection algorithm, suppose $(P_i^{(n)})$ converges actively pointwise to P_i for every index i. Suppose further there is some subsequence (n') of (n) s.t. for every index i,*

$$\alpha_i^{(n')} \longrightarrow \alpha_i \quad \text{and} \quad \lambda_i^{(n')} \longrightarrow \lambda_i$$

for some $\alpha_i \in\,]0, 2]$ and $\lambda_i \in\,]0, 1]$. If the interior of C is nonempty, then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. By Corollary 3.3.(i), $(x^{(n)})$ converges in norm to some point x. We must show that $x \in C$.

Claim: $$P_i^{(n')}x^{(n')} \longrightarrow P_i x \quad \text{for every index } i.$$

Because $\|P_i^{(n')}x^{(n')} - P_i^{(n')}x\| \leq \|x^{(n')} - x\|$, we have $P_i^{(n')}x^{(n')} - P_i^{(n')}x \longrightarrow 0$. Since $\lambda_i^{(n')} \longrightarrow \lambda_i > 0$, we see that i is active at n' for all large n'. The assumption on $(P_i^{(n)})$ implies $P_i^{(n')}x \longrightarrow P_i x$. The claim follows.

Now

$$x^{(n'+1)} = \sum_{i=1}^N \lambda_i^{(n')} \left((1-\alpha_i^{(n')})x^{(n')} + \alpha_i^{(n')} P_i^{(n')} x^{(n')}\right);$$

hence, by taking limits along the subsequence $(x^{(n')})$ and by the claim,

$$x = \sum_{i=1}^N \lambda_i \left((1-\alpha_i)x + \alpha_i P_i x\right)$$

or

$$x = \sum_{i=1}^N \left(\frac{\lambda_i \alpha_i}{\sum_{j=1}^N \lambda_j \alpha_j}\right) P_i x.$$

Proposition 2.12 implies that $x \in C$; the proof is complete. □

EXAMPLE 4.23 (Butnariu and Censor's framework [20, Thm. 4.4]). Suppose X is finite dimensional, the projection algorithm has constant sets, and the relaxation parameters depend only on n, say $\alpha_i^{(n)} \equiv \alpha^{(n)}$ for every index i and all n. Suppose further there is some subsequence (n') of (n) s.t. for every index i, $\lambda_i^{(n')} \longrightarrow \lambda_i$ for some $\lambda_i > 0$.

(i) If there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha^{(n)} \leq 2 - \epsilon$ for all large n, then the sequence $(x^{(n)})$ converges in norm to some point in C.

(ii) If the interior of C is nonempty and there is some subsequence (n'') of (n') s.t. $\alpha^{(n'')} \longrightarrow 2$, then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. (i): The assumption on the weights implies $\sum_n \mu_i^{(n)} = +\infty$ for every index i. Thus (i) follows from Example 4.19. (ii) is immediate from Theorem 4.22. □

REMARK 4.24. Note that the last theorem works especially when $\alpha_i^{(n)} \equiv 2$. Since in this case $\mu_i^{(n)} \equiv 0$, none of the previous results are applicable. If we drop the assumption that $\operatorname{int} C \neq \emptyset$, then the conclusion of the last theorem need not hold; see Example 3.6.

DEFINITION 4.25 (control). We say the projection algorithm *considers remotest sets* if for every n, at least one *remotest index is active*; i.e.,

$$I_{\text{rem}}^{(n)} := \{i : d(x^{(n)}, C_i) = \max\{d(x^{(n)}, C_j) : j = 1 \ldots N\}\} \cap I^{(n)} \neq \emptyset.$$

Following Censor [21], we speak of *remotest set control* if the projection algorithm is singular and considers remotest sets. Obviously,

$$\begin{array}{ccc} \text{remotest set control} & & \\ & \searrow & \\ & & \text{considers remotest sets.} \\ & \nearrow & \\ \text{weighted} & & \end{array}$$

THEOREM 4.26 (weak topology results). *Suppose the projection algorithm is strongly focusing and considers remotest sets. Suppose further that* $(i^{(n)})$ *is a* sequence of active remotest indices; *i.e.,* $i^{(n)} \in I_{\text{rem}}^{(n)}$ *for all* n.

(i) *If* $\sum_n \mu_{i^{(n)}}^{(n)} = +\infty$, *then there is a subsequence* $(x^{(n_k)})_k$ *of* $(x^{(n)})$ *s.t.*

$$\max\{d(x^{(n_k)}, C_j) : j = 1 \ldots N\} \longrightarrow 0,$$

and $(x^{(n_k)})_k$ *converges weakly to the unique weak cluster point of* $(x^{(n)})$ *in* C.

(ii) *If* $\underline{\lim}_n \mu_{i^{(n)}}^{(n)} > 0$, *then* $(x^{(n)})$ *converges weakly to some point in* C *and*

$$\max\{d(x^{(n)}, C_j) : j = 1 \ldots N\} \longrightarrow 0.$$

Proof. (i): By Lemma 3.2.(iv), the series $\sum_n \mu_{i^{(n)}}^{(n)} d^2(x^{(n)}, C_{i^{(n)}}^{(n)})$ is convergent. Hence $\underline{\lim}_n d(x^{(n)}, C_{i^{(n)}}^{(n)}) = 0$. Thus we can extract a subsequence $(x^{(n_k)})_k$ and fix an index i s.t. $d(x^{(n_k)}, C_i^{(n_k)}) \longrightarrow 0$, $i^{(n_k)} \equiv i$, and $(x^{(n_k)})$ converges weakly. Since the algorithm is strongly focusing and considers remotest sets, we conclude that

$$\max\{d(x^{(n_k)}, C_j) : j = 1 \ldots N\} \longrightarrow 0.$$

By weak lower semicontinuity of $d(\cdot, C_j)$ for every index j, the weak limit of $(x^{(n_k)})$ lies in C. By Theorem 2.16.(ii), $(x^{(n)})$ has at most one weak cluster point in C; therefore, (i) is verified. (ii) is proved similarly. □

REMARK 4.27. Remotest set control is an old and successful concept. In 1954, Agmon [1] and Motzkin and Schoenberg [83] studied projection algorithms for solving linear inequalities using remotest set control. Bregman [16] considered the situation when there is an arbitrary collection of intersecting closed convex sets. We will recapture Agmon's main result [1, Thm. 3] and some generalizations in §6.

5. Guaranteeing norm or linear convergence: Regularities. We uphold the notation of the preceding sections; in particular, we remember that $C_1, \ldots, C_N$ are closed convex sets with *nonempty* intersection C.

Norm convergence and (bounded) regularity.

DEFINITION 5.1. We say that the N-tuple of closed convex sets $(C_1, \ldots, C_N)$ is *regular* if

$$\forall_{\epsilon > 0} \; \exists_{\delta > 0} \; \forall_{\substack{x \in X: \\ \max\{d(x, C_j) : j = 1 \ldots N\} \leq \delta}} \quad d(x, C) \leq \epsilon.$$

If this holds only on bounded sets, i.e.,

$$\forall_{X \supseteq S \text{ bounded}} \; \forall_{\epsilon > 0} \; \exists_{\delta > 0} \; \forall_{\substack{x \in S: \\ \max\{d(x, C_j) : j = 1 \ldots N\} \leq \delta}} \quad d(x, C) \leq \epsilon,$$

then we speak of a *boundedly regular* N-tuple $(C_1, \ldots, C_N)$.

Although the definition of (bounded) regularity is independent of the order of the sets, we prefer to think of $C_1, \ldots, C_N$ as a tuple. The geometric idea behind this definition is extremely simple: "If you are close to all sets, then the intersection cannot be too far away." In [13], we utilized this notion to formulate some norm convergence results for von Neumann's alternating projection algorithm for two sets.

The results of this subsection will illustrate the usefulness of this concept in our present framework.

THEOREM 5.2. *Suppose the projection algorithm is strongly focusing and p-intermittent for some positive integer p. Suppose further the N-tuple* $(C_1, \ldots, C_N)$ *is boundedly regular. Let*

$$\nu_n := \min\{\mu_i^{(l)} : np \leq l \leq (n+1)p - 1 \text{ and } i \text{ active at } l\} \quad \textit{for all } n \geq 0.$$

If $\sum_n \nu_n = +\infty$, *then the sequence* $(x^{(n)})$ *converges in norm to some point in* C. *In particular, this happens whenever* $\underline{\lim}_{n: n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index* i.

Proof. By Theorem 3.20.(ii), we obtain a subsequence $(x^{(n_k p)})_k$ of $(x^{(n)})$ s.t. $(x^{(n_k p)})_k$ converges weakly to the unique weak cluster point of $(x^{(n)})$ in C, say x,

$$(*) \qquad \sum_{l=n_k p}^{(n_k+1)p-1} \sum_{i \in I^{(l)}} d(x^{(l)}, C_i^{(l)}) \longrightarrow 0, \quad \text{and}$$

$$(**) \qquad x^{(n_k p + r_k)} - x^{(n_k p)} \longrightarrow 0$$

for all sequences (r_k) in $\{0, \ldots, p-1\}$. Fix any index i. Because the projection algorithm is intermittent, we get a sequence (r_k) in $\{0, \ldots, p-1\}$ s.t. $i \in I^{(n_k p + r_k)}$ for all k. Then, by $(*)$, $d(x^{(n_k p + r_k)}, C_i^{(n_k p + r_k)}) \longrightarrow 0$. Since $(x^{(n_k p + r_k)})_k$ also converges to x (by $(**)$) and the projection algorithm is strongly focusing, we deduce that

$$d(x^{(n_k p + r_k)}, C_i) \longrightarrow 0.$$

Hence, by $(**)$, $d(x^{(n_k p)}, C_i) \longrightarrow 0$. Since i has been chosen arbitrarily, we actually have

$$\max\{d(x^{(n_k p)}, C_j) : j = 1 \ldots N\} \longrightarrow 0.$$

Now $(C_1, \ldots, C_N)$ is boundedly regular and $(x^{(n_k p)})_k$ is bounded; consequently, $d(x^{(n_k p)}, C) \longrightarrow 0$. Therefore, by Corollary 3.3.(ii), $(x^{(n)})$ converges in norm to x. □

THEOREM 5.3. *Suppose the projection algorithm is strongly focusing and considers remotest sets. Suppose further the N-tuple* $(C_1, \ldots, C_N)$ *is boundedly regular and* $(i^{(n)})$ *is a sequence of active remotest indices. If* $\sum_n \mu_{i^{(n)}}^{(n)} = +\infty$, *then the sequence* $(x^{(n)})$ *converges in norm to some point in* C. *In particular, this happens whenever* $\underline{\lim}_n \mu_{i^{(n)}}^{(n)} > 0$.

Proof. By Theorem 4.26.(i), there exists a weakly convergent subsequence $(x^{(n_k)})_k$ of $(x^{(n)})$ with $\max\{d(x^{(n_k)}, C_j) : j = 1 \ldots N\} \longrightarrow 0$. Since $(C_1, \ldots, C_N)$ is boundedly regular, we get $d(x^{(n_k)}, C) \longrightarrow 0$. Now apply Corollary 3.3.(ii). □

In order to make use of these theorems, we must know when an N-tuple $(C_1, \ldots, C_N)$ is boundedly regular. Fortunately, our observations on bounded regularity of a pair in [13] generalize easily to the N-set case.

PROPOSITION 5.4.

(i) *If some set* C_i *is boundedly compact, then the N-tuple* $(C_1, \ldots, C_N)$ *is boundedly regular.*

(ii) *If the N-tuple* $(C_1, \ldots, C_N)$ *is boundedly regular and some set* C_i *is bounded, then* $(C_1, \ldots, C_N)$ *is regular.*

(iii) *If X is finite dimensional, then* every *N-tuple* $(C_1, \ldots, C_N)$ *is boundedly regular.*

Proof. An easy modification of [13, Thm. 3.9 (resp., Thm. 3.15) given for two sets] yields (i) (resp., (ii)). (iii) follows from (i). □

REMARK 5.5. We gave an example [13, Ex. 5.5] of a pair which is not boundedly regular; therefore, bounded regularity requires some assumption.

Linear convergence and (bounded) linear regularity. The following stronger and more quantitative version of (bounded) regularity allows us to discuss *rates of convergence*.

DEFINITION 5.6. We say that the N-tuple of closed convex sets $(C_1, \dots, C_N)$ is *linearly regular* if

$$\underset{\kappa > 0}{\exists} \; \underset{x \in X}{\forall} \; d(x, C) \leq \kappa \max\{d(x, C_j) : j = 1 \dots N\}.$$

Again, if this holds only on bounded sets, i.e.,

$$\underset{X \supseteq S \text{ bounded}}{\forall} \; \underset{\kappa_S > 0}{\exists} \; \underset{x \in S}{\forall} \; d(x, C) \leq \kappa_S \max\{d(x, C_j) : j = 1 \dots N\},$$

then we say that $(C_1, \dots, C_N)$ is *boundedly linearly regular*.

Clearly,

$$\begin{array}{ccc} \text{linearly regular} & \Longrightarrow & \text{boundedly linearly regular} \\ \Downarrow & & \Downarrow \\ \text{regular} & \Longrightarrow & \text{boundedly regular.} \end{array}$$

THEOREM 5.7. *Suppose the projection algorithm is linearly focusing and intermittent. Suppose further the N-tuple $(C_1, \dots, C_N)$ is boundedly linearly regular and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ converges linearly to some point in C; the rate of convergence is independent of the starting point whenever $(C_1, \dots, C_N)$ is linearly regular.*

Proof. Suppose the projection algorithm is p-intermittent. Fix any index i. Then, for all $k \geq 0$, we get m_k with $kp \leq m_k \leq (k+1)p - 1$ and $i \in I^{(m_k)}$. Now $x^{(m_k)} = A^{(m_k - 1)} \dots A^{(kp)} x^{(kp)}$ and, by Lemma 2.4.(iv) and Proposition 2.12.(ii),

$$A^{(n)} \text{ is } \min\{(2 - \alpha_j^{(n)})/\alpha_j^{(n)} : j \text{ active at } n\}\text{-attracting w.r.t. } C \quad \text{for all } n \geq 0.$$

Hence, by the assumption on the relaxation parameters, $A^{(n)}$ is $\epsilon/2$-attracting w.r.t. C for all large n. Thus, by Proposition 2.10.(ii),

$$A^{(m_k - 1)} \dots A^{(kp)} \text{ is } \frac{\epsilon}{2^{p-1}}\text{-attracting w.r.t. } C \quad \text{for all large } k.$$

Since the projection algorithm is linearly focusing, there is some $\beta > 0$ s.t. $\beta d(x^{(n)}, C_j) \leq d(x^{(n)}, C_j^{(n)})$ for all large n and every index j active at n. Now

$$\begin{aligned} d^2(x^{(kp)}, C_i) &\leq (\|x^{(kp)} - x^{(m_k)}\| + d(x^{(m_k)}, C_i))^2 \\ &\leq 2\|x^{(kp)} - x^{(m_k)}\|^2 + 2d^2(x^{(m_k)}, C_i); \end{aligned}$$

here the first inequality follows from the nonexpansivity of $d(\cdot, C_i)$ and the second one is just "$(a+b)^2 \leq 2a^2 + 2b^2$." Fix an arbitrary point $c \in C$. On the one hand, for all large k,

$$\begin{aligned} \|x^{(kp)} - x^{(m_k)}\|^2 &= \|A^{(m_k - 1)} \dots A^{(kp)} x^{(kp)} - x^{(kp)}\|^2 \\ &\leq \frac{2^{p-1}}{\epsilon} (\|x^{(kp)} - c\|^2 - \|x^{(m_k)} - c\|^2) \\ &\leq \frac{2^{p-1}}{\epsilon} (\|x^{(kp)} - c\|^2 - \|x^{((k+1)p)} - c\|^2). \end{aligned}$$

On the other hand, by the assumptions on β, relaxation parameters, weights, and by Lemma 3.2.(ii) and Remark 3.13, we estimate for all large k that

$$\begin{aligned} d^2(x^{(m_k)}, C_i) &\leq \frac{1}{\beta^2} d^2(x^{(m_k)}, C_i^{(m_k)}) \\ &= \frac{1}{\beta^2} \|x^{(m_k)} - P_i^{(m_k)} x^{(m_k)}\|^2 \\ &\leq \frac{1}{\beta^2 \mu_i^{(m_k)}} (\|x^{(m_k)} - c\|^2 - \|x^{(m_k+1)} - c\|^2) \\ &\leq \frac{1}{\beta^2 \epsilon^3} (\|x^{(m_k)} - c\|^2 - \|x^{(m_k+1)} - c\|^2) \\ &\leq \frac{1}{\beta^2 \epsilon^3} (\|x^{(kp)} - c\|^2 - \|x^{((k+1)p)} - c\|^2); \end{aligned}$$

altogether

$$d^2(x^{(kp)}, C_i) \leq \left(\frac{2^p}{\epsilon} + \frac{2}{\beta^2 \epsilon^3} \right) (\|x^{(kp)} - c\|^2 - \|x^{((k+1)p)} - c\|^2),$$

which, after choosing $c := P_C x^{(kp)}$, yields

$$d^2(x^{(kp)}, C_i) \leq \left(\frac{2^p}{\epsilon} + \frac{2}{\beta^2 \epsilon^3} \right) (d^2(x^{(kp)}, C) - d^2(x^{((k+1)p)}, C)).$$

Because i has been chosen arbitrarily, the last estimate is true for every index i, provided that k is large enough. Since $(C_1, \dots, C_N)$ is boundedly linearly regular, we obtain for $S := \{x^{(n)} : n \geq 0\}$ a constant $\kappa_S > 0$ s.t.

$$d(x^{(n)}, C) \leq \kappa_S \max\{d(x^{(n)}, C_j) : j = 1 \dots N\} \quad \text{for all } n \geq 0.$$

Note that if $(C_1, \dots, C_N)$ is linearly regular, then the constant κ_S can be chosen *independent* of S. Combining gives

$$d^2(x^{(kp)}, C) \leq \kappa_S^2 \left(\frac{2^p}{\epsilon} + \frac{2}{\beta^2 \epsilon^3} \right) (d^2(x^{(kp)}, C) - d^2(x^{((k+1)p)}, C)).$$

Therefore, by Theorem 2.16.(vi) applied to $(x^{(kp)})$, the sequence $(x^{(kp)})$ converges linearly to some point x in C. By Theorem 2.16.(i) and Proposition 1.6, the entire sequence $(x^{(n)})$ converges linearly to x; the rate of convergence is independent of the starting point whenever $(C_1, \dots, C_N)$ is linearly regular. □

THEOREM 5.8. *Suppose the projection algorithm is linearly focusing and considers remotest sets. Suppose further the N-tuple $(C_1, \dots, C_N)$ is boundedly linearly regular and $(i^{(n)})$ is a sequence of active remotest indices. If $\underline{\lim}_n \mu_{i^{(n)}}^{(n)} > 0$, then the sequence $(x^{(n)})$ converges linearly to some point in C; the rate of convergence is independent of the starting point whenever $(C_1, \dots, C_N)$ is linearly regular.*

Proof. First, since the projection algorithm is linearly focusing, we get $\beta > 0$ s.t. $\beta d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)})$ for all n and every index i active at n. Second, since $(C_1, \dots, C_N)$ is boundedly linearly regular, we obtain for $S := \{x^{(n)} : n \geq 0\}$ a constant $\kappa_S > 0$ s.t. $d(x^{(n)}, C) \leq \kappa_S \max\{d(x^{(n)}, C_j) : j = 1 \dots N\}$ for all $n \geq 0$. Once more, the constant κ_S can be chosen independent of S whenever $(C_1, \dots, C_N)$ is linearly regular. Third, there is some

$\epsilon > 0$ s.t. $\mu_{i(n)}^{(n)} \geq \epsilon$ for all large n. Putting this together and using Lemma 3.2.(ii), we estimate for all large n that

$$\begin{aligned} d^2(x^{(n)}, C) &\leq \kappa_S^2 \max\{d^2(x^{(n)}, C_j) : j = 1 \dots N\} \\ &= \kappa_S^2 d^2(x^{(n)}, C_{i(n)}) \\ &\leq \left(\frac{\kappa_S}{\beta}\right)^2 \|x^{(n)} - P_{i(n)}^{(n)} x^{(n)}\|^2 \\ &\leq \left(\frac{\kappa_S}{\beta}\right)^2 \frac{1}{\mu_{i(n)}^{(n)}} \left(\|x^{(n)} - P_C x^{(n)}\|^2 - \|x^{(n+1)} - P_C x^{(n)}\|^2\right) \\ &\leq \left(\frac{\kappa_S}{\beta}\right)^2 \frac{1}{\epsilon} \left(d^2(x^{(n)}, C) - d^2(x^{(n+1)}, C)\right). \end{aligned}$$

Therefore, by Theorem 2.16.(vi), $(x^{(n)})$ converges linearly to some point in C (again with a rate independent of the starting point whenever $(C_1, \dots, C_N)$ is linearly regular). □

(Bounded) linear regularity: Examples. Having seen the power of (bounded) linear regularity, we now investigate this concept itself and provide basic prototypes.

PROPOSITION 5.9. *If each set C_i is a closed convex cone, then the following conditions are equivalent.*

(i) *$(C_1, \dots, C_N)$ is regular.*

(ii) *$(C_1, \dots, C_N)$ is linearly regular.*

(iii) *$(C_1, \dots, C_N)$ is boundedly linearly regular.*

Proof. Adapt the proof of [13, Thm. 3.17]. □

REMARK 5.10. It follows that (i), (ii), and (iii) are equivalent if

- $C_1, \dots, C_N$ are closed convex translated cones with a common vertex (a simple translation argument),
- $C_1, \dots, C_N$ are closed affine subspaces with nonempty intersection.

THEOREM 5.11 ((bounded) linear regularity: reduction to pairs). *If each of the $N - 1$ pairs*

$$\begin{aligned} &(C_1, C_2), \\ &(C_1 \cap C_2, C_3), \\ &\vdots \\ &(C_1 \cap C_2 \cap \cdots \cap C_{N-2}, C_{N-1}), \\ &(C_1 \cap C_2 \cap \cdots \cap C_{N-2} \cap C_{N-1}, C_N) \end{aligned}$$

is (boundedly) linearly regular, then so is the N-tuple $(C_1, \dots, C_N)$.

Proof. We consider the case when all pairs are boundedly linearly regular; the case when all pairs are linearly regular is treated analogously. Fix a bounded set S and get (by hypothesis) $\kappa_1, \dots, \kappa_{N-1} > 0$ s.t. for every $x \in S$, we have the estimates

$$\begin{aligned} d(x, C_1 \cap C_2) &\leq \kappa_1 \max\{d(x, C_1), d(x, C_2)\}, \\ d(x, C_1 \cap C_2 \cap C_3) &\leq \kappa_2 \max\{d(x, C_1 \cap C_2), d(x, C_3)\}, \\ &\vdots \\ d(x, C_1 \cap \cdots \cap C_N) &\leq \kappa_{N-1} \max\{d(x, C_1 \cap \cdots \cap C_{N-1}), d(x, C_N)\}; \end{aligned}$$

hence $d(x, C_1 \cap \cdots \cap C_N) \leq \kappa_1 \kappa_2 \cdots \kappa_{N-1} \max\{d(x, C_j) : j = 1 \dots N\}$. □

For bounded linear regularity of pairs, we gave the following sufficient condition (see [13, Cor. 4.5]).

FACT 5.12. Suppose E, F are two closed convex sets. If $0 \in \operatorname{icr}(E - F)$, then the pair (E, F) is boundedly linearly regular. In particular, this happens whenever

(i) $0 \in \operatorname{int}(E - F)$ or

(ii) $E - F$ is a closed subspace.

Combining the two preceding results immediately yields the following.

COROLLARY 5.13. *If*

$$0 \in \operatorname{icr}(C_1 - C_2) \cap \operatorname{icr}((C_1 \cap C_2) - C_3) \cap \cdots \cap \operatorname{icr}((C_1 \cap \cdots \cap C_{N-1}) - C_N),$$

then the N-tuple $(C_1, \ldots, C_N)$ is boundedly linearly regular.

COROLLARY 5.14. *If $C_N \cap \operatorname{int}(C_1 \cap \cdots \cap C_{N-1}) \neq \emptyset$, then $(C_1, \ldots, C_N)$ is boundedly linearly regular.*

REMARK 5.15. These sufficient conditions for bounded linear regularity do depend on the order of the sets, whereas bounded linear regularity does not. Consequently, these conditions might still be applicable after a suitable permutation of the sets.

In applications, the N sets almost always have additional structure. One important case is when all sets are closed subspaces. In the following, we will completely characterize regularity of an N-tuple of closed subspaces. We begin with the case when $N = 2$.

Recall that the *angle $\gamma = \gamma(C_1, C_2) \in [0, \pi/2]$ between two subspaces C_1, C_2* is given by (see Friedrichs [54] or Deutsch [42, 43])

$$\cos\gamma = \sup\{\langle c_1, c_2\rangle : c_1 \in C_1 \cap (C_1 \cap C_2)^\perp, c_2 \in C_2 \cap (C_1 \cap C_2)^\perp, \|c_1\| = \|c_2\| = 1\}.$$

PROPOSITION 5.16. *If C_1, C_2 are two closed subspaces and γ is the angle between them, then the following conditions are equivalent.*

(i) $\gamma > 0$.

(ii) *$C_1 + C_2$ is closed.*

(iii) *$C_1^\perp + C_2^\perp$ is closed.*

(iv) *(C_1, C_2) is linearly regular.*

(v) *(C_1, C_2) is boundedly linearly regular.*

(vi) *(C_1, C_2) is regular.*

(vii) *(C_1, C_2) is boundedly regular.*

Proof. "(i)$\Longleftrightarrow$(ii)" is due to Deutsch [42, Lem. 2.5.(4)] and Simonič ([93], a proof can be found in [13, Lem. 4.10]). "(ii)$\Longleftrightarrow$(iii)" is well known (see Jameson's [70, Cor. 35.6]).

"(ii)$\Longrightarrow$(iv)$\Longleftrightarrow$(v)$\Longleftrightarrow$(vi)$\Longrightarrow$(vii)": Combine Fact 5.12.(ii) and Proposition 5.9.

"(vii)$\Longrightarrow$(i)": Let us prove the contrapositive. Suppose $\gamma = 0$. Then we obtain two sequences $(c_1^{(n)}), (c_2^{(n)})$ with

$$c_1^{(n)} \in C_1, \quad c_2^{(n)} \in C_2, \quad \|c_1^{(n)}\| = \|c_2^{(n)}\| = 1, \quad P_{C_1\cap C_2}c_1^{(n)} = P_{C_1\cap C_2}c_2^{(n)} = 0$$

for every n, and

$$\langle c_1^{(n)}, c_2^{(n)}\rangle \longrightarrow 1 = \cos 0.$$

Expanding $\|c_1^{(n)} - c_2^{(n)}\|^2$ yields $c_1^{(n)} - c_2^{(n)} \longrightarrow 0$. On the one hand, if we define $x^{(n)} := (c_1^{(n)} + c_2^{(n)})/2$, then, by the parallelogram law,

$$\|x^{(n)}\| \longrightarrow 1.$$

On the other hand,

$$P_{C_1\cap C_2}x^{(n)} \equiv 0, \quad x^{(n)} - c_1^{(n)} \longrightarrow 0, \quad x^{(n)} - c_2^{(n)} \longrightarrow 0.$$

Altogether, $(x^{(n)})$ is bounded, $d(x^{(n)}, C_1) \longrightarrow 0$, $d(x^{(n)}, C_2) \longrightarrow 0$, but $d(x^{(n)}, C_1 \cap C_2) = \|x^{(n)}\| \longrightarrow 1$. Therefore, (C_1, C_2) is not boundedly regular and the proof is complete. □

A short excursion into a useful product space. We build—in the spirit of Pierra [85]—the product space

$$\mathbf{X} := \prod_{i=1}^{N}(X, \tfrac{1}{N}\langle\cdot,\cdot\rangle)$$

and define the *diagonal*

$$\boldsymbol{\Delta} := \{(x_1, \ldots, x_N) \in \mathbf{X} : x_1 = x_2 = \cdots = x_N \in X\}$$

and the product

$$\mathbf{B} := \prod_{i=1}^{N} C_i.$$

This allows us to identify the set C with $\boldsymbol{\Delta} \cap \mathbf{B}$. Then (see, for example, [85]) for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, we have

$$\|\mathbf{x} - \mathbf{y}\|^2 = \sum_{i=1}^{N} \tfrac{1}{N}\|x_i - y_i\|^2,$$

and the projections onto $\boldsymbol{\Delta}$ and $\mathbf{B}$ are given by

$$\begin{aligned} P_{\boldsymbol{\Delta}}(x_1, x_2, \ldots, x_N) &= \left(\sum_{i=1}^{N} \tfrac{1}{N}x_i, \sum_{i=1}^{N} \tfrac{1}{N}x_i, \ldots, \sum_{i=1}^{N} \tfrac{1}{N}x_i\right), \\ P_{\mathbf{B}}(x_1, x_2, \ldots, x_N) &= (P_1x_1, P_2x_2, \ldots, P_Nx_N). \end{aligned}$$

PROPOSITION 5.17. *If* $\mathbf{0} \in \operatorname{icr}(\boldsymbol{\Delta} - \mathbf{B})$, *then* $(\boldsymbol{\Delta}, \mathbf{B})$ *is boundedly linearly regular.*

Proof. This is nothing but Fact 5.12 applied in $\mathbf{X}$. □

We now tackle the N-subspace case.

LEMMA 5.18. *If each set C_i is a closed subspace, then the following conditions are equivalent.*

(i) $C_1^\perp + \cdots + C_N^\perp$ *is closed.*

(ii) $\boldsymbol{\Delta} + \mathbf{B}$ *is closed.*

(iii) $(\boldsymbol{\Delta}, \mathbf{B})$ *is (boundedly) (linearly) regular.*

Proof. Denote $C_1^\perp + \cdots + C_N^\perp$ by S and consider

$$T : \mathbf{X} \longrightarrow X : (x_1, \ldots, x_N) \longmapsto x_1 + \cdots + x_N.$$

Clearly, T is onto and kernel $T = \boldsymbol{\Delta}^\perp$. By a useful consequence of the open mapping theorem (see, for example, Holmes's [67, Lem. 17.H]),

$$S \text{ is closed} \iff \boldsymbol{\Delta}^\perp + \prod_{i=1}^{N} C_i^\perp \text{ is closed.}$$

Now apply Proposition 5.16 to $\boldsymbol{\Delta}$ and $\mathbf{B}$ in $\mathbf{X}$. □

THEOREM 5.19 (linear regularity and subspaces). *If each set C_i is a closed subspace, then the following conditions are equivalent.*

(i) $C_1^\perp + \cdots + C_N^\perp$ *is closed.*

(ii) $(C_1, \ldots, C_N)$ *is linearly regular.*

(iii) $(C_1, \ldots, C_N)$ *is boundedly linearly regular.*

(iv) $(C_1, \ldots, C_N)$ *is regular.*

(v) $(C_1, \ldots, C_N)$ *is boundedly regular.*

Proof. "(i)$\Longrightarrow$(ii)": By the last lemma, there is some $\kappa > 0$ s.t.

$$d(\mathbf{x}, \mathbf{\Delta} \cap \mathbf{B}) \leq \kappa \max\{d(\mathbf{x}, \mathbf{\Delta}), d(\mathbf{x}, \mathbf{B})\} \quad \text{for every } \mathbf{x} \in \mathbf{X}.$$

In particular, if $x \in X$ and $\mathbf{x} := (x, \ldots, x) \in \mathbf{\Delta}$, then

$$d^2(x, C) = d^2(\mathbf{x}, \mathbf{\Delta} \cap \mathbf{B}) \leq \kappa^2 d^2(\mathbf{x}, \mathbf{B});$$

therefore, the linear regularity of $(C_1, \ldots, C_N)$ follows from

$$\begin{aligned} d^2(x, C) &\leq \kappa^2 (d^2(x, C_1) + \cdots + d^2(x, C_N))/N \\ &\leq \kappa^2 \max\{d^2(x, C_i) : i = 1 \ldots N\}. \end{aligned}$$

"(ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv)$\Longrightarrow$(v)" follows from Proposition 5.9.

"(v)$\Longrightarrow$(i)": We prove the contrapositive. Suppose $C_1^\perp + \cdots + C_N^\perp$ is not closed. Then, by the preceding lemma, $(\mathbf{\Delta}, \mathbf{B})$ is not boundedly regular. We thus obtain a bounded sequence $(\mathbf{x}^{(n)})$ s.t.

$$d(\mathbf{x}^{(n)}, \mathbf{\Delta}) \longrightarrow 0, \;\; d(\mathbf{x}^{(n)}, \mathbf{B}) \longrightarrow 0, \;\text{but } \underline{\lim}_n d(\mathbf{x}^{(n)}, \mathbf{\Delta} \cap \mathbf{B}) > 0.$$

Define $x^{(n)}$ to be the first coordinate of $P_{\mathbf{\Delta}} \mathbf{x}^{(n)}$. Then the sequence $(x^{(n)})$ is bounded, $\frac{1}{N} \sum_{i=1}^N d^2(x^{(n)}, C_i) \longrightarrow 0$, but $\underline{\lim}_n d^2(x^{(n)}, \bigcap_{i=1}^N C_i) > 0$. Therefore, $(C_1, \ldots, C_N)$ is not boundedly regular and the proof is complete. $\square$

REMARKS 5.20.

- Browder implicitly proved "(i)$\Longrightarrow$(ii)" of the last theorem in [17, §2].
- It is interesting that, unless $N = 2$, the closedness of the sum $C_1 + \cdots + C_N$ is not related to the regularity of the N-tuple $(C_1, \ldots, C_N)$. Indeed, for $N \geq 3$, take two closed subspaces C_1, C_2 with nonclosed sum. (i): Set $C_3 := \cdots := C_N := X$; then $(C_1, \ldots, C_N)$ is not regular (Proposition 5.16), but the sum $C_1 + \cdots + C_N$ is closed. (ii): Set $C_3 := \cdots := C_N := \{0\}$; then $(C_1, \ldots, C_N)$ is regular, but the sum $C_1 + \cdots + C_N$ is not closed. Altogether, *the closedness of the sum $C_1 + \cdots + C_N$ is neither necessary nor sufficient for regularity of the N-tuple $(C_1, \ldots, C_N)$.*
- For closed intersecting affine subspaces, a corresponding version of the last theorem can be formulated (since regularity is preserved under translation of the sets by some fixed vector).
- Applying the last theorem to orthogonal complements yields the following characterization.

  If each set C_i is a closed subspace, then the following conditions are equivalent.

  (i) $C_1 + \cdots + C_N$ is closed.

  (ii) There is some $\kappa > 0$ s.t. for every $x \in X$,

  $$\| P_{\overline{C_1 + \cdots + C_N}} x \| \leq \kappa (\| P_{C_1} x \| + \cdots + \| P_{C_N} x \|).$$

  (iii) For every bounded sequence $(x^{(n)})$,

  $$\max\{\| P_{C_i} x^{(n)} \| : i = 1 \ldots N\} \longrightarrow 0 \;\; \text{implies} \;\; \| P_{\overline{C_1 + \cdots + C_N}} x^{(n)} \| \longrightarrow 0.$$

COROLLARY 5.21. *Suppose each set C_i is a closed subspace. Then the N-tuple $(C_1, \ldots, C_N)$ is linearly regular whenever*

(i) *at least one subspace is finite dimensional or*

(ii) *all subspaces except possibly one have finite co-dimension.*

Proof. (i): Using Proposition 5.16, induction on N, and the well-known fact that *the sum of a closed subspace and a finite-dimensional subspace is closed* (see, for example, Jameson's [70, Prop. 20.1]), we obtain readily that $C_1^\perp + \cdots + C_N^\perp$ is finite co-dimensional and closed. Now apply the last theorem.

(ii): If without loss of generality $C_1, \ldots, C_{N-1}$ are finite co-dimensional, then $C_1^\perp + \cdots + C_{N-1}^\perp$ is finite dimensional. Again, $C_1^\perp + \cdots + C_N^\perp$ is closed and the last theorem applies. □

Once more, an analogous version of the last corollary holds for closed intersecting affine subspaces. We state the most important case.

COROLLARY 5.22 (linear regularity and intersecting hyperplanes). *If each set C_i is a hyperplane, then the N-tuple $(C_1, \ldots, C_N)$ is linearly regular.*

We now give another important class of linearly regular N-tuples.

FACT 5.23 (linear regularity and intersecting halfspaces). If each set C_i is a halfspace, then the N-tuple $(C_1, \ldots, C_N)$ is linearly regular.

REMARK 5.24. In 1952, Hoffman [66] proved this fact, relying on some results by Agmon [1] for Euclidean spaces. It turns out that his proof also works for Hilbert spaces; a detailed proof will appear in the thesis of the first author.

The following result shows how one builds more examples of (boundedly) (linearly) regular tuples.

PROPOSITION 5.25. *Suppose $(C_1, \ldots, C_N)$ is a (boundedly) (linearly) regular N-tuple and $J_1 \cup \cdots \cup J_M = \{1, \ldots, N\}$ is a disjoint decomposition of $\{1, \ldots, N\}$; i.e., $J_m \neq \emptyset$ and $J_m \cap J_{m'} = \emptyset$ for $m, m' \in \{1, \ldots, M\}$ and $m \neq m'$. If we set*

$$D_m := \bigcap_{i \in J_m} C_i \qquad \text{for every } m \in \{1, \ldots, M\},$$

then the M-tuple $(D_1, \ldots, D_M)$ is (boundedly) (linearly) regular.

Proof. Suppose $(C_1, \ldots, C_N)$ is linearly regular. Then there is some $\kappa > 0$ s.t. $d(x, C_1 \cap \cdots \cap C_N) \leq \kappa \max_n d(x, C_n)$ for every $x \in X$; thus

$$\begin{aligned} d(x, D_1 \cap \cdots \cap D_M) &= d(x, C_1 \cap \cdots \cap C_N) \\ &\leq \kappa \max_n d(x, C_n) \\ &= \kappa \max_m \max_{n \in J_m} d(x, C_n) \\ &\leq \kappa \max_m d(x, D_m). \end{aligned}$$

Therefore, $(D_1, \ldots, D_M)$ is linearly regular. The proofs of the remaining cases are similar and thus are omitted. □

COROLLARY 5.26 (linear regularity and intersecting polyhedra). *If each set C_i is a polyhedron, then the N-tuple $(C_1, \ldots, C_N)$ is linearly regular.*

We finish this section with a result on the "frequency" (in the sense of Baire category) of boundedly linear N-tuples. Quite surprisingly, "bounded linear regularity is the rule." Since we will not use this result in what follows, we only sketch a proof. For basic results on the Hausdorff metric, we recommend Klein and Thompson's [76, §4]; for basic results on Baire category see, for example, Holmes's [67, §17].

THEOREM 5.27. *Suppose $\mathcal{T}$ is the set of all N-tuples of the form $(C_1, \ldots, C_N)$, where each set C_i is bounded closed convex and the intersection $\bigcap_{i=1}^N C_i$ is nonempty. Then the subset of all boundedly linearly regular N-tuples is residual in $\mathcal{T}$ (equipped with the Hausdorff metric).*

Sketch of a Proof. We work in the product space $\mathbf{X}$.

Step 1: Show that $\mathcal{T}$ is a closed subset in the complete metric space consisting of all closed subsets of $\mathbf{X}$ equipped with the Hausdorff metric.

Step 2: Denote by $\mathcal{R}$ the subset of $\mathcal{T}$ consisting of all boundedly linearly regular N-tuples. Deduce from Proposition 5.17 that if $\mathbf{B} = (C_1, \ldots, C_N) \in \mathcal{T}$ and $0 \in \operatorname{int}(\mathbf{\Delta} - \mathbf{B})$, then $\mathbf{B} \in \mathcal{R}$.

Step 3: Define $\mathcal{O} := \{B \in \mathcal{T} : 0 \in \operatorname{int}(\mathbf{\Delta} - \mathbf{B})\}$ and show that $\mathcal{O}$ is dense in $\mathcal{T}$ (given $\mathbf{B} = (C_1, \ldots, C_N) \in \mathcal{T}$, consider the "nearby" $(C_1 + \epsilon B_{\mathbf{X}}, \ldots, C_N + \epsilon B_{\mathbf{X}})$ in $\mathcal{O}$ for small $\epsilon > 0$).

Step 4: Prove that $\mathcal{O}$ is open in $\mathcal{T}$. Indeed, denote the Hausdorff metric by h, fix $\mathbf{B} \in \mathcal{O}$, and get $\epsilon > 0$ s.t. $\epsilon B_{\mathbf{X}} \subseteq \mathbf{\Delta} - \mathbf{B}$. Suppose $\mathbf{B}' \in \mathcal{T}$ with $h(\mathbf{B}, \mathbf{B}') < \epsilon/2$. Then

$$\frac{\epsilon}{2} B_{\mathbf{X}} + \frac{\epsilon}{2} B_{\mathbf{X}} = \epsilon B_{\mathbf{X}} \subseteq \mathbf{\Delta} - \mathbf{B} \subseteq \mathbf{\Delta} - \mathbf{B}' + \frac{\epsilon}{2} B_{\mathbf{X}}.$$

By Rådström's cancellation lemma [87, Lem. 1],

$$\frac{\epsilon}{2} B_{\mathbf{X}} \subseteq \mathbf{\Delta} - \mathbf{B}';$$

thus $\mathbf{B}' \in \mathcal{O}$.

Conclusion. $\mathcal{R}$ is residual in $\mathcal{T}$ because $\mathcal{R} \supseteq \mathcal{O}$ and $\mathcal{O}$ is open and dense in $\mathcal{T}$.

REMARK 5.28. In view of Theorems 5.7, 5.8, and 5.27, we can loosely say that "linear convergence is the rule for certain algorithms." The restriction that every C_i be bounded is not really severe since a reduction to this case can be made as soon as the starting point is chosen.

6. Harvest time II: Examples. In this section, numerous examples for our results are given. *To demonstrate the applicability of our framework, we mainly chose examples that are closely related to known results and only occasionally comment on (sometimes very substantial) possible generalizations.*

"Fairly" general sets.

Random control.

EXAMPLE 6.1. Suppose the projection algorithm is linearly focusing and some set C_j is boundedly compact. Suppose further that

(i) $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for every index i or

(ii) there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\sum_{n:n \text{ active for } i} \mu_i^{(n)} = +\infty$ for every index i and all large n active for i.

Then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. By Lemma 3.2.(iv), the sum $\sum_{n:n \text{ active for } j} \mu_j^{(n)} d^2(x^{(n)}, C_j^{(n)})$ is finite. Since the projection algorithm is linearly focusing, the assumptions on $(\mu_i^{(n)})$ imply the existence of a subsequence $(x^{(n')})$ of $(x^{(n)})$ with

$$\|x^{(n')} - P_j x^{(n')}\| = d(x^{(n')}, C_j) \longrightarrow 0 \quad \text{and} \quad j \text{ is active at } n'.$$

After passing to a subsequence if necessary, we can assume that $(P_j x^{(n')})$ is norm convergent; hence, so is $(x^{(n')})$. Therefore, the sequence $(x^{(n)})$ has a norm cluster point. If (i) holds, then the result follows from Theorem 3.10. Otherwise (ii) holds and then the result follows from Theorems 4.15 and 3.20.(iii). ☐

REMARKS 6.2.

- This result improves Examples 4.6, 4.18.(i), and 4.19.
- As we commented in Remarks 4.7, the problem becomes much harder without a compactness assumption. Nevertheless, some interesting results were obtained by Bruck [18], Youla [107], and Dye and Reich [48].

An immediate consequence of Example 6.1 is the following.

EXAMPLE 6.3 (Bruck's [18, Cor. 1.2]). Suppose the projection algorithm is singular and has constant sets where (at least) one is boundedly compact. If there is some $\epsilon > 0$ s.t.

$\epsilon \leq \alpha_i^{(n)} \equiv \alpha_i \leq 2 - \epsilon$ for every index i, then the sequence $(x^{(n)})$ converges in norm to some point in C.

REMARK 6.4. Bruck's proof is quite different and is highly recommended.

Intermittent control. We start with some results on linear convergence.

EXAMPLE 6.5 (Browder's [17, Thm. 3]). Suppose the projection algorithm is almost cyclic, unrelaxed, and has constant sets. Suppose further that $0 \in C$ and for every $r > 0$, there is some $\kappa_r > 0$ s.t.

$$(*) \qquad \|x\| \leq \kappa_r \max\{d(x, C_i) : i = 1 \dots N\} \quad \text{for all } x \in rB_X.$$

Then $C = \{0\}$ and the sequence $(x^{(n)})$ converges linearly to 0.

Proof. By $(*)$, obviously $C \cap rB_X = \{0\}$. Since r can be chosen arbitrarily large, it follows that $C = \{0\}$. Thus $(*)$ states that $d(x, C) = \|x\| \leq \max\{d(x, C_i) : i = 1 \dots N\}$; i.e., the N-tuple $(C_1, \dots, C_N)$ is boundedly linearly regular. The result now follows from Theorem 5.7. □

EXAMPLE 6.6 (Youla and Webb's [108, Thm. 3]). Suppose the projection algorithm is cyclic and has constant sets. Suppose further the relaxation parameters satisfy $0 < \alpha_i^{(n)} \equiv \alpha_i < 2$ for every index i and all n active for i. If there is some index $j \in \{1, \dots, N\}$ s.t.

$$C_j \cap \operatorname{int} \bigcap_{i \in \{1,\dots,N\} \setminus \{j\}} C_i \neq \emptyset,$$

then the sequence $(x^{(n)})$ converges linearly to some point in C.

Proof. By Corollary 5.14 (cf. Remark 5.15), the N-tuple $(C_1, \dots, C_N)$ is boundedly linearly regular. Now apply Theorem 5.7. □

REMARK 6.7. An extended version of Youla and Webb's well-written paper is Youla's [106].

Analogously, we can prove the following.

EXAMPLE 6.8 (Gubin, Polyak, and Raik's [60, Thm. 1.(a)]). Suppose the projection algorithm is cyclic and has constant sets. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for every index i and all n active for i. If there is some index $j \in \{1, \dots, N\}$ s.t.

$$C_j \cap \operatorname{int} \bigcap_{i \in \{1,\dots,N\} \setminus \{j\}} C_i \neq \emptyset,$$

then the sequence $(x^{(n)})$ converges in norm (in fact, linearly) to some point in C.

REMARK 6.9. Gubin, Polyak, and Raik's paper [60] is a cornerstone for this field and contains many original results and applications.

REMARK 6.10. The preceding examples all followed from Theorem 5.7 and the results on bounded linear regularity. Since Theorem 5.7 allows more general control, other iterations are covered as well. For example, the conclusions of the last three examples remain valid if we replace "(almost) cyclic" by "weighted." Similarly, adjusting Theorem 5.2 yields various examples on norm converge.

The following results on weak convergence follow readily from Theorem 3.20.(i).

EXAMPLE 6.11 (Browder's [17, Thm. 2] for finitely many sets). Suppose the projection algorithm is almost cyclic, unrelaxed, and has constant sets. Then the sequence $(x^{(n)})$ converges weakly to some point in C.

REMARK 6.12 (cyclic projections). If in the last example "almost cyclic" is replaced by "cyclic," then one obtains *the method of cyclic projections*; the conclusion of the last example becomes Bregman's [16, Thm. 1]. The case when the sets C_i do not necessarily intersect is discussed in some detail in [14].

EXAMPLE 6.13 (Crombez's [38, Thm. 3]). Suppose the projection algorithm is weighted and has constant sets. Suppose further the relaxation parameters and weights satisfy $0 < \alpha_i^{(n)} \equiv \alpha_i < 2$ and $0 < \lambda_i^{(n)} \equiv \lambda_i$ for every index i and all n. Then the sequence $(x^{(n)})$ converges weakly to some point in C.

REMARK 6.14. Crombez [38] assumed in addition that one of the sets is the entire space (which has the identity as projection).

Consideration of remotest sets control.

EXAMPLE 6.15 (Gubin, Polyak, and Raik's [60, Thm. 1.(a)] for finitely many sets). Suppose the projection algorithm has remotest set control and constant sets. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for every index i and all n active for i. If there is some index $j \in \{1, \ldots, N\}$ s.t.

$$C_j \cap \operatorname{int} \bigcap_{i \in \{1,\ldots,N\} \setminus \{j\}} C_i \neq \emptyset,$$

then the sequence $(x^{(n)})$ converges linearly to some point in C.

Proof. The projection algorithm is linearly focusing and the N-tuple $(C_1, \ldots, C_N)$ is boundedly linearly regular (Corollary 5.14 and Remark 5.15). The result follows from Theorem 5.8. □

EXAMPLE 6.16 (Bregman's [16, Thm. 2] for finitely many sets). Suppose the projection algorithm is unrelaxed and has remotest set control and constant sets. Then the sequence $(x^{(n)})$ converges weakly to some point in C.

Proof. It is immediate from Theorem 4.26.(ii). □

Subspaces.

EXAMPLE 6.17 (Browder's [17, Cor. to Thm. 3]). Suppose the projection algorithm is almost cyclic, unrelaxed, and has constant sets that are closed subspaces. If

$$C_1^{\perp} + \cdots + C_N^{\perp}$$

is closed, then the sequence $(x^{(n)})$ converges linearly.

Proof. Combine Theorems 5.7 and 5.19. □

EXAMPLE 6.18 (a remark on Smith, Solmon, and Wagner's [94, Thm. 2.2]). Suppose the projection algorithm is cyclic, unrelaxed, and has constant sets that are closed subspaces. If the angle between

$$C_i \text{ and } C_{i+1} \cap \cdots \cap C_N$$

is positive for every index $i \in \{1, \ldots, N-1\}$, then the sequence $(x^{(n)})$ converges linearly.

Proof. Combine Theorems 5.7 and 5.11 and Proposition 5.16. □

REMARKS 6.19.

- In the last two examples, the two quite different looking hypotheses on the subspaces turned out to be *special instances of bounded linear regularity*. This, together with Theorem 5.7, explained linear convergence.
- It follows from Amemiya and Ando's work [5] that the limits of the sequences of the two previous examples equal

$$P_C x^{(0)}.$$

- The grandfather of these results on subspaces is the following.

Suppose the projection algorithm is cyclic, unrelaxed, and has constant sets that are closed subspaces. Then the sequence $(x^{(n)})$ converges in norm to $P_C x^{(0)}$.

The 2-subspace version is due to von Neumann [103, Thm. 13.7]; Halperin [61, Thm. 1] proved the N-subspace version. The reader will note that there is no hypothesis on the subspaces (and, however, no conclusion on the rate of convergence). Since bounded regularity and linear regularity of an N-tuple of subspaces are the same (Theorem 5.19), our framework is incapable of recapturing the von Neumann/Halperin result. For applications, though, one is often interested in *linear* convergence results. Those follow under additional hypotheses that imply regularity (see the last two examples) and are thus covered by our framework. The best and most complete reference on the von Neumann/Halperin framework and its impressive applications is Deutsch's survey article [44]; see also Deutsch and Hundal's recent [45].

- Although mathematically intriguing, controls that are different from intermittent or remotest set control seem to be of little use for applications; consider, for example, two closed subspaces with closed sum and with intersection $\{0\}$. A singular unrelaxed projection algorithm for these two sets converges linearly whenever its control is intermittent or considers remotest sets (cf. Theorems 5.7 and 5.8). However, if we consider, for example, the random control version where we project onto the first subspace whenever n is a power of 2, and onto the second subspace otherwise, then the resulting sequence $(x^{(n)})$ is *not* linearly convergent.

Hyperplanes. Hyperplanes play an important role in applications for two reasons. First, the solution of a system of linear equations is nothing but the intersection of the corresponding hyperplanes. Second, projections onto hyperplanes can be calculated easily. In fact, if a hyperplane C_i is given by

$$C_i = \{x \in X : \langle a_i, x\rangle = b_i\}$$

for some $a_i \in X \setminus \{0\}$ and $b_i \in \mathbb{R}$, then, for every $x \in X$,

$$P_i x = P_{C_i} x = x - \frac{(\langle a_i, x\rangle - b_i)}{\|a_i\|^2} a_i \quad \text{and} \quad d(x, C_i) = \frac{|\langle a_i, x\rangle - b_i|}{\|a_i\|}.$$

Intermittent control.

EXAMPLE 6.20. Suppose the projection algorithm is intermittent and has constant sets that are hyperplanes. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all large n and every index i active at n. Then the sequence converges linearly to some point in C with a rate independent of the starting point.

Proof. Combine Theorem 5.7 and Corollary 5.22. □

The following special cases of the last example are well known.

EXAMPLE 6.21 (Herman, Lent, and Lutz's [64, Cor. 1], Trummer's [97, Thm. 5]). Suppose X is finite dimensional and the projection algorithm is cyclic and has constant sets that are hyperplanes. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

EXAMPLE 6.22 (Kaczmarz [71], Gordon, Bender, and Herman [59]). Suppose X is finite dimensional and the projection algorithm is cyclic, unrelaxed, and has constant sets that are hyperplanes. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

REMARKS 6.23.

- The precursor of these results is certainly the last example, which was discovered by Kaczmarz as early as 1937.
- Kaczmarz's method is well understood even in the *infeasible* case; we refer the interested reader to Tanabe's [96] and Trummer's [97, 99].
- The iteration described in Example 6.21 is also known as "ART" (algebraic reconstruction technique).

EXAMPLE 6.24 (Trummer's [97, first part of Thm. 1]). Suppose X is finite dimensional and the projection algorithm is cyclic and has constant sets that are hyperplanes. If ν_n is defined as in Theorem 5.2 and $\sum_n \nu_n = +\infty$, then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. It is immediate from Theorem 5.2. □

REMARKS 6.25.

- Trummer also investigated the infeasible case; see [97, 99].
- Using Theorem 5.2, we can similarly recapture Trummer's [97, second part of Thm. 1], where he describes an iteration that yields a *nonnegative* solution (assuming there exists at least one).

REMARK 6.26. Herman et al. [65] used block control variants of Example 6.20 for image reconstruction. Their algorithms are based on a (more matrix-theoretic) framework by Eggermont, Herman, and Lent [49].

Weighted control.

EXAMPLE 6.27 (Trummer's [98, Thm. 8]). Suppose X is finite dimensional and the projection algorithm is weighted, unrelaxed, and has constant sets that are hyperplanes $C_i = \{x \in X : \langle a_i, x\rangle = b_i\}$. If the weights are given by

$$\lambda_i^{(n)} \equiv \frac{\|a_i\|^2}{\sum_{j=1}^N \|a_j\|^2},$$

then the sequence $(x^{(n)})$ converges linearly to some point in C.

Proof. The control is 1-intermittent; thus, the result follows from Example 6.20. □

REMARK 6.28. Trummer even allowed infeasible systems and identified the limit; see [98].

THEOREM 6.29. *Suppose the projection algorithm is weighted and has constant sets that are hyperplanes $C_i = \{x \in X : \langle a_i, x\rangle = b_i\}$. Suppose further there exists a subsequence (n') of (n) and some $\epsilon > 0$ s.t.*

$$\epsilon \leq \alpha_i^{(n')},\ \lambda_i^{(n')} \quad \textit{for all } n' \textit{ and every index } i.$$

If span $\{a_1, \dots, a_N\}$ *is at least two dimensional, then the sequence $(x^{(n)})$ converges in norm to some point in C.*

Proof. Without loss of generality, we assume $\|a_i\| = 1$ for every index i. Fix $x \in C$. Then, by Lemma 3.2.(i),

$$\|x^{(n)} - x\|^2 - \|x^{(n+1)} - x\|^2 \geq \sum_{i<j} \lambda_i^{(n)}\lambda_j^{(n)}\alpha_i^{(n)}\alpha_j^{(n)}\|P_i x^{(n)} - P_j x^{(n)}\|^2$$

for all $n \geq 0$. Summing over n and remembering that each set C_i is a hyperplane, we obtain a convergent series whose general term

$$\sum_{i<j} \lambda_i^{(n)}\lambda_j^{(n)}\alpha_i^{(n)}\alpha_j^{(n)}\|(\langle a_j, x^{(n)}\rangle - b_j)a_j - (\langle a_i, x^{(n)}\rangle - b_i)a_i\|^2$$

tends to 0. Hence, along the subsequence $(x^{(n')})$, we have

$$(*) \qquad (\langle a_i, x^{(n')}\rangle - b_i)a_i - (\langle a_j, x^{(n')}\rangle - b_j)a_j \longrightarrow 0 \quad \text{whenever } i \neq j.$$

Now fix any index i and obtain another index $j \neq i$ s.t. $\{a_i, a_j\}$ are *linearly independent.* Then, by $(*)$,

$$\langle a_i, x^{(n')}\rangle - b_i \longrightarrow 0.$$

Thus $d(x^{(n')}, C_i) = |\langle a_i, x^{(n')}\rangle - b_i| \longrightarrow 0$. Since i has been chosen arbitrarily, we conclude that

$$\max\{d(x^{(n')}, C_i) : i = 1 \dots N\} \longrightarrow 0,$$

and further, by linear regularity of $(C_1, \dots, C_N)$ (Corollary 5.22),

$$d(x^{(n')}, C) \longrightarrow 0.$$

The result follows from Corollary 3.3.(ii). $\square$

The following classical example is now obvious.

EXAMPLE 6.30 (Cimmino's method [29] in Hilbert space). Suppose the projection algorithm is weighted and has constant sets that are hyperplanes $C_i = \{x \in X : \langle a_i, x\rangle = b_i\}$. Suppose further the relaxation parameters and weights satisfy

$$\alpha_i^{(n)} \equiv 2, \quad \lambda_i^{(n)} \equiv \lambda_i > 0$$

for all $n \geq 0$ and every index i. If span $\{a_1, \dots, a_N\}$ is at least two dimensional, then the sequence $(x^{(n)})$ converges in norm to some point in C.

REMARKS 6.31.

- For Euclidean spaces, the last example was known to Cimmino as far back as 1938. His method has a nice geometric interpretation: one obtains $x^{(n+1)}$ from $x^{(n)}$ by reflecting $x^{(n)}$ in all N hyperplanes and then taking a weighted average.
- As Example 3.6 shows, the assumption on span $\{a_1, \dots, a_N\}$ is essential.
- Due to their parallelizability, Cimmino's and related methods with weighted control are currently used with great success; see Censor's survey article [23].

We present a variation of Cimmino's method that includes a method suggested by Ansorge.

EXAMPLE 6.32 (a generalization of Ansorge's method [6]). Suppose X is finite dimensional and the projection algorithm has constant sets where $C_N = X$. Suppose further that

$$\alpha_N^{(n)} \equiv 1, \; \lambda_N^{(n)} \equiv \lambda_N > 0, \; \alpha_1^{(n)} \equiv \cdots \equiv \alpha_{N-1}^{(n)} \equiv 2,$$

and

$$\lambda_i^{(n)} := \begin{cases} \dfrac{(1-\lambda_N) f(d(x^{(n)}, C_i))}{\sum_{j=1}^{N} f(d(x^{(n)}, C_j))} & \text{if } x^{(n)} \notin C, \\[2ex] \dfrac{1-\lambda_N}{N-1} & \text{otherwise} \end{cases}$$

for some strictly increasing continuous function $f : [0, +\infty[\to [0, +\infty[$ with $f(0) = 0$. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. Clearly, the projection algorithm is strongly focusing and considers remotest sets. The N-tuple $(C_1, \dots, C_N)$ is boundedly regular (Proposition 5.4.(iii)). Suppose that $(i^{(n)})$ is a sequence of active remotest indices. Then $i^{(n)} \in \{1, \dots, N-1\}$ and

$$\mu_{i^{(n)}}^{(n)} = \lambda_{i^{(n)}}^{(n)} 2\lambda_N.$$

If $\sum_n \lambda_{i(n)}^{(n)} = +\infty$, then we are done by Theorem 5.3. Otherwise $\sum_n \lambda_{i(n)}^{(n)} < +\infty$ and hence $\lambda_{i(n)}^{(n)} \longrightarrow 0$. Now $(x^{(n)})$ is bounded and f is continuous; thus $(\sum_j f(d(x^{(n)}, C_j)))_n$ is a bounded sequence. Consequently,

$$f(d(x^{(n)}, C_i)) \leq f(d(x^{(n)}, C_{i(n)})) \longrightarrow 0 \quad \text{for every index } i,$$

which implies that

$$\max\{d(x^{(n)}, C_i) : i = 1 \ldots N\} \longrightarrow 0.$$

Because of the bounded regularity of $(C_1, \ldots, C_N)$, we get $d(x^{(n)}, C) \longrightarrow 0$; now Corollary 3.3.(ii) completes the proof. ☐

REMARK 6.33. Ansorge's method [6] arises when the sets $C_1, \ldots, C_{N-1}$ are hyperplanes and $f = |\cdot|^\gamma$ for $\gamma > 0$.

Halfspaces. Halfspaces play an important role for essentially the same reasons hyperplanes do: their intersection describes the solution of the corresponding system of linear inequalities (this problem is also referred to as the *linear feasibility problem*) and the projections are easy to calculate. Indeed, if a halfspace C_i is given by

$$C_i = \{x \in X : \langle a_i, x\rangle \leq b_i\}$$

for some $a_i \in X \setminus \{0\}$ and $b_i \in \mathbb{R}$, then, for every $x \in X$,

$$P_i x = P_{C_i} x = x - \frac{(\langle a_i, x\rangle - b_i)^+}{\|a_i\|^2} a_i \quad \text{and} \quad d(x, C_i) = \frac{(\langle a_i, x\rangle - b_i)^+}{\|a_i\|}.$$

Some of the algorithms for finding a solution of the linear feasibility problem discussed below have been used with great success in radiation therapy treatment planning; we refer the reader to Censor, Altschuler, and Powlis's interesting survey article [25].

Intermittent control.

EXAMPLE 6.34. Suppose the projection algorithm is intermittent and has constant sets that are halfspaces. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

Proof. Combine Theorem 5.7 and Fact 5.23. ☐

We deduce readily the next two examples.

EXAMPLE 6.35 (Gubin, Polyak, and Raik's [60, Thm. 1.(d)], Herman, Lent, and Lutz's [64, Thm. 1]). Suppose the projection algorithm is cyclic and has constant sets that are halfspaces. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all large n and every index i active at n. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

REMARKS 6.36. By Example 6.34, the rate of convergence of the sequence $(x^{(n)})$ is actually linear and independent of the starting point. Herman, Lent, and Lutz assumed additionally that X is finite dimensional. Mandel [78, Thm. 3.1] offered an upper bound for the rate of convergence for the case when X is finite dimensional and $0 < \alpha_i^{(n)} \equiv \alpha < 2$.

EXAMPLE 6.37 (Censor, Altschuler, and Powlis's [25, Alg. 3]). Suppose the projection algorithm considers only blocks and has constant sets that are hyperplanes. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all $n \geq 0$ and every index i. Suppose finally that for every index i, there is some $\lambda_i > 0$ s.t. $\lambda_i^{(n)} = \lambda_i$ for all n active for i. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

REMARKS 6.38. Censor, Altschuler, and Powlis [25] offered no results on convergence; however, Aharoni and Censor's [3, Thm. 1] yields norm convergence of $(x^{(n)})$ in Euclidean spaces. We thus add two features. First, we remove the restriction on finite dimensionality. Second, we establish linear convergence.

Weighted control. The following two examples are also consequences of Example 6.34.

EXAMPLE 6.39 (Eremin's [52, Cor. to Thm. 1.2]). Suppose the projection algorithm is weighted and has constant sets that are halfspaces. Suppose further the relaxation parameters and weights satisfy

$$0 < \alpha_i^{(n)} \equiv \alpha_i < 2, \;\; 0 < \lambda_i^{(n)} \equiv \lambda_i, \;\; \alpha_i \lambda_i < \frac{2}{N}$$

for every index i and all n. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

EXAMPLE 6.40 (the feasible case of De Pierro and Iusem's [39, Lem. 8]). Suppose X is finite dimensional and the projection algorithm is weighted and has constant sets that are halfspaces. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \le \alpha_i^{(n)} \equiv \alpha^{(n)} \le 2 - \epsilon$, $0 < \lambda_i^{(n)} \equiv \lambda_i$ for every index i and all n. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

REMARKS 6.41. In the last example, the rate of convergence of the sequence $(x^{(n)})$ is actually linear and independent of the starting point. For a slightly more restrictive scheme, De Pierro and Iusem could also identify the limit of $(x^{(n)})$ in the infeasible case as a least squares solution; see [39].

Consideration of remotest sets control.

EXAMPLE 6.42. Suppose the projection algorithm considers remotest sets and has constant sets that are halfspaces. Suppose further that $(i^{(n)})$ is a sequence of active remotest indices. If $\underline{\lim}_n \mu_{i^{(n)}}^{(n)} > 0$, then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

Proof. Combine Theorem 5.8 and Fact 5.23. □

EXAMPLE 6.43 (Gubin, Polyak, and Raik's [60, Thm. 1.(d)]). Suppose the projection algorithm has remotest set control and constant sets that are halfspaces. If there is some $\epsilon > 0$ s.t. $\epsilon \le \alpha_i^{(n)} \le 2 - \epsilon$ for all n and every index i active at n, then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

Proof. We have for any index i active at n,

$$\mu_i^{(n)} = \lambda_i^{(n)} \alpha_i^{(n)} (2 - \textstyle\sum_{j=1}^N \lambda_j^{(n)} \alpha_j^{(n)}) = \alpha_i^{(n)} (2 - \alpha_i^{(n)}) \ge \epsilon^2;$$

the result thus follows from the previous example. □

The basic result in this subsection is due to Agmon and Motzkin and Schoenberg. It dates back to as early as 1954.

EXAMPLE 6.44 (Agmon's [1, Thm. 3], Motzkin and Schoenberg's [83, Case 1 in Thm. 1 and Thm. 2]). Suppose X is finite dimensional and the projection algorithm has remotest set control and constant sets that are halfspaces. If $0 < \alpha_i^{(n)} \equiv \alpha < 2$ for all n and every index i active at n, then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. This is a special case of the preceding example. □

REMARKS 6.45.

- While Agmon considered only the case when $\alpha = 1$, he already obtained linear convergence of $(x^{(n)})$ with a rate independent of the starting point.
- Motzkin and Schoenberg did not establish linear convergence; they discussed, however, the case when $\alpha = 2$.
- It follows from Example 6.43 that the rate of convergence is linear and independent of the starting point. Again, Mandel provided an upper bound for the rate; see [78, Thm. 2.2].

THEOREM 6.46. *Suppose $N \ge 2$, the projection algorithm has constant sets that are halfspaces $C_i = \{x \in X : \langle a_i, x \rangle \le b_i\}$, and* violated constraints correspond exactly to active indices; *i.e., if* $x^{(n)} \notin C$, *then*

$$x^{(n)} \notin C_i \iff i \in I^{(n)}.$$

Suppose further that there is some $\epsilon > 0$ *s.t.* $\epsilon \le \alpha_i^{(n)},\ \lambda_i^{(n)}$ *for all large n and every index i active at n and that*

$$I^{(n)} = \{i\} \text{ implies } \alpha_i^{(n)} \le 2 - \epsilon.$$

Suppose finally that if i and j are two distinct indices, then

$$(*) \qquad \left\{ \begin{array}{l} \text{either } \{a_i, a_j\} \textit{ is linearly independent} \\ \text{or } X \setminus C_i \subseteq C_j \textit{ (equivalently, } X \setminus C_j \subseteq C_i\textit{),} \\ \text{but never both.} \end{array} \right.$$

Then the sequence $(x^{(n)})$ *converges in norm to some point in* C.

Proof. We assume without loss that $\|a_i\| = 1$ for every index i and that $x^{(n)} \notin C$ for all n (otherwise, the projection algorithm becomes constant anyway). Clearly, the projection algorithm is linearly focusing and considers remotest sets, so let $(i^{(n)})$ be a sequence of active remotest indices. Since the N-tuple $(C_1, \ldots, C_N)$ is linearly regular (Fact 5.23), we can also assume that $\sum_n \mu_{i^{(n)}}^{(n)} < +\infty$ (otherwise, we are done by Theorem 5.3). Hence

$$\mu_{i^{(n)}}^{(n)} = \lambda_{i^{(n)}}^{(n)} \alpha_{i^{(n)}}^{(n)} (2 - \textstyle\sum_{j=1}^{N} \lambda_j^{(n)} \alpha_j^{(n)}) \longrightarrow 0.$$

Claim 1: $\qquad I^{(n)}$ is not a singleton for all large n.

Otherwise, there would be a subsequence (n') of (n) s.t. $I^{(n')} = \{i^{(n')}\}$. On the other hand, $\mu_{i^{(n')}}^{(n')} = 1\alpha_{i^{(n')}}^{(n')}(2 - 1\alpha_{i^{(n')}}^{(n')}) \ge \epsilon^2$, which would contradict $\mu_{i^{(n)}}^{(n)} \longrightarrow 0$. Hence Claim 1 is verified.

By Claim 1, we can find a subsequence (n') of (n) and two distinct indices i, j s.t.

$$i^{(n')} \equiv i \quad \text{and} \quad \{i, j\} \subseteq I^{(n')} \quad \text{for all } n'.$$

Claim 2: $\qquad \{a_i, a_j\}$ is linearly independent.

Otherwise, $X \setminus C_i \subseteq C_j$. Since $x^{(n')} \notin C_i$, we would conclude $x^{(n')} \in C_j$, which would contradict $j \in I^{(n')}$. Thus Claim 2 holds.

Similarly to the proof of Theorem 6.29, we get

$$P_i x^{(n')} - P_j x^{(n')} \longrightarrow 0$$

or

$$\Big(\langle a_i, x^{(n')}\rangle - b_i\Big) a_i - \Big(\langle a_j, x^{(n')}\rangle - b_j\Big) a_j \longrightarrow 0.$$

Now $\langle a_i, x^{(n')}\rangle - b_i = d(x^{(n')}, C_i)$ and $\langle a_j, x^{(n')}\rangle - b_j = d(x^{(n')}, C_j)$; hence Claim 2 implies in particular that $d(x^{(n')}, C_i) \longrightarrow 0$, or, recalling that $i^{(n')} \equiv i$,

$$\max\{d(x^{(n')}, C_l) : l = 1 \ldots N\} \longrightarrow 0.$$

The linear regularity of $(C_1, \ldots, C_N)$ yields $d(x^{(n')}, C) \longrightarrow 0$. Now apply Corollary 3.3.(ii). $\square$

EXAMPLE 6.47 (Censor and Elfving's framework [26, Alg. 1]). Suppose X is finite dimensional, $N \ge 2$, and the projection algorithm has constant sets that are halfspaces. Define $I_{\text{out}}^{(n)} := \{i \in \{1, \ldots, N\} : x^{(n)} \notin C_i\}$ for all $n \ge 0$, and let $m_1, \ldots, m_N > 0$ be given constants with $\sum_{i=1}^{N} m_i = 1$. Suppose further the relaxation parameters and weights are chosen according to the following cases.

0: $I_{\text{out}}^{(n)}$ is empty. Then choose the relaxation parameters and weights as you wish (the projection algorithm becomes constant anyway).

1: $I_{\text{out}}^{(n)} = \{i^{(n)}\}$ is a singleton. Then set

$$\alpha_i^{(n)} :\equiv 2m_{i^{(n)}} \text{ and } I^{(n)} := \{i^{(n)}\}.$$

2: $I_{\text{out}}^{(n)}$ contains at least two indices. Then set

$$\alpha_i^{(n)} :\equiv 2 \text{ and } \lambda_i^{(n)} = \begin{cases} 0 & \text{if } x^{(n)} \in C_i, \\ \dfrac{m_i}{\sum_{j:j\notin C_j} m_j} & \text{otherwise.} \end{cases}$$

Suppose finally that if i and j are two distinct indices, then

$$(*) \qquad \begin{cases} \text{either } \{a_i, a_j\} \text{ is linearly independent} \\ \text{or } X \setminus C_i \subseteq C_j \text{ (equivalently, } X \setminus C_j \subseteq C_i), \\ \text{but never both.} \end{cases}$$

Then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. This is a special case of the previous example. □

REMARKS 6.48.

- Censor and Elfving also investigated an iteration [26, Alg. 2] that is more general than the iteration in Example 6.47. Their method of proof is more matrix theoretic and is quite different from ours.
- They claimed that the last example does not need the hypothesis $(*)$. This is, however, false since otherwise a suitable modification of Example 3.6 would yield a counterexample.
- It is possible to recapture *Cimmino's method* (Example 6.30) *for pairwise distinct hyperplanes* by describing each hyperplane $\{x \in X : \langle a_i, x\rangle = b_i\}$ by the corresponding two halfspaces $\{x \in X : \langle a_i, x\rangle \leq b_i\}$, $\{x \in X : \langle -a_i, x\rangle \leq -b_i\}$ and then applying the previous example. This nice observation is due to Censor and Elfving. The assumption that the hyperplanes are pairwise distinct is not really severe; it merely means that "each hyperplane should be counted only once."

REMARK 6.49. More algorithms for solving the linear feasibility problem are given in §7.

Polyhedra. The class of polyhedra is large: it contains the class of halfspaces, the class of hyperplanes, and the class of finite-co-dimensional affine subspaces. It is generally not easy to calculate projections onto polyhedra; there are, however, besides the examples discussed in the previous subsections, two additional important exceptions—hyperslabs and the finite-dimensional positive cone.

A *hyperslab* C_i is given by

$$C_i = \{x \in X : c_i \leq \langle a_i, x\rangle \leq b_i\}$$

for some $a_i \in X \setminus \{0\}$ and two real numbers $c_i \leq b_i$. Then, for every x,

$$P_i x = P_{C_i} x = x - \frac{(\langle a_i, x\rangle - b_i)^+ - (c_i - \langle a_i, x\rangle)^+}{\|a_i\|^2} a_i$$

and

$$d(x, C_i) = \frac{\left|(\langle a_i, x\rangle - b_i)^+ - (c_i - \langle a_i, x\rangle)^+\right|}{\|a_i\|}.$$

The *positive cone* in $X := \mathbb{R}^d$ is denoted X^+ and is given by $X^+ = \{x \in X : x_i \geq 0 \text{ for } i = 1, \ldots, d\}$. Its projection is given by $x = (x_i)_{i=1}^d \longmapsto x^+ := (x_i^+)_{i=1}^d$ for every $x \in X$.

EXAMPLE 6.50 (Censor, Altschuler, and Powlis's [25, Alg. 4]). Suppose X is finite dimensional and the projection algorithm has constant sets that are hyperslabs except $C_N = X^+$. Suppose further the projection algorithm considers only blocks, where the number of blocks is M and $J_M = \{N\}$. If there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all n and every index i active at n, then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

Proof. By Corollary 5.26, the N-tuple $(C_1, \ldots, C_N)$ is linearly regular. Now apply Theorem 5.7. $\square$

REMARK 6.51. Again, Aharoni and Censor's [3, Thm. 1] guarantees norm convergence. We obtain in addition *linear convergence*.

7. Harvest time III: Subgradient algorithms.

Theory. We return to the setting of §4, where we defined projection algorithms. Loosely speaking, "a projection algorithm that for at least one index i chooses its supersets $C_i^{(n)}$ of C_i to be halfspaces constructed from subgradients of a fixed convex function" is called a subgradient algorithm. Before we make this "construction" precise, we collect some basic facts on subgradients.

DEFINITION 7.1. Suppose $f : X \longrightarrow \mathbb{R}$ is a convex function. Given a point $x_0 \in X$, the set

$$\{x^* \in X : \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) \text{ for all } x \in X\}$$

is called the *subdifferential of f at x_0* and is denoted $\partial f(x_0)$. The elements of $\partial f(x_0)$ are called *subgradients of f at x_0*. If $\partial f(x_0)$ is nonempty, then f is said to be *subdifferentiable at x_0*.

The importance of this concept stems from the easy-to-verify fact that

$$x_0 \text{ is a minimizer of } f \iff 0 \in \partial f(x_0).$$

Deeper are the following facts: for proofs see, for example, Ekeland and Temam's [50, Chap. I: Cor. 2.5, Prop. 5.3, Prop. 5.2, and Cor. 2.3].

FACTS 7.2. Suppose $f : X \longrightarrow \mathbb{R}$ is a convex function and $x_0 \in X$. Then

(i) f is continuous at x_0 and $\partial f(x_0)$ is a singleton if and only if f is lower semicontinuous and Gâteaux differentiable at x_0. In this case, the unique subgradient of f at x_0 coincides with the Gâteaux derivative of f at x_0.

(ii) If f is continuous at x_0, then f is subdifferentiable at x_0.

(iii) If X is finite dimensional, then f is continuous and subdifferentiable everywhere.

LEMMA 7.3. *Suppose $f : X \longrightarrow \mathbb{R}$ is a convex function, $x_0 \in X$, and f is subdifferentiable at x_0. Suppose further $S := \{x \in X : f(x) \leq 0\}$ is nonempty. For any $g(x_0) \in \partial f(x_0)$, define the closed convex set H by*

$$H := H(f, x_0, g(x_0)) := \{x \in X : f(x_0) + \langle g(x_0), x - x_0 \rangle \leq 0\}.$$

Then

(i) *$H \supseteq S$. If $g(x_0) \neq 0$, then H is a halfspace; otherwise, $H = X$.*

(ii) $$P_H x_0 = \begin{cases} x_0 - \dfrac{f(x_0)}{\|g(x_0)\|^2} g(x_0) & \textit{if } f(x_0) > 0, \\ x_0 & \textit{otherwise.} \end{cases}$$

(iii) $d(x_0, H) = \begin{cases} \dfrac{f(x_0)}{\|g(x_0)\|} & \textit{if } f(x_0) > 0, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (i): If $s \in S$, then $f(x_0) + \langle g(x_0), s - x_0 \rangle \leq f(s) \leq 0$ and hence $s \in H$.

(ii): Use Facts 1.5.(ii) to verify the candidate for $P_H x_0$. (iii) follows immediately from (ii). □

REMARK 7.4. The importance of the halfspace defined in the last lemma is explained by the following. Suppose we want to find a point in S; i.e., we look for a solution of the convex feasibility problem $f(x) \leq 0$. If $f(x_0) \leq 0$, then we are done. Otherwise $f(x_0) > 0$. It is usually "hard" to solve $f(x) = 0$ (otherwise, we would just solve); therefore, we instead consider a *first-order approximation of* f, say

$$f(x) \approx \tilde{f}(x) := f(x_0) + \langle g(x_0), x - x_0 \rangle \quad \text{for some } g(x_0) \in \partial f(x_0),$$

and solve $\tilde{f}(x_0) = 0$, to which a solution is given by

$$P_H x_0 = x_0 - \frac{f(x_0)}{\|g(x_0)\|^2} g(x_0).$$

We now give the precise definition of a subgradient algorithm.

DEFINITION 7.5. Suppose for some index $i \in \{1, \ldots, N\}$ and for all n every set $C_i^{(n)}$ of a given projection algorithm is of the form

$$\begin{aligned} C_i^{(n)} &= H(f_i, x^{(n)}, g_i(x^{(n)})) \\ &= \{x \in X : f_i(x^{(n)}) + \langle g_i(x^{(n)}), x - x^{(n)} \rangle \leq 0\} \end{aligned}$$

for some fixed convex function $f_i : X \longrightarrow \mathbb{R}$, where f_i is subdifferentiable at every $x^{(n)}$ and $g_i(x^{(n)}) \in \partial f_i(x^{(n)})$. Suppose further that

$$C_i = \{x \in X : f_i(x) \leq 0\}.$$

Then we call this projection algorithm a *subgradient algorithm*. Every such index i is called a *subgradient index*; the set of all subgradient indices is denoted I_∂.

REMARKS 7.6.

- Subgradient algorithms and projection algorithms are closely related in the following sense.

  (i) Every subgradient algorithm is a projection algorithm (by definition).

  (ii) Every *projection algorithm with constant sets* can be viewed as a subgradient algorithm. To see this, one chooses $f_i := d(\cdot, C_i)$ and takes into account that

$$\partial f_i(x) = \partial d(x, C_i) = \begin{cases} \dfrac{x - P_i x}{\|x - P_i x\|} & \text{if } x \notin C_i, \\ N_{C_i}(x) \cap B_X & \text{otherwise,} \end{cases}$$

  where $N_{C_i}(x) = \{x^* \in X : \langle C_i - x, x^* \rangle \leq 0\}$ is the *normal cone of* C_i *at* x.

- The aim of subgradient algorithms is to solve convex feasibility problems. For a good survey on subgradient algorithms and other methods for solving convex feasibility methods, see Censor's [22].
- The reader should be warned that our use of the term "subgradient algorithm" is not quite standard. In the literature, "subgradient algorithms" may refer to considerably more general algorithms; see, for example, Shor's [92].

We now provide a fairly large class of *focusing* subgradient algorithms to which our previous results are applicable.

THEOREM 7.7 (prototype of a focusing subgradient algorithm). *Given a subgradient algorithm, suppose the subdifferentials of f_i are nonempty and uniformly bounded on bounded sets for every index $i \in I_\partial$. Suppose further $(P_i^{(n)})$ converges actively pointwise to P_i for every index $i \in \{1, \ldots, N\} \setminus I_\partial$. Then the subgradient algorithm is focusing.*

Proof. Fix an index $i \in \{1, \ldots, N\}$. Suppose $(x^{(n_k)})$ is a subsequence of $(x^{(n)})$ with $x^{(n_k)} \rightharpoonup x$, $x^{(n_k)} - P_i^{(n_k)} x^{(n_k)} \to 0$, and i is active at n_k for all k. We must show that $x \in C_i$. In view of Lemma 4.2, we need only consider the case when $i \in I_\partial$. Then, by weak lower semicontinuity of f_i,

$$f_i(x) \leq \underline{\lim}_k f_i(x^{(n_k)}).$$

If $f_i(x^{(n_k)}) \leq 0$ infinitely often, then clearly $f_i(x) \leq 0$ and so $x \in C_i$. Otherwise $f_i(x^{(n_k)}) > 0$ for all large k. Since the sequence $(x^{(n)})$ is bounded, there is some $M > 0$ s.t. the norm of every subgradient of f_i at $x^{(n)}$ is at most M. Thus, by Lemma 7.3.(iii),

$$0 \longleftarrow d(x^{(n_k)}, C_i^{(n_k)}) = \frac{f_i(x^{(n_k)})}{\|g_i(x^{(n_k)})\|} \geq \frac{f_i(x^{(n_k)})}{M};$$

hence $f_i(x^{(n_k)}) \longrightarrow 0$. Therefore, $f_i(x) \leq 0$ and $x \in C_i$. □

The property that ∂f_i is uniformly bounded on bounded sets is a standard assumption for theorems on subgradient algorithms; see, for example, [52, 86, 28, 4, 40]. We now *characterize* this property.

PROPOSITION 7.8 (uniform boundedness of subdifferentials on bounded sets). *Suppose $f : X \longrightarrow \mathbb{R}$ is a convex function. Then the following conditions are equivalent.*

(i) *f is bounded on bounded sets.*

(ii) *f is (globally) Lipschitz continuous on bounded sets.*

(iii) *The subdifferentials of f are nonempty and uniformly bounded on bounded sets.*

Proof. "(i)$\Longrightarrow$(ii)" can be found in Roberts and Varberg's [88, Proof of Thm. 41.B].

"(ii)$\Longrightarrow$(iii)": By Facts 7.2.(ii), f is subdifferentiable everywhere. It is enough to show that the subgradients of f are uniformly bounded on open balls centered at 0. So fix $r > 0$ and obtain (by assumption) a Lipschitz constant for f on $\operatorname{int} rB_X$, say L. Now fix $x \in \operatorname{int} rB_X$ and get $s > 0$ s.t. $x + sB_X \subseteq \operatorname{int} rB_X$. Pick any $x^* \in \partial f(x)$ and $b \in B_X$. Then

$$\langle x^*, sb \rangle \leq f(x + sb) - f(x) \leq Ls\|b\|;$$

thus $\|x^*\| = \sup\langle x^*, B_X \rangle \leq L$ and therefore the subgradients of f are uniformly bounded on $\operatorname{int} rB_X$ by L.

"(iii)$\Longrightarrow$(i)": It is enough to show that f is bounded on rB_X for every $r > 0$. By assumption, there is some $M > 0$ s.t. the norm of any subgradient of f at any point in rB_X is at most M. Fix $x \in rB_X$. On the one hand, pick $x^* \in \partial f(x)$. Then $\langle x^*, 0 - x \rangle \leq f(0) - f(x)$; thus $f(x) \leq f(0) + \langle x^*, x \rangle \leq f(0) + Mr$. Hence f is bounded above on rB_X by $f(0) + Mr$. On the other hand, picking $x_0^* \in \partial f(0)$ shows similarly that f is bounded below on rB_X by $f(0) - Mr$. Altogether, f is bounded on rB_X and the proof is complete. □

COROLLARY 7.9. *If X is finite dimensional, then every convex function from X to $\mathbb{R}$ is subdifferentiable everywhere and its subdifferentials are uniformly bounded on bounded sets.*

Proof. By Facts 7.2.(iii), any convex function from X to $\mathbb{R}$ is continuous everywhere. Since X is finite dimensional, this function attains its minimum and maximum on bounded closed sets; in particular, it is bounded on bounded closed sets. The result now follows from the previous proposition. □

REMARKS 7.10.

- The last corollary implies that if X is finite dimensional, then *every* convex function from X to $\mathbb{R}$ can be used in Theorem 7.7.
- As the following example demonstrates, the assumption that X is finite dimensional in the last corollary cannot be dropped. Consequently, the convex function constructed below cannot be used in Theorem 7.7.

EXAMPLE 7.11. Define the function f by

$$f \ : \ X := \ell_2 \ \longrightarrow \ \mathbb{R} \ : \ \mathbf{x} = (x_n) \ \longmapsto \ \textstyle\sum_{n=1}^{\infty} n x_n^{2n}.$$

Then f is everywhere finite, convex, continuous, and subdifferentiable. However, on B_X, neither is the function f bounded nor are the subdifferentials of f uniformly bounded.

Proof. Fix an arbitrary $\mathbf{x} = (x_n) \in X$. Then, on the one hand, $x_n \longrightarrow 0$. On the other hand, $\sqrt[n]{n} \longrightarrow 1$. Hence, eventually

$$x_n^2 \leq \frac{1}{(\sqrt[n]{n})^3} \quad \Longleftrightarrow \quad n x_n^{2n} \leq \frac{1}{n^2};$$

thus $f(\mathbf{x})$ is finite. Also, f is, as the supremum of convex and lower semicontinuous functions

$$f = \sup_m \textstyle\sum_{n=1}^{m} n x_n^{2n},$$

convex and lower semicontinuous too. Therefore, f is everywhere continuous (see, for example, [50, Chap. I: Cor. 2.5]) and subdifferentiable (Fact 7.2.(ii)). Choosing $\mathbf{x} = n$th unit vector in X shows that

$$\sup f(B_X) \geq f(\mathbf{x}) = n;$$

thus f is unbounded on B_X. The proof of "(iii)$\Longrightarrow$(i)" in the last proposition shows that the subgradients of f are not uniformly bounded on B_X. $\square$

Under a Slater-type constraint qualification, we even obtain *linearly* focusing subgradient algorithms.

THEOREM 7.12 (prototype of a linearly focusing subgradient algorithm). *Given a subgradient algorithm, suppose that there is some* Slater point $\hat{x} \in X$ *s.t.*

$$f_i(\hat{x}) < 0$$

and that the subdifferentials of f_i are nonempty and uniformly bounded on bounded sets for every subgradient index $i \in I_\partial$. Suppose further there is some $\beta > 0$ s.t.

$$\beta d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)})$$

for every index $i \in \{1, \ldots, N\} \setminus I_\partial$ and all large n active for i. Then the subgradient algorithm is linearly focusing.

Proof. Fix any index $i \in \{1, \ldots, N\}$. It is sufficient to show that there is some $\beta_i > 0$ s.t.

$$(*) \qquad \beta_i d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)}) \quad \text{for all large } n \text{ active for } i.$$

Case 1: $i \in \{1, \ldots, N\} \setminus I_\partial$. Then $\beta_i = \beta$ does the job for $(*)$.

Case 2: $i \in I_\partial$. Since $(x^{(n)})$ is bounded, there is some $M > 0$ s.t. for all $n \geq 0$, $\|\hat{x} - x^{(n)}\| \leq M$ and the norm of every subgradient of f_i at every $x^{(n)}$ is at most M. Now fix n active for i. Without loss, we assume that $f_i(x^{(n)}) > 0$ (otherwise, $(*)$ holds trivially). Define

$$\epsilon := -f_i(\hat{x}) > 0, \quad \lambda := \frac{\epsilon}{f_i(x^{(n)}) + \epsilon} \in]0, 1[$$

and set

$$y := (1-\lambda)\hat{x} + \lambda x^{(n)}.$$

Then

$$f_i(y) = f_i((1-\lambda)\hat{x} + \lambda x^{(n)}) \leq (1-\lambda) f_i(\hat{x}) + \lambda f_i(x^{(n)}) = 0;$$

hence $y \in C_i$. We estimate

$$\begin{aligned} d^2(x^{(n)}, C_i) &\leq \|x^{(n)} - y\|^2 = (1-\lambda)^2 \|\hat{x} - x^{(n)}\|^2 \\ &= \left(\frac{f_i(x^{(n)})}{f_i(x^{(n)}) + \epsilon}\right)^2 \|\hat{x} - x^{(n)}\|^2 \\ &\leq \left(\frac{f_i(x^{(n)})}{\epsilon}\right)^2 M^2 \\ &= \left(\frac{d(x^{(n)}, C_i^{(n)}) \|g_i(x^{(n)})\|}{\epsilon}\right)^2 M^2 \\ &\leq \frac{M^4}{\epsilon^2} d^2(x^{(n)}, C_i^{(n)}). \end{aligned}$$

Therefore, $(*)$ holds with $\beta_i = \epsilon / M^2$ and the proof is complete. $\square$

Examples.

Censor and Lent's framework. We investigate in this subsection a framework essentially suggested by Censor and Lent [28]. They considered (cf. Example 7.14) subgradient algorithms where *every* index is a subgradient index; i.e., $I_\partial = \{1, \ldots, N\}$. Then

$$C = \bigcap_{i=1}^{N} C_i = \bigcap_{i=1}^{N} \{x \in X : f_i(x) \leq 0\}$$

is the set of solutions of the convex feasibility problem

$$f_i(x) \leq 0 \qquad \text{for } i = 1, \ldots, N,$$

where each f_i is a continuous convex function from X to $\mathbb{R}$.

THEOREM 7.13 (Censor and Lent's framework in Euclidean spaces). *Suppose X is finite dimensional. Then the sequence $(x^{(n)})$ converges in norm to some point in C whenever one of the following conditions holds.*

(i) *(random control)* $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index i.*

(ii) *(intermittent control) The subgradient algorithm is p-intermittent and* $\sum_n \nu_n = +\infty$ *(where ν_n is defined as in Theorem* 3.20.(ii)).

Proof. By Theorem 7.7 and Corollary 7.9, the subgradient algorithm is focusing. Now (i) follows from Corollary 3.12, whereas (ii) is immediate from Corollary 3.25. $\square$

We now obtain Censor and Lent's fundamental result as a special case of the last theorem.

EXAMPLE 7.14 (Censor and Lent's [28, Thm. 1]). Suppose X is finite dimensional and the subgradient algorithm is almost cyclic. Suppose further there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all n and every index i active at n. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

THEOREM 7.15 (Censor and Lent's framework in Hilbert spaces). *Suppose the projection algorithm is p-intermittent and the functions f_i have nonempty and uniformly bounded subdifferentials on bounded sets.*

(i) *If* $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ *for every index i, then the sequence $(x^{(n)})$ converges weakly to some point in C.*

(ii) *If* $\sum_n \nu_n = +\infty$ *(where ν_n is defined as in Theorem* 3.20.(ii)), *then the sequence $(x^{(n)})$ has a (unique) weak cluster point in C.*

Proof. By Theorem 7.7, the subgradient algorithm is focusing. The result is now immediate from Theorem 3.20. □

EXAMPLE 7.16 (Eremin's [52, Thm. 1.1 for convex functions and subgradients]). Suppose $N = 1$ and f_1 has nonempty and uniformly bounded subdifferentials on bounded sets.

(i) If there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_1^{(n)} \leq 1$, then the sequence $(x^{(n)})$ converges weakly to some point in C.

(ii) If $\overline{\lim}_n\ \alpha_1^{(n)} < 2$ and $\sum_n \alpha_1^{(n)} = +\infty$, then the sequence $(x^{(n)})$ converges weakly to some point in C.

REMARKS 7.17.

- Eremin considered a more abstract iteration scheme.
- In view of Theorem 7.15, the assumptions in Example 7.16 can be weakened to "there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_1^{(n)} \leq 2-\epsilon$" for (i), and "$\sum_n \alpha_1^{(n)}(2-\alpha_1^{(n)}) = +\infty$" for (ii).

THEOREM 7.18 (Censor and Lent's framework with a Slater point). *Suppose each function f_i has nonempty and uniformly bounded subdifferentials on bounded sets and there is some Slater point $\hat{x} \in C$, i.e., $f_i(\hat{x}) < 0$, for every index i. Then the sequence $(x^{(n)})$ converges in norm to some point in X, say x.*

(i) *If* $\sum_n \mu_i^{(n)} = +\infty$ *for every index i, then $x \in C$.*

(ii) *If the subgradient algorithm is intermittent and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all large n and every index i active at n, then $x \in C$ and the sequence* $(x^{(n)})$ converges linearly *to x.*

Proof. By Theorem 7.12, the subgradient algorithm is linearly focusing. The Slater point $\hat{x}$ lies in the interior of $C_i = \{x \in X : f_i(x) \leq 0\}$; thus $\hat{x} \in \operatorname{int} C$ and $(C_1, \dots, C_N)$ is boundedly linearly regular (Corollary 5.14). (i) follows from Theorem 3.20.(iii), whereas (ii) follows from Theorem 5.7. □

EXAMPLE 7.19 (De Pierro and Iusem's [40, Thm. 2]). Suppose X is finite dimensional and there is some Slater point $\hat{x} \in C$. Suppose further the subgradient algorithm is almost cyclic and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ for all n and every index i active at n. Then the sequence $(x^{(n)})$ converges linearly to some point in C.

Proof. Combine Corollary 7.9 and Theorem 7.18.(ii). □

REMARK 7.20. De Pierro and Iusem's proof is different from ours. They obtain Example 7.19 via an investigation of an iteration that converges *finitely* when a Slater point exists (but may diverge otherwise).

EXAMPLE 7.21 (Eremin's [52, Thm. 1.3]). Suppose each function f_i has nonempty and uniformly bounded subdifferentials on bounded sets and there is some Slater point $\hat{x} \in C$. If $0 < \lambda_i^{(n)} \equiv \lambda_i$ and $0 < \alpha_i^{(n)} \equiv \alpha_i < 2$ for every index i, then the sequence $(x^{(n)})$ converges linearly to some point in C.

Proof. The subgradient algorithm is weighted; hence it is 1-intermittent and Theorem 7.18.(ii) applies. □

Polyak's framework. In this subsection, we concentrate on a framework suggested by Polyak [86]. He considered a subgradient algorithm where

$$N = 2, \quad I_\partial = \{1\}, \quad \text{and} \quad C_2^{(n)} \equiv C_2.$$

Hence the set $C = C_1 \cap C_2 = \{x \in C_2 : f_1(x) \leq 0\}$ describes the solutions of the convex feasibility problem

$$f_1(x) \leq 0, \quad x \in C_2,$$

where f_1 is a continuous convex function from X to $\mathbb{R}$.

We can view Polyak's framework as a special case of Censor and Lent's framework by setting $N = 2$ and letting $f_2 = d(\cdot, C_2)$. We now "translate" some results obtained in the last subsection to this framework.

For example, Theorem 7.15.(i) becomes the following theorem.

THEOREM 7.22. *Suppose the projection algorithm is 2-intermittent and the function f_1 has nonempty and uniformly bounded subdifferentials on bounded sets. If $\underline{\lim}_{n:n \text{ active for } i}\, \mu_i^{(n)} > 0$ for $i = 1$ and 2, then the sequence $(x^{(n)})$ converges weakly to some point in C.*

EXAMPLE 7.23 (Polyak's [86, Thm. 1]). Suppose the function f_1 has nonempty and uniformly bounded subdifferentials on bounded sets. If the subgradient algorithm is cyclic and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_1^{(n)} \leq 2 - \epsilon$, $\alpha_2^{(n)} \equiv 1$ for all n, then the sequence $(x^{(n)})$ converges weakly to some point in C.

A "translation" of Theorem 7.13.(ii) yields the following theorem.

THEOREM 7.24. *Suppose X is finite dimensional. If the subgradient algorithm is 2-intermittent and $\sum_n \nu_n = +\infty$ (where ν_n is defined as in Theorem* 3.20.(ii)), *then the sequence $(x^{(n)})$ converges in norm to some point in C.*

EXAMPLE 7.25 (a special case of Allen et al.'s [4, Prop. 7]). Suppose X is finite dimensional, the subgradient algorithm is cyclic, and there is some $\epsilon > 0$ s.t. $0 < \alpha_1^{2n} \leq 2 - \epsilon$ and $\alpha_2^{(n)} \equiv \alpha_1^{(2n+1)} \equiv 1$ for all $n \geq 0$. If $\sum_n \alpha_1^{2n} = +\infty$, then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. The subgradient algorithm is certainly 2-intermittent; hence define ν_n as in Theorem 3.20.(ii) and check that $\nu_n = \alpha_1^{(2n)}(2 - \alpha_1^{(2n)}) \geq \alpha_1^{(2n)}\epsilon$. Therefore, $\sum_n \nu_n = +\infty$ and the result follows from Theorem 7.24. $\square$

REMARKS 7.26.

- An inspection of the proof shows that we can replace the assumptions on $(\alpha_1^{(2n)})$ by the more general "$\sum_n \alpha_1^{(2n)}(2 - \alpha_1^{(2n)}) = +\infty$."
- Allen et al. [4] also investigated the situation where it is allowed that f_1 takes the value $+\infty$ and C is empty.

The next theorem does not follow from Censor and Lent's framework. The necessary work, however, is modest.

THEOREM 7.27. *Suppose the subgradient algorithm is intermittent and there is some $\hat{x} \in C_2$ with $f_1(\hat{x}) < 0$. Suppose further f_1 has nonempty and uniformly bounded subdifferentials on bounded sets. If there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all large n and every index i active at n, then the sequence $(x^{(n)})$ converges linearly to some point in C.*

Proof. By Theorem 7.12, the subgradient algorithm is linearly focusing. Since $\hat{x} \in C_2 \cap \operatorname{int} C_1$, the pair (C_1, C_2) is boundedly linearly regular (Corollary 5.14). Now apply Theorem 5.7. $\square$

EXAMPLE 7.28 (a case of Polyak's [86, Thm. 4]). Suppose the subgradient algorithm is cyclic and there is some $\hat{x} \in C_2$ with $f_1(\hat{x}) < 0$. Suppose further f_1 has nonempty and uniformly bounded subdifferentials on bounded sets. If there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_1^{(2n)} \leq$

$2-\epsilon$ and $\alpha_1^{(2n+1)} \equiv \alpha_2^{(n)} \equiv 1$ for all n, then the sequence $(x^{(n)})$ converges linearly to some point in C.

REMARKS 7.29. On first sight, Polyak's framework looks fairly special since it deals only with *one* function. A standard trick, however, allows one to handle finitely many convex functions. Suppose we are given M continuous convex functions $\phi_1, \ldots, \phi_M$ from X to $\mathbb{R}$. If we want to solve the convex feasibility problem

$$x \in C_2 \qquad \text{and} \qquad \phi_i(x) \leq 0, \quad i = 1, \ldots, M,$$

then we simply set

$$f_1 := \max\{\phi_1, \ldots, \phi_M\}, \;\; C_1 := \{x \in X : f_1(x) \leq 0\},$$

and we see that $C = C_1 \cap C_2$ are precisely the solutions of the above problem. Hence all methods discussed in this subsection are applicable. For example, the reformulation of the last theorem to this situation yields a partial generalization of Polyak's [86, Thm. 6]. It only remains to describe ∂f_1. The reader can find a formula in Ioffe and Tihomirov's book [68, Thm. 3 on p. 201f] that becomes in our setting

$$\partial f_1(x) = \overline{\text{conv}} \bigcup_{j \,:\, f_1(x) = \phi_j(x)} \partial \phi_j(x).$$

A generalization of Dos Santos's framework. In this section, we discuss a generalization of a framework due to Dos Santos (cf. Example 7.34). On first sight, this framework looks like a subgradient algorithm; it is, however, actually a projection algorithm as defined in §4. It works as follows.

Suppose we are given M continuous convex functions $\phi_{i,k}$ from X to $\mathbb{R}$ that are partitioned into N $(\leq M)$ "blocks," where the ith block consists of M_i functions

$$\{\phi_{i,1}, \ldots, \phi_{i,M_i}\} \quad \text{for every index } i,$$

so that $M_1 + \cdots + M_N = M$. Let

$$C_i = \{x \in X : \phi_{i,k}(x) \leq 0 \;\text{ for }\; k = 1, \ldots, M_i\} \quad \text{for every index } i;$$

then $C = \bigcap_{i=1}^N C_i$ is the set of solutions of the convex feasibility problem

$$\phi_{i,k}(x) \leq 0, \qquad i = 1, \ldots, N, \quad k = 1, \ldots, M_i.$$

As always, we assume feasibility; i.e., C is nonempty.

Given a point $x^{(n)}$, we define N continuous convex functions

$$f_i^{(n)} : X \longrightarrow \mathbb{R} : x \longmapsto \sum_{k=1}^{M_i} \omega_{i,k}^{(n)} \frac{\phi_{i,k}^+(x^{(n)})}{\|\psi_{i,k}(x^{(n)})\|^2} \phi_{i,k}^+(x),$$

where we use the convention that

$$\frac{0}{0} := 0,$$

and where $\psi_{i,k}(x^{(n)}) \in \partial\phi_{i,k}^+(x^{(n)})$ and $\omega_{i,k}^{(n)}$ are nonnegative real numbers with $\sum_{k=1}^{M_i} \omega_{i,k}^{(n)} = 1$ for every index i and $k = 1, \ldots, M_i$. We further set

$$g_i^{(n)}(x^{(n)}) := \sum_{k=1}^{M_i} \omega_{i,k}^{(n)} \frac{\phi_{i,k}^+(x^{(n)})}{\|\psi_{i,k}(x^{(n)})\|^2} \psi_{i,k}(x^{(n)}) \quad \text{for every index } i$$

and note that $g_i^{(n)}(x^{(n)}) \in \partial f_i^{(n)}(x^{(n)})$. Sticking to the notation of Lemma 7.3, we finally define N closed convex sets by

$$C_i^{(n)} := H(f_i^{(n)}, x^{(n)}, g_i^{(n)}(x^{(n)})) \quad \text{for every index } i.$$

Then Lemma 7.3 (and the convention $\frac{0}{0} := 0$) yields the following lemma.

LEMMA 7.30. *For every index i and all $n \geq 0$,*

$$\text{(i)} \qquad \begin{aligned} C_i^{(n)} &\supseteq \{x \in X : f_i^{(n)}(x) \leq 0\} \\ &\supseteq \bigcap_{k=1}^{M_i} \{x \in X : \phi_{i,k}(x) \leq 0\} \\ &= C_i. \end{aligned}$$

(ii) $P_i^{(n)} x^{(n)} = x^{(n)} - \left(\dfrac{f_i^{(n)}(x^{(n)})}{\|g_i^{(n)}(x^{(n)})\|^2} \right) g_i^{(n)}(x^{(n)})$.

(iii) $d(x^{(n)}, C_i^{(n)}) = \dfrac{f_i^{(n)}(x^{(n)})}{\|g_i^{(n)}(x^{(n)})\|}$.

By Lemma 7.30.(i), we are thus given a projection algorithm to which we refer as the *generalized DS algorithm*. Dos Santos [47, §6] gave an excellent motivation for a special case of the generalized DS algorithm. Of course, now we wish to bring our convergence results into play; hence, we must know what makes a generalized DS algorithm (linearly) focusing.

DEFINITION 7.31 (control). We say that the generalized DS algorithm *considers most violated constraints* if there is some $\tau > 0$ s.t. for every index i, there is some $k \in \{1, \ldots, M_i\}$ with

$$\phi_{i,k}^+(x^{(n)}) = \max_l \phi_{i,l}^+(x^{(n)}) \quad \text{and} \quad \omega_{i,k}^{(n)} \geq \tau \quad \text{for all } n \geq 0.$$

THEOREM 7.32 (prototype of a (linearly) focusing generalized DS algorithm). *Suppose the generalized DS algorithm considers most violated constraints and the functions $\phi_{i,k}$ have nonempty and uniformly bounded subdifferentials on bounded sets.*

(i) *Then the generalized DS algorithm is focusing.*

(ii) *Suppose that in addition for every index i at least one of the following conditions holds.*

1. *There is some Slater point $\hat{x}_i \in C_i$ s.t.*

$$\phi_{i,k}(\hat{x}_i) < 0 \quad \textit{for every } k = 1, \ldots, M_i.$$

2. *Each $\phi_{i,k}$ is a distance function to some closed convex set $C_{i,k}$ and the M_i-tuple*

$$(C_{i,1}, \ldots, C_{i,M_i}) \quad \textit{is boundedly linearly regular.}$$

Then the generalized DS algorithm is linearly focusing.

Proof. First, we get $L_1 > 0$ s.t. $\|\psi_{i,k}(x^{(n)})\| \leq L_1$ for all $n \geq 0$, every index i, and all $k = 1, \ldots, M_i$. Second, we get $\tau > 0$ s.t. for all $n \geq 0$ and every index i, $\omega_{i,k^*}^{(n)} \geq \tau$ for some $k^* \in \{1, \ldots, M_i\}$ with $\phi_{i,k^*}^+(x^{(n)}) = \max_k \phi_{i,k}^+(x^{(n)})$. Now fix an index i and $n \geq 0$ and assume (without loss, as we will see) that $x^{(n)} \notin C$. It is convenient to abbreviate

$$\omega_k := \omega_{i,k}^{(n)}, \quad q_k := \frac{\phi_{i,k}^+(x^{(n)})}{\|\psi_{i,k}(x^{(n)})\|}, \quad z_k := \frac{\psi_{i,k}(x^{(n)})}{\|\psi_{i,k}(x^{(n)})\|}$$

and to let any appearing k's range in $\{k \in \{1, \ldots, M_i\} : z_k \neq 0\}$. Using the convexity of $\|\cdot\|$ and $(\cdot)^2$, we estimate

$$d(x^{(n)}, C_i^{(n)}) = \frac{f_i^{(n)}(x^{(n)})}{\|g_i^{(n)}(x^{(n)})\|} = \frac{\sum_k \omega_k q_k^2}{\|\sum_k \omega_k q_k z_k\|}$$

$$\geq \frac{\sum_k \omega_k q_k^2}{\sum_k \omega_k q_k \|z_k\|} = \frac{\sum_k \omega_k q_k^2}{\sum_k \omega_k q_k}$$

$$\geq \frac{(\sum_k \omega_k q_k)^2}{\sum_k \omega_k q_k} = \sum_k \omega_k q_k.$$

By choice of τ and L_1, we conclude that

$$(*) \qquad d(x^{(n)}, C_i^{(n)}) \geq \frac{\tau}{L_1} \max_k \phi_{i,k}^+(x^{(n)}) \quad \text{for every index } i \text{ and all } n \geq 0.$$

Note that if $x^{(n)} \in C$, then $(*)$ holds trivially.

(i): If $(x^{(n')})$ is a weakly convergent subsequence of $(x^{(n)})$ with weak limit x and $d(x^{(n')}, C_i^{(n')}) \longrightarrow 0$, then, by $(*)$, $\overline{\lim}_{n'} \max_k \phi_{i,k}^+(x^{(n')}) \leq 0$. Since $\phi_{i,k}^+$ is weakly lower semicontinuous, this implies $\phi_{i,k}^+(x) \leq 0$ for $k = 1, \dots, M_i$. Hence $x \in C_i$ and the generalized DS algorithm is focusing.

(ii): *Case* 1: Condition 1 holds. Get $L_2 > 0$ s.t. $\|\hat{x}_i - x^{(n)}\| \leq L_2$ for all $n \geq 0$. Now fix an index i and n. Define

$$y := (1 - \lambda)\hat{x} + \lambda x^{(n)},$$

where

$$\lambda := \frac{\min_k\{-\phi_{i,k}(\hat{x})\}}{\min_k\{-\phi_{i,k}(\hat{x})\} + \max_k\{\phi_{i,k}^+(x^{(n)})\}} \quad \in]0, 1].$$

Then one readily verifies that $\phi_{i,k}(y) \leq 0$ for all k; thus $y \in C_i$. We estimate

$$d^2(x^{(n)}, C_i) \leq \|x^{(n)} - y\|^2 = (1 - \lambda)^2 \|\hat{x} - x^{(n)}\|^2$$
$$\leq \left(\frac{\max_k\{\phi_{i,k}^+(x^{(n)})\}}{\min_k\{-\phi_{i,k}(\hat{x})\}}\right)^2 L_2^2.$$

Combining the previous estimate with $(*)$ yields

$$\frac{\tau \min_k\{-\phi_{i,k}(\hat{x})\}}{L_1 L_2} d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)}) \quad \text{for all } n \geq 0.$$

Case 2: Condition 2 holds. Because $(C_{i,1}, \dots, C_{i,M_i})$ is boundedly linearly regular, there exists some $L_3 > 0$ s.t.

$$d(x^{(n)}, C_i) \leq L_3 \max_k d(x^{(n)}, C_{i,k}) = L_3 \max_k \phi_{i,k}(x^{(n)}) \quad \text{for all } n \geq 0.$$

Fix an index i. Combining the last estimate with $(*)$ this time yields

$$\frac{\tau}{L_1 L_3} d(x^{(n)}, C_i) \leq d(x^{(n)}, C_i^{(n)}) \quad \text{for all } n \geq 0.$$

In both cases, we have found an inequality that makes the generalized DS algorithm linearly focusing. $\square$

Having identified nice classes of (linearly) focusing generalized DS algorithms, we could now systematically "translate" our previous results to this situation; again, we opt for a small selection.

Dos Santos's original framework. Dos Santos considered the situation when $N = 2$, and we set $L := M_1$,

$$\phi_1 := \phi_{1,1}, \ldots, \phi_L := \phi_{1,M_1},$$

$M_2 = 1$, and $\phi_{2,1} = d(\cdot, C_2)$ for some closed convex nonempty set C_2. Thus $C = C_1 \cap C_2$ is the set of solutions of the convex feasibility problem

$$x \in C_2 \quad \text{and} \quad \phi_k(x) \leq 0 \quad \text{for } k = 1, \ldots, L.$$

We refer to the generalized DS algorithm in this framework as the *DS algorithm*.

THEOREM 7.33 (Dos Santos's original framework). *Suppose the DS algorithm is intermittent and considers most violated constraints and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all n and every index i active at n. Suppose further each ϕ_k has nonempty and uniformly bounded subdifferentials on bounded sets.*

(i) *Then the sequence $(x^{(n)})$ converges weakly to some point in C. Consequently, if X is finite dimensional, then the sequence $(x^{(n)})$ converges in norm to some point in C.*

(ii) *If there is some $\hat{x} \in C_2$ s.t. $\phi_k(\hat{x}) < 0$ for $k = 1, \ldots, L$, then the sequence $(x^{(n)})$ converges linearly to some point in C.*

Proof. (i): By Theorem 7.32.(i), the DS algorithm is focusing. The result now follows from Remark 3.13 and Theorem 3.20.(i).

(ii): On the one hand, the DS algorithm is linearly focusing (Theorem 7.32.(ii)). On the other hand, $\hat{x} \in C_2 \cap \operatorname{int} C_1$, so (C_1, C_2) is boundedly linearly regular (Corollary 5.14). Altogether (Theorem 5.7), the sequence $(x^{(n)})$ converges linearly to some point in C. ☐

EXAMPLE 7.34 (Dos Santos's [47, Thm.]). Suppose X is finite dimensional, the DS algorithm is cyclic, and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_1^{(2n)} \leq 2 - \epsilon$ and $\alpha_2^{(n)} \equiv 1$ for all $n \geq 0$. Suppose further $0 < \omega_{1,k}^{(2n)} \equiv \omega_k$ for $k = 1, \ldots, L$. Then the sequence $(x^{(n)})$ converges in norm to some point in C.

Proof. Since $\omega_k > 0$ for all k, the DS algorithm certainly considers most violated constraints. Now combine Corollary 7.9 and Theorem 7.33.(i). ☐

REMARKS 7.35.

- For $L = 1$, Dos Santos's and Polyak's frameworks coincide.
- Dos Santos reports good numerical results on his algorithm. Theorem 7.33.(ii) shows that the qualitative performances of his and Censor and Lent's frameworks are comparable (cf. Theorem 7.18 and Example 7.19).

The polyhedral framework. The polyhedral framework is the special case of the generalized Dos Santos framework, where $\phi_{i,k}$ is the distance function to some polyhedron $C_{i,k}$ for every index i and all $k = 1, \ldots, M_i$. Throughout this subsection, we investigate this situation.

THEOREM 7.36 (polyhedron framework). *In the polyhedral framework, suppose the generalized DS algorithm considers most violated constraints. Suppose further it is intermittent or considers remotest sets. Suppose finally there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \leq 2 - \epsilon$ and $\epsilon \leq \lambda_i^{(n)}$ for all n and every index i active at n. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.*

Proof. For every index i, the M_i-tuple $(C_{i,1}, \ldots, C_{i,M_i})$ is linearly regular (Corollary 5.26). Hence, by Theorem 7.32.(ii), the generalized DS algorithm is linearly focusing. Now each C_i is also a polyhedron; thus by Corollary 5.26, $(C_1, \ldots, C_N)$ is linearly regular. Therefore, the result follows from Theorem 5.7 (for intermittent control) or Theorem 5.8 (if the algorithm considers remotest sets). ☐

REMARK 7.37. If $N = M$, each $C_{i,1}$ is a halfspace, $\omega_{i,1}^{(n)} \equiv \omega_i > 0$, and there is some $\epsilon > 0$ s.t. $\epsilon \leq \alpha_i^{(n)} \equiv \alpha^{(n)} \leq 2 - \epsilon$ for all n and every index i, then we recapture Example 6.40 (which is due to De Pierro and Iusem [39]).

We register two more special cases.

EXAMPLE 7.38 (Merzlyakov's [80, Thm.]). In the polyhedral framework, suppose X is finite dimensional, $N = 1$, each $C_{1,i}$ is a halfspace, and the generalized DS algorithm considers most violated constraints. Suppose further $0 < \alpha_1^{(n)} \equiv \alpha_1 < 2$. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point.

REMARK 7.39. Merzlyakov [80] actually considered a more general version, where the $\omega_{1,i}^{(n)}$ need not necessarily sum up to 1.

EXAMPLE 7.40 (Yang and Murty's [105]). In the polyhedral framework, suppose X is finite dimensional, each $C_{i,k}$ is a halfspace, and there is some $\epsilon > 0$ s.t. the generalized DS algorithm satisfies

$$\omega_{i,k}^{(n)} \begin{cases} = 0 & \text{if } x^{(n)} \in C_{i,k}, \\ \geq \epsilon & \text{otherwise} \end{cases}$$

for all n, every index i, and all $k = 1, \ldots, M_i$. Suppose further the relaxation parameters satisfy $0 < \alpha_i^{(n)} \equiv \alpha < 2$ for all n and every index i. Then the sequence $(x^{(n)})$ converges linearly to some point in C with a rate independent of the starting point whenever one of the following conditions holds.

1. (basic surrogate constraint method: [105, §3]) $N = 1$.
2. (sequential surrogate constraint method: [105, §4]) The generalized DS algorithm is cyclic.
3. (parallel surrogate constraint method: [105, §5]) There is some $\epsilon' > 0$ s.t. $x^{(n)} \notin C_i$ implies $\lambda_i^{(n)} \geq \epsilon'$ for all n and every index i.

Proof. Obviously, the generalized DS algorithm considers most violated constraints. The first condition is a special case of the second one, which in turn follows from Theorem 7.36. The assumption in the third condition guarantees that the algorithm considers remotest sets; hence, this case is also covered by Theorem 7.36. □

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Final remarks. This attempt to review and unify is necessarily incomplete—it is impossible to keep track of all relevant manuscripts in the field. We thus wish to apologize to the contributors who should have been included here but are not. The manuscript is merely a snapshot of what the authors knew in mid-1993; time, of course, has not stood still. The manuscripts sent to us recently by Combettes [30–37], García-Palomares [55], and Kiwiel [72–75] deal with exciting new generalizations and deserve much attention. A synthesis of a selection of these results may be found in the first author's Ph.D. thesis (*Projection Algorithms and Monotone Operators*, Simon Fraser University, 1996).

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Notes on non-convex Lions-Mercier iterations

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1 Introduction

Let $R_A(x) := 2\,P_A(x) - x, R_B(x) := 2\,P_B(x) - x$, where P_A, P_B denote the Euclidean metric projections, or nearest point maps, on closed sets A and B. In our setting, the *Lions-Mercier* (LM) iteration (which can be given many other names [?] such as *Douglas-Rachford* or *Feinup*'s algorithm) is the procedure: *reflect, reflect and average*:

$$x \mapsto T(x) := \frac{x + R_A\left(R_B(x)\right)}{2}. \tag{1}$$

Note that a fixed point z of T produces precisely a point w such that $w := P_B(z) = P_A\left(R_B(z)\right)$ is an element of $A \cap B$. Moreover, if one shows that $\|T(z_n) - z_n\| \to 0$ (known as *asymptotic regularity* of $z_{n+1} := T(z_n)$) then every cluster point of the corresponding orbit produces a fixed point z.

The consequent theory of this and related iterations is well understood in the convex case [?, ?, ?]. In the non-convex case the iteration, also called "divide-and-concur" [?], has been very successful in a variety of reconstruction problems [?, ?] but the theory to explain why is largely absent.

In this note we look at a simple but illustrative special case. The subtlety of this prototype indicates a good deal about the behaviour of the general iteration. Since (LM) has performed much better than other projection iterations on a variety of hard problems [?, ?] we focus on its behaviour.[1]

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[1]In optical abberation correction as required on the Hubble telescope, however, cyclic projection and its variants have worked well.

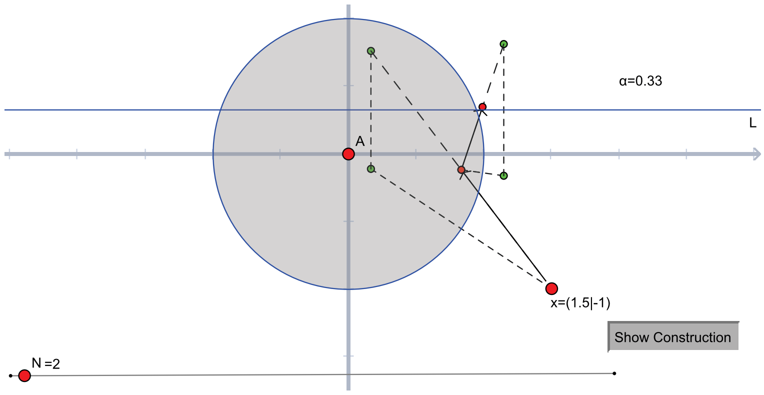

Figure 1: **Two steps showing the construction.**

2 Dynamics with the circle

In the simplest non-convex case where B is the unit circle and A is a horizontal line of height α the recursion becomes $x_0 := x$, $y_0 := y$ and

$$x_{n+1} \quad := \quad \frac{x_n}{r_n} = \cos\theta_n, \tag{2}$$

$$y_{n+1} \quad := \quad = y_n + \alpha - \frac{y_n}{r_n} = y_n + \alpha - \sin\theta_n, \tag{3}$$

where $\theta_n := \arctan(y_n/x_n)$ and $r_n = \sqrt{x_n^2 + y_n^2}$.

Figure **??** shows two steps of the underlying geometric construction. All figures were constructed in *Cinderella* (`www.cinderella.de`). A web applet version of the underlying Cinderella construction is available at `http://kortenkamps.net/material/IterationBorwein.html`. Indeed, many of the insights for the proofs below came from examining the constructions (the number of iterations N, the height of the line, and the the initial point are all dynamic—changing one changes the entire visible trajectory).

Let $z_n := (x_n, y_n)$. By symmetry we restrict to $\alpha \geq 0$. It is easy to see that if $x_0 = 0$ then the iteration remains on the vertical axis. We leave this case for the next section.

Thus, we assume that $x_0 > 0$ and it follows from equation (**??**) that we have $0 < x_n < 1$ for all $n \geq 1$.

We distinguish four cases:

1. $\alpha = 0$. In this case we prove in Theorem **??** below that $z_n \to (1, 0)$. (See Figure **??**.)

2. $0 < \alpha < 1$. In this case we conjecture that $z_n \to (\sqrt{1-\alpha^2}, \alpha)$. (See Figure **??**.)

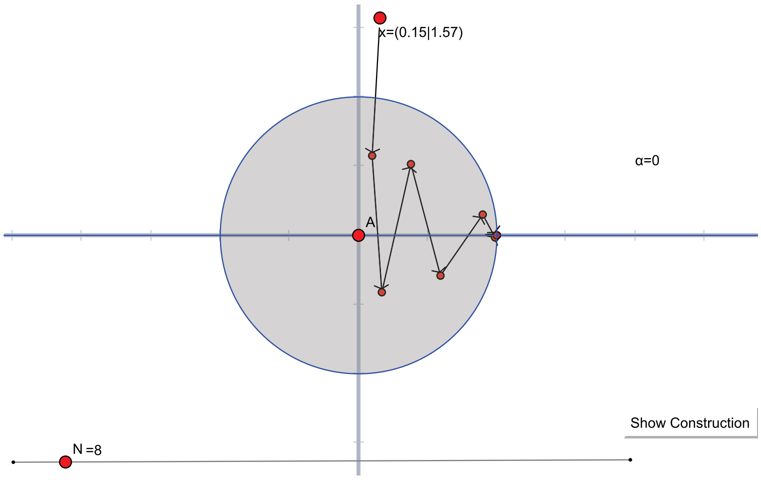

Figure 2: **Case with $\alpha = 0$.**

3. $\alpha = 1$. In this case we prove in Theorem **??** below that $z_n \to (0, \overline{y})$ for some finite $\overline{y} > 1$. (See Figure **??**.)

4. $\alpha > 1$. In this infeasible case we prove in Theorem **??** below that $y_n \to \infty$ at linear rate and $x_n \to 0$.

Theorem 1 (Infeasible case) *If $\alpha > 1$ then $y_n \to \infty$ at linear rate as $n \to \infty$, and $x_n \to 0$.*

Proof. An easy estimate from equation (**??**) is $y_{n+1} - y_n \geq \alpha - 1 > 0$. The assertion about y_n follows and the behaviour of x_n is left as an exercise. ∎

For the remaining feasible cases the following preliminary computation is useful. We write

$$\begin{aligned} r_{n+1}^2 &= \frac{x_n^2}{r_n^2} + \frac{y_n^2}{r_n^2} + (y_n + \alpha)^2 - 2\,\frac{y_n}{r_n}\,(y_n + \alpha) \\ &= 1 + \alpha^2 + y_n^2 \left(1 - \frac{2}{r_n}\right) + 2\alpha y_n \left(1 - \frac{1}{r_n}\right). \end{aligned}$$

Thus,

$$r_{n+1}^2 - 1 = \alpha^2 + y_n^2 \left(1 - \frac{2}{r_n}\right) + 2\alpha y_n \left(1 - \frac{1}{r_n}\right). \tag{4}$$

Proposition 1 *Suppose that $\alpha = 0$ and that $n > 0$ and $r_n > 1$. Then $r_{n+1} < r_n$.*

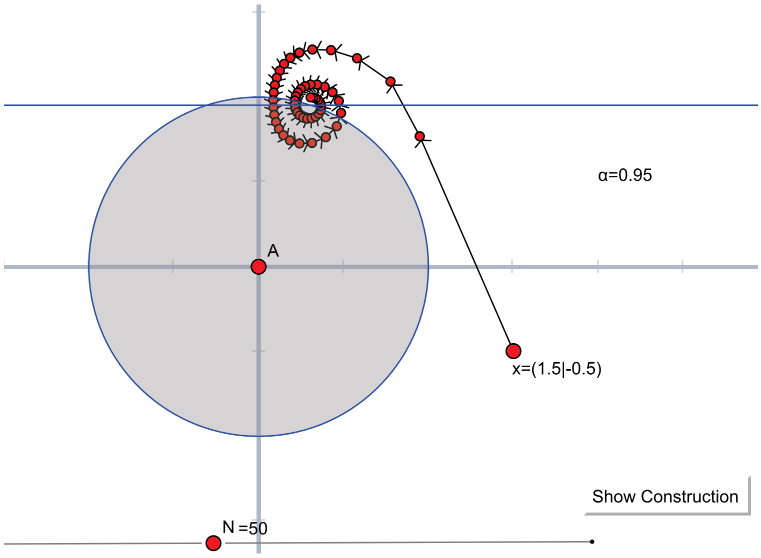

Figure 3: **Case with $\alpha = 0.9$.**

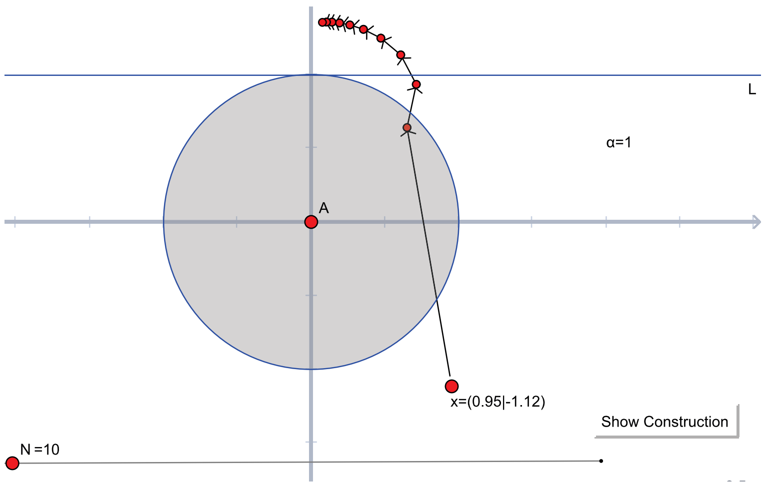

Figure 4: **Case with $\alpha = 1$.**

Proof. Equation (**??**) becomes

$$r_{n+1}^2 - 1 \quad = \quad \frac{y_n^2}{r_n^2}\left(r_n^2 - 2\,r_n + 1\right) - \frac{y_n^2}{r_n^2} \le (r_n - 1)^2 \,.$$

Hence $r_{n+1}^2 - 1 \le (r_n - 1)^2$. Thus, either $r_{n+1} < 1$ or $0 < r_{n+1} - 1 \le r_n - 1$. In either case we are done. ■

Proposition 2 *Suppose that $\alpha = 0$ and that $n > 0$ and $r_n < 1$. Then $r_{n+1} < 1$.*

Proof. This time we use Equation (**??**) in the form

$$1 - r_{n+1}^2 \quad = \quad \frac{y_n^2}{r_n}\,(2 - r_n) > 0,$$

since $r_n < 1$. ■

Proposition 3 *Suppose that $\alpha = 1$ and that $n > 0$ and $r_n > 2$. Then $r_{n+1} < r_n$.*

Proof. Equation (**??**) rewrites as

$$r_{n+1}^2 - 1 \quad = \quad 1 + \frac{y_n^2}{r_n^2}\left(r_n^2 - 2\,r_n\right) + 2\,\frac{y_n}{r_n}\,(r_n - 1)\,.$$

Hence $r_{n+1}^2 - 1 < 1 + \left(r_n^2 - 2\,r_n\right) + 2\,(r_n - 1) = r_n^2 - 1$, and $r_{n+1} < r_n$ as required. ■

Theorem 2 (Equatorial case) *If $\alpha = 0$ then $z_n \to (1, 0)$.*

Proof. By Proposition **??** either (a) r_n strictly decreases to $r \ge 1$, which is easily seen to be impossible, or (b) in finitely many steps $r_n < 1$. We appeal to Proposition **??** to conclude that $r_m < 1$ for all $m < n$.

We note that

$$|\tan(\theta_{n+1})| = |1 - r_n|\,|\tan(\theta_n)| < |\tan(\theta_n)|, \tag{5}$$

and so $\tan(\theta_n)$ is decreasing in modulus. It follows, on taking limits in formula (**??**) that (a) $r_n \to 1$ or (b) $\theta_n \to 0$. In case (a) we see from equation (**??**) that $y_n \to 0$ and from (**??**) that $x_n \to 1$.

Thus, we are left only with the case that $\theta_n \to 0$. But now $x_{n+1} = \cos(theta_n) \to 1$ and $asy_{n+1}/x_{n+1} \to 0$, the proof is complete. ■

Theorem 3 (Tangent case) *If $\alpha = 1$ then then $z_n \to \overline{z} := (0, \overline{y})$ for some finite $\overline{y} > 1$ (and the projection on the sphere of $\overline{z}$ is the intersection point of the two sets).*

Proof. An easy estimate from equation (**??**) is $y_{n+1} - y_n \geq 0$. Thence y_n is nondecreasing with possibly infinite limit $\overline{y}$. If $\overline{y}$ is finite then taking limits in (**??**) shows $\lim_{n\to\infty} r_n = \lim_{n\to\infty} y_n$, which completes the proof—as $r \leq 1$ is easy to rule out.

In the remaining case, by relabeling, we may assume that $r_n > y_n > 2$ for all n. Thence Proposition **??** inductively shows that r_n decreases to some finite $\overline{r} > \overline{y}$. This contradiction concludes the proof. ■

It remains to consider $0 < \alpha < 1$ and it seems probable that similar but more careful arguments using Equation (**??**) are key to showing the ubiquitous behaviour shown in Figure **??**.

3 Behaviour on the vertical axis

It is clear both geometrically and analytically that the vertical axis is left invariant by the iteration (**??**,**??**). Even so, starting with $x_0 = 0$ leads to quite varied behaviour. We note that $P_B(0)$ is the entire unit disk, and so the mapping is intrinsically multivalued at zero.

Again we distinguish four cases:

1. $\alpha = 0$. In this case the mapping has period two for y in $[-1, 1]$. For $|y| > 1$, however $T^{(2)}([0, y]) = [0, y - 2\,\mathrm{sign}(y)]$.

2. $0 < \alpha < 1$. In this case, the behaviour of the map is quite subtle and depends on the the starting point and α. It exhibits periodicity of varied orders when both are rational.

3. $\alpha = 1$. In this case $T([0, y]) = [0, y]$ for $y > 0$ and $T([0, y]) = [0, y + 2]$ for $y < 0$. Hence after a finite number of iterations the iteration terminates.

4. $\alpha > 1$. In this infeasible case we again see simpler translational behaviour of T.

4 Extensions

Several natural extensions to study (graphically and analytically) take B as the sphere in n-dimensional space E and consider:

- A as an affine subspace in E of dimension $2 < m < n$;
- A as a polyhedron (or polyhedral cone) with $n = 2$ or $n = 3$.

Remark 1 Note, even in two dimensions, alternating projections, alternating reflections, project-project and average, and reflect-reflect and average will all often converge to (locally nearest) infeasible points even when A is simply the ray $R := \{[x, 0] : x \geq -1/2\}$ and B is the circle as before. They can also behave quite 'chaotically'. (See Figure **??** for a periodic illustration in *Maple*.) So the affine nature of the convex set seems quite important.

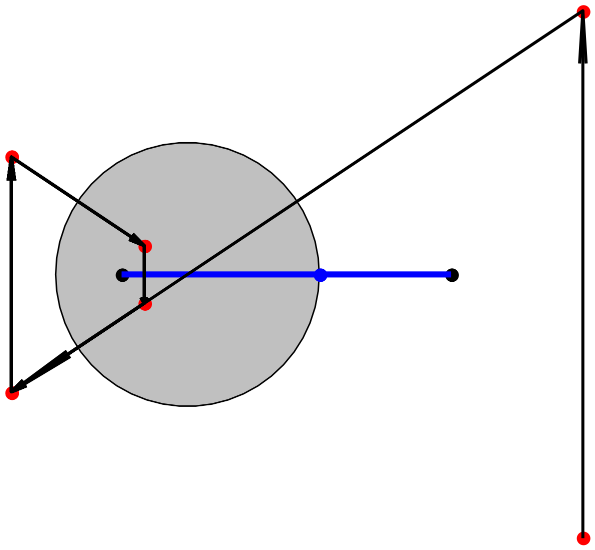

Figure 5: **Iterated reflection on the ray R.**

Remark 2 (Nearest point to an ellipse) Consider the ellipse

$$E := \left\{(x,y)\colon \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right\}$$

in standard form. The best approximation $P_E(u,v) = \left(\frac{a^2u}{a^2-t}, \frac{b^2v}{b^2-t}\right)$ where t solves $\frac{a^2u^2}{(a^2-t)^2}+\frac{b^2v^2}{(b^2-t)^2}=1$. This generalizes neatly to a hyperbola (one solves the general quartic $x^4-ux^3+vx-1=0$ and $[x,1/x]$ is the nearest point.)

Remark 3 (Nearest point to the p-sphere) For $0<p<\infty$, consider the p-sphere in two dimensions

$$S_p := \{(x,y)\colon |x|^p+|y|^p=1\}\,.$$

Let $z^* := (1-z^p)^{1/p}$. For $uv \neq 0$, the best approximation $P_{S_p}(u,v) = (\mathrm{sign}(u)z, \mathrm{sign}(v)z^*)$ where either $z=0,1$ or $0<z<1$ solves

$$z^{*p-1}(z-|u|)-z^{p-1}(z^*-|v|)=0.$$

[Then one computes the two or three distances and select the point yielding the least value. It is instructive to make a plot, say for $p=1/2$.] This extends to the case where $uv=0$. Note that this also yields the nearest point formula for the p-ball.

It should be possible to consider local convergence by linearization of T from Equation (**??**). This makes it important to understand approximate solution of a point in the intersection of two hyperplanes.

For the hyperplane $H_a := \{x \colon \langle a, x\rangle = b\}$ the projection is

$$x \mapsto x + \{\langle a, x\rangle - b\}\,\frac{a}{\|a\|^2}.$$

The consequent averaged-reflection version of the Douglas-Rachford or Lions-Mercier recursion for a point in the intersection of N distinct hyperplanes is:

$$x \mapsto x + \frac{2}{N}\sum_{k=1}^{N}\{\langle a_k, x\rangle - b_k\}\,\frac{a_k}{\|a_k\|^2}. \tag{6}$$

The corresponding-averaged projection algorithm is:

$$x \mapsto x + \frac{1}{N}\sum_{k=1}^{N}\{\langle a_k, x\rangle - b_k\}\,\frac{a_k}{\|a_k\|^2} \tag{7}$$

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