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Jonathan M. Borwein
Adrian S. Lewis

# Convex Analysis and Nonlinear Optimization

## Theory and Examples

*Second Edition*

# Contents

# Chapter 1

# Background

## 1.1 Euclidean Spaces

We begin by reviewing some of the fundamental algebraic, geometric and analytic ideas we use throughout the book. **Our setting, for most of the book, is an arbitrary Euclidean space $\mathbf{E}$**, by which we mean a finite-dimensional vector space over the reals $\mathbf{R}$, equipped with an inner product $\langle \cdot, \cdot \rangle$. We would lose no generality if we considered only the space $\mathbf{R}^n$ of real (column) $n$-vectors (with its standard inner product), but a more abstract, coordinate-free notation is often more flexible and elegant.

We define the *norm* of any point $x$ in $\mathbf{E}$ by $\|x\| = \sqrt{\langle x, x \rangle}$, and the *unit ball* is the set

$$B = \{x \in \mathbf{E} \mid \|x\| \leq 1\}.$$

Any two points $x$ and $y$ in $\mathbf{E}$ satisfy the *Cauchy–Schwarz inequality*

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

We define the sum of two sets $C$ and $D$ in $\mathbf{E}$ by

$$C + D = \{x + y \mid x \in C,\ y \in D\}.$$

The definition of $C - D$ is analogous, and for a subset $\Lambda$ of $\mathbf{R}$ we define

$$\Lambda C = \{\lambda x \mid \lambda \in \Lambda,\ x \in C\}.$$

Given another Euclidean space $\mathbf{Y}$, we can consider the Cartesian product Euclidean space $\mathbf{E} \times \mathbf{Y}$, with inner product defined by $\langle (e, x), (f, y) \rangle = \langle e, f \rangle + \langle x, y \rangle$.

We denote the nonnegative reals by $\mathbf{R}_+$. If $C$ is nonempty and satisfies $\mathbf{R}_+ C = C$ we call it a *cone*. (Notice we require that cones contain the origin.) Examples are the positive orthant

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid \text{each } x_i \geq 0\},$$

and the cone of vectors with nonincreasing components

$$\mathbf{R}^n_{\geq} = \{x \in \mathbf{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n\}.$$

The smallest cone containing a given set $D \subset \mathbf{E}$ is clearly $\mathbf{R}_+ D$.

The fundamental geometric idea of this book is *convexity*. A set $C$ in $\mathbf{E}$ is *convex* if the line segment joining any two points $x$ and $y$ in $C$ is contained in $C$: algebraically, $\lambda x + (1-\lambda)y \in C$ whenever $0 \leq \lambda \leq 1$. An easy exercise shows that intersections of convex sets are convex.

Given any set $D \subset \mathbf{E}$, the *linear span* of $D$, denoted $\operatorname{span}(D)$, is the smallest linear subspace containing $D$. It consists exactly of all linear combinations of elements of $D$. Analogously, the *convex hull* of $D$, denoted $\operatorname{conv}(D)$, is the smallest convex set containing $D$. It consists exactly of all *convex combinations* of elements of $D$, that is to say points of the form $\sum_{i=1}^m \lambda_i x^i$, where $\lambda_i \in \mathbf{R}_+$ and $x^i \in D$ for each $i$, and $\sum \lambda_i = 1$ (see Exercise 2).

The language of elementary point-set topology is fundamental in optimization. A point $x$ lies in the *interior* of the set $D \subset \mathbf{E}$ (denoted $\operatorname{int} D$) if there is a real $\delta > 0$ satisfying $x + \delta B \subset D$. In this case we say $D$ is a *neighbourhood* of $x$. For example, the interior of $\mathbf{R}^n_+$ is

$$\mathbf{R}^n_{++} = \{x \in \mathbf{R}^n \mid \text{each } x_i > 0\}.$$

We say the point $x$ in $\mathbf{E}$ is the *limit* of the sequence of points $x^1, x^2, \ldots$ in $\mathbf{E}$, written $x^j \to x$ as $j \to \infty$ (or $\lim_{j\to\infty} x^j = x$), if $\|x^j - x\| \to 0$. The *closure* of $D$ is the set of limits of sequences of points in $D$, written $\operatorname{cl} D$, and the *boundary* of $D$ is $\operatorname{cl} D \setminus \operatorname{int} D$, written $\operatorname{bd} D$. The set $D$ is *open* if $D = \operatorname{int} D$, and is *closed* if $D = \operatorname{cl} D$. Linear subspaces of $\mathbf{E}$ are important examples of closed sets. Easy exercises show that $D$ is open exactly when its complement $D^c$ is closed, and that arbitrary unions and finite intersections of open sets are open. The interior of $D$ is just the largest open set contained in $D$, while $\operatorname{cl} D$ is the smallest closed set containing $D$. Finally, a subset $G$ of $D$ is *open in* $D$ if there is an open set $U \subset \mathbf{E}$ with $G = D \cap U$.

Much of the beauty of convexity comes from *duality* ideas, interweaving geometry and topology. The following result, which we prove a little later, is both typical and fundamental.

**Theorem 1.1.1 (Basic separation)** *Suppose that the set $C \subset \mathbf{E}$ is closed and convex, and that the point $y$ does not lie in $C$. Then there exist real $b$ and a nonzero element $a$ of $\mathbf{E}$ satisfying $\langle a, y\rangle > b \geq \langle a, x\rangle$ for all points $x$ in $C$.*

Sets in $\mathbf{E}$ of the form $\{x \mid \langle a, x\rangle = b\}$ and $\{x \mid \langle a, x\rangle \leq b\}$ (for a nonzero element $a$ of $\mathbf{E}$ and real $b$) are called *hyperplanes* and *closed halfspaces*, respectively. In this language the above result states that the point $y$ is *separated* from the set $C$ by a hyperplane. In other words, $C$ is contained in a certain closed halfspace whereas $y$ is not. Thus there is a "dual" representation of $C$ as the intersection of all closed halfspaces containing it.

The set $D$ is *bounded* if there is a real $k$ satisfying $kB \supset D$, and it is *compact* if it is closed and bounded. The following result is a central tool in real analysis.

**Theorem 1.1.2 (Bolzano–Weierstrass)** *Bounded sequences in* $\mathbf{E}$ *have convergent subsequences.*

Just as for sets, geometric and topological ideas also intermingle for the functions we study. Given a set $D$ in $\mathbf{E}$, we call a function $f : D \to \mathbf{R}$ *continuous* (on $D$) if $f(x^i) \to f(x)$ for any sequence $x^i \to x$ in $D$. In this case it easy to check, for example, that for any real $\alpha$ the *level set* $\{x \in D \mid f(x) \leq \alpha\}$ is closed providing $D$ is closed.

Given another Euclidean space $\mathbf{Y}$, we call a map $A : \mathbf{E} \to \mathbf{Y}$ *linear* if any points $x$ and $z$ in $\mathbf{E}$ and any reals $\lambda$ and $\mu$ satisfy $A(\lambda x + \mu z) = \lambda Ax + \mu Az$. In fact any linear function from $\mathbf{E}$ to $\mathbf{R}$ has the form $\langle a, \cdot \rangle$ for some element $a$ of $\mathbf{E}$. Linear maps and *affine* functions (linear functions plus constants) are continuous. Thus, for example, closed halfspaces are indeed closed. A *polyhedron* is a finite intersection of closed halfspaces, and is therefore both closed and convex. The *adjoint* of the map $A$ above is the linear map $A^* : \mathbf{Y} \to \mathbf{E}$ defined by the property

$$\langle A^* y, x \rangle = \langle y, Ax \rangle \text{ for all points } x \text{ in } \mathbf{E} \text{ and } y \text{ in } \mathbf{Y}$$

(whence $A^{**} = A$). The *null space* of $A$ is $N(A) = \{x \in \mathbf{E} \mid Ax = 0\}$. The *inverse image* of a set $H \subset \mathbf{Y}$ is the set $A^{-1}H = \{x \in \mathbf{E} \mid Ax \in H\}$ (so for example $N(A) = A^{-1}\{0\}$). Given a subspace $G$ of $\mathbf{E}$, the *orthogonal complement* of $G$ is the subspace

$$G^\perp = \{y \in \mathbf{E} \mid \langle x, y \rangle = 0 \text{ for all } x \in G\},$$

so called because we can write $\mathbf{E}$ as a direct sum $G \oplus G^\perp$. (In other words, any element of $\mathbf{E}$ can be written uniquely as the sum of an element of $G$ and an element of $G^\perp$.) Any subspace $G$ satisfies $G^{\perp\perp} = G$. The range of any linear map $A$ coincides with $N(A^*)^\perp$.

Optimization studies properties of minimizers and maximizers of functions. Given a set $\Lambda \subset \mathbf{R}$, the *infimum* of $\Lambda$ (written $\inf \Lambda$) is the greatest lower bound on $\Lambda$, and the *supremum* (written $\sup \Lambda$) is the least upper bound. To ensure these are always defined, it is natural to append $-\infty$ and $+\infty$ to the real numbers, and allow their use in the usual notation for open and closed intervals. Hence, $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$, and for example

$(-\infty, +\infty]$ denotes the interval $\mathbf{R} \cup \{+\infty\}$. We try to avoid the appearance of $+\infty - \infty$, but when necessary we use the convention $+\infty - \infty = +\infty$, so that any two sets $C$ and $D$ in $\mathbf{R}$ satisfy $\inf C + \inf D = \inf(C + D)$. We also adopt the conventions $0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$. A *(global) minimizer* of a function $f : D \to \mathbf{R}$ is a point $\bar{x}$ in $D$ at which $f$ attains its infimum

$$\inf_D f = \inf f(D) = \inf\{f(x) \mid x \in D\}.$$

In this case we refer to $\bar{x}$ as an *optimal solution* of the *optimization problem* $\inf_D f$.

For a positive real $\delta$ and a function $g : (0, \delta) \to \mathbf{R}$, we define

$$\liminf_{t \downarrow 0} g(t) = \lim_{t \downarrow 0} \inf_{(0,t)} g$$

and

$$\limsup_{t \downarrow 0} g(t) = \lim_{t \downarrow 0} \sup_{(0,t)} g.$$

The limit $\lim_{t \downarrow 0} g(t)$ exists if and only if the above expressions are equal.

The question of *attainment*, or in other words the *existence* of an optimal solution for an optimization problem is typically topological. The following result is a prototype. The proof is a standard application of the Bolzano–Weierstrass theorem above.

**Proposition 1.1.3 (Weierstrass)** *Suppose that the set $D \subset \mathbf{E}$ is nonempty and closed, and that all the level sets of the continuous function $f : D \to \mathbf{R}$ are bounded. Then $f$ has a global minimizer.*

Just as for sets, convexity of functions will be crucial for us. Given a convex set $C \subset \mathbf{E}$, we say that the function $f : C \to \mathbf{R}$ is *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all points $x$ and $y$ in $C$ and $0 \leq \lambda \leq 1$. The function $f$ is *strictly convex* if the inequality holds strictly whenever $x$ and $y$ are distinct in $C$ and $0 < \lambda < 1$. It is easy to see that a strictly convex function can have at most one minimizer.

Requiring the function $f$ to have bounded level sets is a "growth condition". Another example is the stronger condition

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > 0, \tag{1.1.4}$$

where we define

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = \lim_{r \to +\infty} \inf \left\{ \frac{f(x)}{\|x\|} \;\middle|\; x \in C \cap rB^c \right\}.$$

Surprisingly, for *convex* functions these two growth conditions are equivalent.

**Proposition 1.1.5** *For a convex set $C \subset \mathbf{E}$, a convex function $f : C \to \mathbf{R}$ has bounded level sets if and only if it satisfies the growth condition (1.1.4).*

The proof is outlined in Exercise 10.

## Exercises and Commentary

Good general references are [177] for elementary real analysis and [1] for linear algebra. Separation theorems for convex sets originate with Minkowski [142]. The theory of the relative interior (Exercises 11, 12, and 13) is developed extensively in [167] (which is also a good reference for the recession cone, Exercise 6).

1. Prove the intersection of an arbitrary collection of convex sets is convex. Deduce that the convex hull of a set $D \subset \mathbf{E}$ is well-defined as the intersection of all convex sets containing $D$.

2. (a) Prove that if the set $C \subset \mathbf{E}$ is convex and if

$$x^1, x^2, \ldots, x^m \in C,\ 0 \le \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{R},$$

and $\sum \lambda_i = 1$ then $\sum \lambda_i x^i \in C$. Prove, furthermore, that if $f : C \to \mathbf{R}$ is a convex function then $f(\sum \lambda_i x^i) \le \sum \lambda_i f(x^i)$.

   (b) We see later (Theorem 3.1.11) that the function $-\log$ is convex on the strictly positive reals. Deduce, for any strictly positive reals $x^1, x^2, \ldots, x^m$, and any nonnegative reals $\lambda_1, \lambda_2, \ldots, \lambda_m$ with sum 1, the *arithmetic-geometric mean inequality*

$$\sum_i \lambda_i x^i \ge \prod_i (x^i)^{\lambda_i}.$$

   (c) Prove that for any set $D \subset \mathbf{E}$, $\operatorname{conv} D$ is the set of all convex combinations of elements of $D$.

3. Prove that a convex set $D \subset \mathbf{E}$ has convex closure, and deduce that $\operatorname{cl}(\operatorname{conv} D)$ is the smallest closed convex set containing $D$.

4. **(Radstrom cancellation)** Suppose sets $A, B, C \subset \mathbf{E}$ satisfy

$$A + C \subset B + C.$$

   (a) If $A$ and $B$ are convex, $B$ is closed, and $C$ is bounded, prove

$$A \subset B.$$

   (Hint: Observe $2A + C = A + (A + C) \subset 2B + C$.)

   (b) Show this result can fail if $B$ is not convex.

5. * **(Strong separation)** Suppose that the set $C \subset \mathbf{E}$ is closed and convex, and that the set $D \subset \mathbf{E}$ is compact and convex.

   (a) Prove the set $D - C$ is closed and convex.

   (b) Deduce that if in addition $D$ and $C$ are disjoint then there exists a nonzero element $a$ in $\mathbf{E}$ with $\inf_{x \in D} \langle a, x \rangle > \sup_{y \in C} \langle a, y \rangle$. Interpret geometrically.

   (c) Show part (b) fails for the closed convex sets in $\mathbf{R}^2$,

$$\begin{aligned} D &= \{x \mid x_1 > 0,\ x_1 x_2 \geq 1\}, \\ C &= \{x \mid x_2 = 0\}. \end{aligned}$$

6. ** **(Recession cones)** Consider a nonempty closed convex set $C \subset \mathbf{E}$. We define the *recession cone* of $C$ by

$$0^+(C) = \{d \in \mathbf{E} \mid C + \mathbf{R}_+ d \subset C\}.$$

   (a) Prove $0^+(C)$ is a closed convex cone.

   (b) Prove $d \in 0^+(C)$ if and only if $x + \mathbf{R}_+ d \subset C$ for some point $x$ in $C$. Show this equivalence can fail if $C$ is not closed.

   (c) Consider a family of closed convex sets $C_\gamma$ $(\gamma \in \Gamma)$ with nonempty intersection. Prove $0^+(\cap C_\gamma) = \cap 0^+(C_\gamma)$.

   (d) For a unit vector $u$ in $\mathbf{E}$, prove $u \in 0^+(C)$ if and only if there is a sequence $(x^r)$ in $C$ satisfying $\|x^r\| \to \infty$ and $\|x^r\|^{-1} x^r \to u$. Deduce $C$ is unbounded if and only if $0^+(C)$ is nontrivial.

   (e) If $\mathbf{Y}$ is a Euclidean space, the map $A : \mathbf{E} \to \mathbf{Y}$ is linear, and $N(A) \cap 0^+(C)$ is a linear subspace, prove $AC$ is closed. Show this result can fail without the last assumption.

   (f) Consider another nonempty closed convex set $D \subset \mathbf{E}$ such that $0^+(C) \cap 0^+(D)$ is a linear subspace. Prove $C - D$ is closed.

7. For any set of vectors $a^1, a^2, \ldots, a^m$ in $\mathbf{E}$, prove the function $f(x) = \max_i \langle a^i, x \rangle$ is convex on $\mathbf{E}$.

8. Prove Proposition 1.1.3 (Weierstrass).

9. **(Composing convex functions)** Suppose that the set $C \subset \mathbf{E}$ is convex and that the functions $f_1, f_2, \ldots, f_n : C \to \mathbf{R}$ are convex, and define a function $f : C \to \mathbf{R}^n$ with components $f_i$. Suppose further that $f(C)$ is convex and that the function $g : f(C) \to \mathbf{R}$ is convex and *isotone*: any points $y \leq z$ in $f(C)$ satisfy $g(y) \leq g(z)$. Prove the composition $g \circ f$ is convex.

10. * **(Convex growth conditions)**

   (a) Find a function with bounded level sets which does not satisfy the growth condition (1.1.4).

   (b) Prove that any function satisfying (1.1.4) has bounded level sets.

   (c) Suppose the convex function $f : C \to \mathbf{R}$ has bounded level sets but that (1.1.4) fails. Deduce the existence of a sequence $(x^m)$ in $C$ with $f(x^m) \leq \|x^m\|/m \to +\infty$. For a fixed point $\bar{x}$ in $C$, derive a contradiction by considering the sequence

$$\bar{x} + \frac{m}{\|x^m\|}(x^m - \bar{x}).$$

   Hence complete the proof of Proposition 1.1.5.

## The relative interior

Some arguments about finite-dimensional convex sets $C$ simplify and lose no generality if we assume $C$ contains 0 and spans $\mathbf{E}$. The following exercises outline this idea.

11. ** **(Accessibility lemma)** Suppose $C$ is a convex set in $\mathbf{E}$.

   (a) Prove $\mathrm{cl}\, C \subset C + \epsilon B$ for any real $\epsilon > 0$.

   (b) For sets $D$ and $F$ in $\mathbf{E}$ with $D$ open, prove $D + F$ is open.

   (c) For $x$ in $\mathrm{int}\, C$ and $0 < \lambda \leq 1$, prove $\lambda x + (1 - \lambda)\mathrm{cl}\, C \subset C$. Deduce $\lambda\, \mathrm{int}\, C + (1 - \lambda)\mathrm{cl}\, C \subset \mathrm{int}\, C$.

   (d) Deduce $\mathrm{int}\, C$ is convex.

   (e) Deduce further that if $\mathrm{int}\, C$ is nonempty then $\mathrm{cl}\,(\mathrm{int}\, C) = \mathrm{cl}\, C$. Is convexity necessary?

12. ** **(Affine sets)** A set $L$ in $\mathbf{E}$ is *affine* if the entire line through any distinct points $x$ and $y$ in $L$ lies in $L$: algebraically, $\lambda x + (1 - \lambda)y \in L$ for *any* real $\lambda$. The *affine hull* of a set $D$ in $\mathbf{E}$, denoted $\mathrm{aff}\, D$, is the smallest affine set containing $D$. An *affine combination* of points $x^1, x^2, \ldots, x^m$ is a point of the form $\sum_1^m \lambda_i x^i$, for reals $\lambda_i$ summing to one.

   (a) Prove the intersection of an arbitrary collection of affine sets is affine.

   (b) Prove that a set is affine if and only if it is a translate of a linear subspace.

   (c) Prove $\mathrm{aff}\, D$ is the set of all affine combinations of elements of $D$.

   (d) Prove $\mathrm{cl}\, D \subset \mathrm{aff}\, D$ and deduce $\mathrm{aff}\, D = \mathrm{aff}\,(\mathrm{cl}\, D)$.

(e) For any point $x$ in $D$, prove $\operatorname{aff} D = x + \operatorname{span}(D - x)$, and deduce the linear subspace $\operatorname{span}(D - x)$ is independent of $x$.

13. ** **(The relative interior)** (We use Exercises 11 and 12.) The *relative interior* of a convex set $C$ in $\mathbf{E}$, denoted $\operatorname{ri} C$, is its interior relative to its affine hull. In other words, a point $x$ lies in $\operatorname{ri} C$ if there is a real $\delta > 0$ with $(x + \delta B) \cap \operatorname{aff} C \subset C$.

   (a) Find convex sets $C_1 \subset C_2$ with $\operatorname{ri} C_1 \not\subset \operatorname{ri} C_2$.

   (b) Suppose $\dim \mathbf{E} > 0$, $0 \in C$ and $\operatorname{aff} C = \mathbf{E}$. Prove $C$ contains a basis $\{x^1, x^2, \ldots, x^n\}$ of $\mathbf{E}$. Deduce $(1/(n+1)) \sum_1^n x^i \in \operatorname{int} C$. Hence deduce that any nonempty convex set in $\mathbf{E}$ has nonempty relative interior.

   (c) Prove that for $0 < \lambda \leq 1$ we have $\lambda \operatorname{ri} C + (1 - \lambda) \operatorname{cl} C \subset \operatorname{ri} C$, and hence $\operatorname{ri} C$ is convex with $\operatorname{cl}(\operatorname{ri} C) = \operatorname{cl} C$.

   (d) Prove that for a point $x$ in $C$, the following are equivalent:

      (i) $x \in \operatorname{ri} C$.

      (ii) For any point $y$ in $C$ there exists a real $\epsilon > 0$ with $x + \epsilon(x - y)$ in $C$.

      (iii) $\mathbf{R}_+(C - x)$ is a linear subspace.

   (e) If $\mathbf{F}$ is another Euclidean space and the map $A : \mathbf{E} \to \mathbf{F}$ is linear, prove $\operatorname{ri} AC \supset A \operatorname{ri} C$.

## 1.2 Symmetric Matrices

Throughout most of this book our setting is an abstract Euclidean space $\mathbf{E}$. This has a number of advantages over always working in $\mathbf{R}^n$: the basis-independent notation is more elegant and often clearer, and it encourages techniques which extend beyond finite dimensions. But more concretely, identifying $\mathbf{E}$ with $\mathbf{R}^n$ may obscure properties of a space beyond its simple Euclidean structure. As an example, in this short section we describe a Euclidean space which "feels" very different from $\mathbf{R}^n$: the space $\mathbf{S}^n$ of $n \times n$ real symmetric matrices.

The nonnegative orthant $\mathbf{R}^n_+$ is a cone in $\mathbf{R}^n$ which plays a central role in our development. In a variety of contexts the analogous role in $\mathbf{S}^n$ is played by the cone of positive semidefinite matrices, $\mathbf{S}^n_+$. (We call a matrix $X$ in $\mathbf{S}^n$ *positive semidefinite* if $x^T X x \geq 0$ for all vectors $x$ in $\mathbf{R}^n$, and *positive definite* if the inequality is strict whenever $x$ is nonzero.) These two cones have some important differences; in particular, $\mathbf{R}^n_+$ is a polyhedron, whereas the cone of positive semidefinite matrices $\mathbf{S}^n_+$ is not, even for $n = 2$. The cones $\mathbf{R}^n_+$ and $\mathbf{S}^n_+$ are important largely because of the orderings they induce. (The latter is sometimes called the *Loewner ordering*.) For points $x$ and $y$ in $\mathbf{R}^n$ we write $x \leq y$ if $y - x \in \mathbf{R}^n_+$, and $x < y$ if $y - x \in \mathbf{R}^n_{++}$ (with analogous definitions for $\geq$ and $>$). The cone $\mathbf{R}^n_+$ is a *lattice cone*: for any points $x$ and $y$ in $\mathbf{R}^n$ there is a point $z$ satisfying

$$w \geq x \text{ and } w \geq y \;\Leftrightarrow\; w \geq z.$$

(The point $z$ is just the componentwise maximum of $x$ and $y$.) Analogously, for matrices $X$ and $Y$ in $\mathbf{S}^n$ we write $X \preceq Y$ if $Y - X \in \mathbf{S}^n_+$, and $X \prec Y$ if $Y - X$ lies in $\mathbf{S}^n_{++}$, the set of positive definite matrices (with analogous definitions for $\succeq$ and $\succ$). By contrast, it is straightforward to see $\mathbf{S}^n_+$ is *not* a lattice cone (Exercise 4).

We denote the identity matrix by $I$. The *trace* of a square matrix $Z$ is the sum of the diagonal entries, written $\operatorname{tr} Z$. It has the important property $\operatorname{tr}(VW) = \operatorname{tr}(WV)$ for any matrices $V$ and $W$ for which $VW$ is well-defined and square. We make the vector space $\mathbf{S}^n$ into a Euclidean space by defining the inner product

$$\langle X, Y \rangle = \operatorname{tr}(XY) \;\text{ for } X, Y \in \mathbf{S}^n.$$

Any matrix $X$ in $\mathbf{S}^n$ has $n$ real eigenvalues (counted by multiplicity), which we write in nonincreasing order $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$. In this way we define a function $\lambda : \mathbf{S}^n \to \mathbf{R}^n$. We also define a linear map $\operatorname{Diag} : \mathbf{R}^n \to \mathbf{S}^n$, where for a vector $x$ in $\mathbf{R}^n$, $\operatorname{Diag} x$ is an $n \times n$ diagonal matrix with diagonal entries $x_i$. This map embeds $\mathbf{R}^n$ as a subspace of $\mathbf{S}^n$ and the cone $\mathbf{R}^n_+$ as a subcone of $\mathbf{S}^n_+$. The determinant of a square matrix $Z$ is written $\det Z$.

We write $\mathbf{O}^n$ for the group of $n \times n$ *orthogonal* matrices (those matrices $U$ satisfying $U^T U = I$). Then any matrix $X$ in $\mathbf{S}^n$ has an *ordered spectral decomposition* $X = U^T(\mathrm{Diag}\,\lambda(X))U$, for some matrix $U$ in $\mathbf{O}^n$. This shows, for example, that the function $\lambda$ is *norm-preserving*: $\|X\| = \|\lambda(X)\|$ for all $X$ in $\mathbf{S}^n$. For any $X$ in $\mathbf{S}^n_+$, the spectral decomposition also shows there is a unique matrix $X^{1/2}$ in $\mathbf{S}^n_+$ whose square is $X$.

The Cauchy–Schwarz inequality has an interesting refinement in $\mathbf{S}^n$ which is crucial for variational properties of eigenvalues, as we shall see.

**Theorem 1.2.1 (Fan)** *Any matrices $X$ and $Y$ in $\mathbf{S}^n$ satisfy the inequality*

$$\mathrm{tr}\,(XY) \leq \lambda(X)^T \lambda(Y). \tag{1.2.2}$$

*Equality holds if and only if $X$ and $Y$ have a* **simultaneous ordered spectral decomposition**: *there is a matrix $U$ in $\mathbf{O}^n$ with*

$$X = U^T(\mathrm{Diag}\,\lambda(X))U \quad \textit{and} \quad Y = U^T(\mathrm{Diag}\,\lambda(Y))U. \tag{1.2.3}$$

A standard result in linear algebra states that matrices $X$ and $Y$ have a simultaneous (*unordered*) spectral decomposition if and only if they commute. Notice condition (1.2.3) is a stronger property.

The special case of Fan's inequality where both matrices are diagonal gives the following classical inequality. For a vector $x$ in $\mathbf{R}^n$, we denote by $[x]$ the vector with the same components permuted into nonincreasing order. We leave the proof of this result as an exercise.

**Proposition 1.2.4 (Hardy–Littlewood–Pólya)** *Any vectors $x$ and $y$ in $\mathbf{R}^n$ satisfy the inequality*

$$x^T y \leq [x]^T [y].$$

We describe a proof of Fan's theorem in the exercises, using the above proposition and the following classical relationship between the set $\mathbf{\Gamma}^n$ of *doubly stochastic* matrices (square matrices with all nonnegative entries, and each row and column summing to one) and the set $\mathbf{P}^n$ of *permutation* matrices (square matrices with all entries zero or one, and with exactly one entry of one in each row and in each column).

**Theorem 1.2.5 (Birkhoff)** *Doubly stochastic matrices are convex combinations of permutation matrices.*

We defer the proof to a later section (Section 4.1, Exercise 22).

## Exercises and Commentary

Fan's inequality (1.2.2) appeared in [73], but is closely related to earlier work of von Neumann [184]. The condition for equality is due to [180]. The Hardy–Littlewood–Pólya inequality may be found in [91]. Birkhoff's theorem [15] was in fact proved earlier by König [115].

1. Prove $\mathbf{S}^n_+$ is a closed convex cone with interior $\mathbf{S}^n_{++}$.

2. Explain why $\mathbf{S}^2_+$ is not a polyhedron.

3. **($\mathbf{S}^3_+$ is not strictly convex)** Find nonzero matrices $X$ and $Y$ in $\mathbf{S}^3_+$ such that $\mathbf{R}_+X \neq \mathbf{R}_+Y$ and $(X+Y)/2 \notin \mathbf{S}^3_{++}$.

4. **(A nonlattice ordering)** Suppose the matrix $Z$ in $\mathbf{S}^2$ satisfies

$$W \succeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } W \succeq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Leftrightarrow W \succeq Z.$$

   (a) By considering diagonal $W$, prove

$$Z = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

   for some real $a$.

   (b) By considering $W = I$, prove $Z = I$.

   (c) Derive a contradiction by considering

$$W = \tfrac{2}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

5. **(Order preservation)**

   (a) Prove any matrix $X$ in $\mathbf{S}^n$ satisfies $(X^2)^{1/2} \succeq X$.

   (b) Find matrices $X \succeq Y$ in $\mathbf{S}^2_+$ such that $X^2 \not\succeq Y^2$.

   (c) For matrices $X \succeq Y$ in $\mathbf{S}^n_+$, prove $X^{1/2} \succeq Y^{1/2}$. (Hint: Consider the relationship

$$\langle (X^{1/2}+Y^{1/2})x, (X^{1/2}-Y^{1/2})x\rangle = \langle (X-Y)x, x\rangle \geq 0,$$

   for eigenvectors $x$ of $X^{1/2} - Y^{1/2}$.)

6. * **(Square-root iteration)** Suppose a matrix $A$ in $\mathbf{S}^n_+$ satisfies $I \succeq A$. Prove that the iteration

$$Y_0 = 0, \quad Y_{n+1} = \frac{1}{2}\big(A + Y_n^2\big) \quad (n = 0, 1, 2, \ldots)$$

   is nondecreasing (that is, $Y_{n+1} \succeq Y_n$ for all $n$) and converges to the matrix $I - (I-A)^{1/2}$. (Hint: Consider diagonal matrices $A$.)

7. **(The Fan and Cauchy–Schwarz inequalities)**

   (a) For any matrices $X$ in $\mathbf{S}^n$ and $U$ in $\mathbf{O}^n$, prove $\|U^T XU\| = \|X\|$.

   (b) Prove the function $\lambda$ is norm-preserving.

   (c) Explain why Fan's inequality is a refinement of the Cauchy–Schwarz inequality.

8. Prove the inequality $\operatorname{tr} Z + \operatorname{tr} Z^{-1} \geq 2n$ for all matrices $Z$ in $\mathbf{S}^n_{++}$, with equality if and only if $Z = I$.

9. Prove the Hardy–Littlewood–Pólya inequality (Proposition 1.2.4) directly.

10. Given a vector $x$ in $\mathbf{R}^n_+$ satisfying $x_1 x_2 \ldots x_n = 1$, define numbers $y_k = 1/x_1 x_2 \ldots x_k$ for each index $k = 1, 2, \ldots, n$. Prove

$$x_1 + x_2 + \ldots + x_n = \frac{y_n}{y_1} + \frac{y_1}{y_2} + \ldots \frac{y_{n-1}}{y_n}.$$

   By applying the Hardy–Littlewood–Pólya inequality (1.2.4) to suitable vectors, prove $x_1 + x_2 + \ldots + x_n \geq n$. Deduce the inequality

$$\frac{1}{n} \sum_1^n z_i \geq \Big( \prod_1^n z_i \Big)^{1/n}$$

   for any vector $z$ in $\mathbf{R}^n_+$.

11. For a fixed column vector $s$ in $\mathbf{R}^n$, define a linear map $A : \mathbf{S}^n \to \mathbf{R}^n$ by setting $AX = Xs$ for any matrix $X$ in $\mathbf{S}^n$. Calculate the adjoint map $A^*$.

12. * **(Fan's inequality)** For vectors $x$ and $y$ in $\mathbf{R}^n$ and a matrix $U$ in $\mathbf{O}^n$, define

$$\alpha = \langle \operatorname{Diag} x, U^T (\operatorname{Diag} y) U \rangle.$$

   (a) Prove $\alpha = x^T Z y$ for some doubly stochastic matrix $Z$.

   (b) Use Birkhoff's theorem and Proposition 1.2.4 to deduce the inequality $\alpha \leq [x]^T [y]$.

   (c) Deduce Fan's inequality (1.2.2).

13. **(A lower bound)** Use Fan's inequality (1.2.2) for two matrices $X$ and $Y$ in $\mathbf{S}^n$ to prove a *lower* bound for $\operatorname{tr}(XY)$ in terms of $\lambda(X)$ and $\lambda(Y)$.

14. * **(Level sets of perturbed log barriers)**
   (a) For $\delta$ in $\mathbf{R}_{++}$, prove the function
   $$t \in \mathbf{R}_{++} \mapsto \delta t - \log t$$
   has compact level sets.
   (b) For $c$ in $\mathbf{R}^n_{++}$, prove the function
   $$x \in \mathbf{R}^n_{++} \mapsto c^T x - \sum_{i=1}^{n} \log x_i$$
   has compact level sets.
   (c) For $C$ in $\mathbf{S}^n_{++}$, prove the function
   $$X \in \mathbf{S}^n_{++} \mapsto \langle C, X \rangle - \log \det X$$
   has compact level sets. (Hint: Use Exercise 13.)

15. * **(Theobald's condition)** Assuming Fan's inequality (1.2.2), complete the proof of Fan's theorem (1.2.1) as follows. Suppose equality holds in Fan's inequality (1.2.2), and choose a spectral decomposition
   $$X + Y = U^T (\operatorname{Diag} \lambda(X + Y)) U$$
   for some matrix $U$ in $\mathbf{O}^n$.
   (a) Prove $\lambda(X)^T \lambda(X + Y) = \langle U^T (\operatorname{Diag} \lambda(X)) U, X + Y \rangle$.
   (b) Apply Fan's inequality (1.2.2) to the two inner products
   $$\langle X, X + Y \rangle \ \text{ and } \ \langle U^T (\operatorname{Diag} \lambda(X)) U, Y \rangle$$
   to deduce $X = U^T (\operatorname{Diag} \lambda(X)) U$.
   (c) Deduce Fan's theorem.

16. ** **(Generalizing Theobald's condition [122])** Consider a set of matrices $X^1, X^2, \ldots, X^m$ in $\mathbf{S}^n$ satisfying the conditions
   $$\operatorname{tr}(X^i X^j) = \lambda(X^i)^T \lambda(X^j) \ \text{ for all } i \text{ and } j.$$
   Generalize the argument of Exercise 15 to prove the entire set of matrices $\{X^1, X^2, \ldots, X^m\}$ has a simultaneous ordered spectral decomposition.

17. ** **(Singular values and von Neumann's lemma)** Let $\mathbf{M}^n$ denote the vector space of $n \times n$ real matrices. For a matrix $A$ in $\mathbf{M}^n$ we define the *singular values* of $A$ by $\sigma_i(A) = \sqrt{\lambda_i(A^T A)}$ for $i = 1, 2, \ldots, n$, and hence define a map $\sigma : \mathbf{M}^n \to \mathbf{R}^n$. (Notice zero may be a singular value.)

(a) Prove

$$\lambda \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix} = \begin{bmatrix} \sigma(A) \\ [-\sigma(A)] \end{bmatrix}$$

(b) For any other matrix $B$ in $\mathbf{M}^n$, use part (a) and Fan's inequality (1.2.2) to prove

$$\operatorname{tr}(A^T B) \le \sigma(A)^T \sigma(B).$$

(c) If $A$ lies in $\mathbf{S}^n_+$, prove $\lambda(A) = \sigma(A)$.

(d) By considering matrices of the form $A + \alpha I$ and $B + \beta I$, deduce Fan's inequality from von Neumann's lemma (part (b)).

# Chapter 3

# Fenchel Duality

## 3.1 Subgradients and Convex Functions

We have already seen, in the First order sufficient condition (2.1.2), one benefit of convexity in optimization: critical points of convex functions are global minimizers. In this section we extend the types of functions we consider in two important ways:

(i) We do not require $f$ to be differentiable.

(ii) We allow $f$ to take the value $+\infty$.

Our derivation of first order conditions in Section 2.3 illustrates the utility of considering nonsmooth functions even in the context of smooth problems. Allowing the value $+\infty$ lets us rephrase a problem like

$$\inf\{g(x) \mid x \in C\}$$

as $\inf(g + \delta_C)$, where the *indicator function* $\delta_C(x)$ is 0 for $x$ in $C$ and $+\infty$ otherwise.

The *domain* of a function $f : \mathbf{E} \to (\infty, +\infty]$ is the set

$$\operatorname{dom} f = \{x \in \mathbf{E} \mid f(x) < +\infty\}.$$

We say $f$ is *convex* if it is convex on its domain, and *proper* if its domain is nonempty. We call a function $g : \mathbf{E} \to [-\infty, +\infty)$ *concave* if $-g$ is convex, although for reasons of simplicity we will consider primarily convex functions. If a convex function $f$ satisfies the stronger condition

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \quad \text{for all } x, y \in \mathbf{E},\ \lambda, \mu \in \mathbf{R}_+$$

we say $f$ is *sublinear*. If $f(\lambda x) = \lambda f(x)$ for all $x$ in $\mathbf{E}$ and $\lambda$ in $\mathbf{R}_+$ then $f$ is *positively homogeneous*: in particular this implies $f(0) = 0$. (Recall the convention $0 \cdot (+\infty) = 0$.) If $f(x+y) \leq f(x) + f(y)$ for all $x$ and $y$ in $\mathbf{E}$ then we say $f$ is *subadditive*. It is immediate that if the function $f$ is sublinear then $-f(x) \leq f(-x)$ for all $x$ in $\mathbf{E}$. The *lineality space* of a sublinear function $f$ is the set

$$\operatorname{lin} f = \{x \in \mathbf{E} \mid -f(x) = f(-x)\}.$$

The following result (whose proof is left as an exercise) shows this set is a subspace.

**Proposition 3.1.1 (Sublinearity)** *A function $f : \mathbf{E} \to (\infty, +\infty]$ is sublinear if and only if it is positively homogeneous and subadditive. For a sublinear function $f$, the lineality space* $\operatorname{lin} f$ *is the largest subspace of* $\mathbf{E}$ *on which $f$ is linear.*

As in the First order sufficient condition (2.1.2), it is easy to check that if the point $\bar{x}$ lies in the domain of the convex function $f$ then the directional derivative $f'(\bar{x}; \cdot)$ is well-defined and positively homogeneous, taking values in $[-\infty, +\infty]$. The *core* of a set $C$ (written $\operatorname{core}(C)$) is the set of points $x$ in $C$ such that for any direction $d$ in $\mathbf{E}$, $x + td$ lies in $C$ for all small real $t$. This set clearly contains the interior of $C$, although it may be larger (Exercise 2).

**Proposition 3.1.2 (Sublinearity of the directional derivative)** *If the function $f : \mathbf{E} \to (\infty, +\infty]$ is convex then, for any point $\bar{x}$ in* $\operatorname{core}(\operatorname{dom} f)$, *the directional derivative $f'(\bar{x}; \cdot)$ is everywhere finite and sublinear.*

**Proof.** For $d$ in $\mathbf{E}$ and nonzero $t$ in $\mathbf{R}$, define

$$g(d; t) = \frac{f(\bar{x} + td) - f(\bar{x})}{t}.$$

By convexity we deduce, for $0 < t \leq s \in \mathbf{R}$, the inequality

$$g(d; -s) \leq g(d; -t) \leq g(d; t) \leq g(d; s).$$

Since $\bar{x}$ lies in $\operatorname{core}(\operatorname{dom} f)$, for small $s > 0$ both $g(d; -s)$ and $g(d; s)$ are finite, so as $t \downarrow 0$ we have

$$+\infty > g(d; s) \geq g(d; t) \downarrow f'(\bar{x}; d) \geq g(d; -s) > -\infty. \tag{3.1.3}$$

Again by convexity we have, for any directions $d$ and $e$ in $\mathbf{E}$ and real $t > 0$,

$$g(d + e; t) \leq g(d; 2t) + g(e; 2t).$$

Now letting $t \downarrow 0$ gives subadditivity of $f'(\bar{x}; \cdot)$. The positive homogeneity is easy to check. □

The idea of the derivative is fundamental in analysis because it allows us to approximate a wide class of functions using *linear functions*. In optimization we are concerned specifically with the minimization of functions, and hence often a *one-sided approximation* is sufficient. In place of the gradient we therefore consider *subgradients*, those elements $\phi$ of $\mathbf{E}$ satisfying

$$\langle \phi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) \text{ for all points } x \text{ in } \mathbf{E}. \tag{3.1.4}$$

We denote the set of subgradients (called the *subdifferential*) by $\partial f(\bar{x})$, defining $\partial f(\bar{x}) = \emptyset$ for $\bar{x}$ not in $\operatorname{dom} f$. The subdifferential is always a closed convex set. We can think of $\partial f(\bar{x})$ as the value at $\bar{x}$ of the "multifunction" or "set-valued map" $\partial f : \mathbf{E} \to \mathbf{E}$. The importance of such maps is another of our themes. We define its *domain*

$$\operatorname{dom} \partial f = \{x \in \mathbf{E} \mid \partial f(x) \neq \emptyset\}$$

(Exercise 19). We say $f$ is *essentially strictly convex* if it is strictly convex on any convex subset of $\operatorname{dom} \partial f$.

The following very easy observation suggests the fundamental significance of subgradients in optimization.

**Proposition 3.1.5 (Subgradients at optimality)** *For any proper function $f : \mathbf{E} \to (\infty, +\infty]$, the point $\bar{x}$ is a (global) minimizer of $f$ if and only if the condition $0 \in \partial f(\bar{x})$ holds.*

Alternatively put, minimizers of $f$ correspond exactly to "zeroes" of $\partial f$.

The derivative is a local property whereas the subgradient definition (3.1.4) describes a global property. The main result of this section shows that the set of subgradients of a convex function is usually *nonempty*, and that we can describe it locally in terms of the directional derivative. We begin with another simple exercise.

**Proposition 3.1.6 (Subgradients and directional derivatives)** *If the function $f : \mathbf{E} \to (\infty, +\infty]$ is convex and the point $\bar{x}$ lies in $\operatorname{dom} f$, then an element $\phi$ of $\mathbf{E}$ is a subgradient of $f$ at $\bar{x}$ if and only if it satisfies $\langle \phi, \cdot \rangle \leq f'(\bar{x}; \cdot)$.*

The idea behind the construction of a subgradient for a function $f$ that we present here is rather simple. We recursively construct a decreasing sequence of sublinear functions which, after translation, minorize $f$. At each step we guarantee one extra direction of linearity. The basic step is summarized in the following exercise.

**Lemma 3.1.7** *Suppose that the function $p : \mathbf{E} \to (\infty, +\infty]$ is sublinear and that the point $\bar{x}$ lies in $\operatorname{core}(\operatorname{dom} p)$. Then the function $q(\cdot) = p'(\bar{x}; \cdot)$ satisfies the conditions* (*i*) $q(\lambda\bar{x}) = \lambda p(\bar{x})$ *for all real* $\lambda$,

(*ii*) $q \le p$, *and*

(*iii*) $\operatorname{lin} q \supset \operatorname{lin} p + \operatorname{span}\{\bar{x}\}$.

With this tool we are now ready for the main result, which gives conditions guaranteeing the existence of a subgradient. Proposition 3.1.6 showed how to identify subgradients from directional derivatives; this next result shows how to move in the reverse direction.

**Theorem 3.1.8 (Max formula)** *If the function* $f : \mathbf{E} \to (\infty, +\infty]$ *is convex then any point* $\bar{x}$ *in* core (dom $f$) *and any direction* $d$ *in* $\mathbf{E}$ *satisfy*

$$f'(\bar{x}; d) = \max\{\langle \phi, d\rangle \mid \phi \in \partial f(\bar{x})\}. \tag{3.1.9}$$

*In particular, the subdifferential* $\partial f(\bar{x})$ *is nonempty.*

**Proof.** In view of Proposition 3.1.6, we simply have to show that for any fixed $d$ in $\mathbf{E}$ there is a subgradient $\phi$ satisfying $\langle \phi, d\rangle = f'(\bar{x}; d)$. Choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $\mathbf{E}$ with $e_1 = d$ if $d$ is nonzero. Now define a sequence of functions $p_0, p_1, \ldots, p_n$ recursively by $p_0(\cdot) = f'(\bar{x}; \cdot)$, and $p_k(\cdot) = p'_{k-1}(e_k; \cdot)$ for $k = 1, 2, \ldots, n$. We essentially show that $p_n(\cdot)$ is the required subgradient.

First note that, by Proposition 3.1.2, each $p_k$ is everywhere finite and sublinear. By part (iii) of Lemma 3.1.7 we know

$$\operatorname{lin} p_k \supset \operatorname{lin} p_{k-1} + \operatorname{span}\{e_k\} \quad \text{for } k = 1, 2, \ldots, n,$$

so $p_n$ is linear. Thus there is an element $\phi$ of $\mathbf{E}$ satisfying $\langle \phi, \cdot\rangle = p_n(\cdot)$.

Part (ii) of Lemma 3.1.7 implies $p_n \le p_{n-1} \le \ldots \le p_0$, so certainly, by Proposition 3.1.6, any point $x$ in $\mathbf{E}$ satisfies

$$p_n(x - \bar{x}) \le p_0(x - \bar{x}) = f'(\bar{x}; x - \bar{x}) \le f(x) - f(\bar{x}).$$

Thus $\phi$ is a subgradient. If $d$ is zero then we have $p_n(0) = 0 = f'(\bar{x}; 0)$. Finally, if $d$ is nonzero then by part (i) of Lemma 3.1.7 we see

$$\begin{aligned} p_n(d) \le p_0(d) = p_0(e_1) = -p'_0(e_1; -e_1) = \\ -p_1(-e_1) = -p_1(-d) \le -p_n(-d) = p_n(d), \end{aligned}$$

whence $p_n(d) = p_0(d) = f'(\bar{x}; d)$. □

**Corollary 3.1.10 (Differentiability of convex functions)** *Suppose the function* $f : \mathbf{E} \to (\infty, +\infty]$ *is convex and the point* $\bar{x}$ *lies in* core (dom $f$). *Then* $f$ *is Gâteaux differentiable at* $\bar{x}$ *exactly when* $f$ *has a unique subgradient at* $\bar{x}$ *(in which case this subgradient is the derivative).*

We say the convex function $f$ is *essentially smooth* if it is Gâteaux differentiable on $\operatorname{dom} \partial f$. (In this definition, we also require $f$ to be "lower semicontinuous"; we defer discussion of lower semicontinuity until we need it, in Section 4.2.) We see later (Section 4.1, Exercise 21) that a function is essentially smooth if and only if its subdifferential is always singleton or empty.

The Max formula (Theorem 3.1.8) shows that convex functions typically have subgradients. In fact this property characterizes convexity (Exercise 12). This leads to a number of important ways of recognizing convex functions, one of which is the following example. Notice how a locally defined analytic condition results in a global geometric conclusion. The proof is outlined in the exercises.

**Theorem 3.1.11 (Hessian characterization of convexity)** *Given an open convex set $S \subset \mathbf{R}^n$, suppose the continuous function $f : \operatorname{cl} S \to \mathbf{R}$ is twice continuously differentiable on $S$. Then $f$ is convex if and only if its Hessian matrix is positive semidefinite everywhere on $S$.*

## Exercises and Commentary

The algebraic proof of the Max formula we follow here is due to [22]. The exercises below develop several standard characterizations of convexity—see for example [167]. The convexity of $-\log\det$ (Exercise 21) may be found in [99], for example. We shall see that the core and interior of a convex set in fact coincide (Theorem 4.1.4).

1. Prove Proposition 3.1.1 (Sublinearity).

2. **(Core versus interior)** Consider the set in $\mathbf{R}^2$

$$D = \{(x, y) \mid y = 0 \text{ or } |y| \geq x^2\}.$$

Prove $0 \in \operatorname{core}(D) \setminus \operatorname{int}(D)$.

3. Prove the subdifferential is a closed convex set.

4. **(Subgradients and normal cones)** If a point $\bar{x}$ lies in a set $C \subset \mathbf{E}$, prove $\partial \delta_C(\bar{x}) = N_C(\bar{x})$.

5. Prove the following functions $x \in \mathbf{R} \mapsto f(x)$ are convex and calculate $\partial f$:

   (a) $$|x|$$

   (b) $$\delta_{\mathbf{R}_+}$$

   (c) $$\begin{cases} -\sqrt{x} & \text{if } x \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

(d)
$$\begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

6. Prove Proposition 3.1.6 (Subgradients and directional derivatives).

7. Prove Lemma 3.1.7.

8. **(Subgradients of norm)** Calculate $\partial \|\cdot\|$. Generalize your result to an arbitrary sublinear function.

9. **(Subgradients of maximum eigenvalue)** Prove
$$\partial \lambda_1(0) = \{Y \in \mathbf{S}^n_+ \mid \operatorname{tr} Y = 1\}.$$

10. ** For any vector $\mu$ in the cone $\mathbf{R}^n_{\geq}$, prove
$$\partial \langle \mu, [\cdot] \rangle (0) = \operatorname{conv}(\mathbf{P}^n \mu)$$
(see Section 2.2, Exercise 9 (Schur-convexity)).

11. * Define a function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x_1, x_2, \ldots, x_n) = \max_j \{x_j\}$, let $\bar{x} = 0$ and $d = (1, 1, \ldots, 1)^T$, and let $e_k = (1, 1, \ldots, 1, 0, \ldots, 0)^T$ (ending in $(k-1)$ zeroes). Calculate the functions $p_k$ defined in the proof of Theorem 3.1.8 (Max formula), using Proposition 2.3.2 (Directional derivatives of max functions).

12. * **(Recognizing convex functions)** Suppose the set $S \subset \mathbf{R}^n$ is open and convex, and consider a function $f : S \to \mathbf{R}$. For points $x \notin S$, define $f(x) = +\infty$.

    (a) Prove $\partial f(x)$ is nonempty for all $x$ in $S$ if and only if $f$ is convex. (Hint: For points $u$ and $v$ in $S$ and real $\lambda$ in $[0, 1]$, use the subgradient inequality (3.1.4) at the points $\bar{x} = \lambda u + (1 - \lambda) v$ and $x = u, v$ to check the definition of convexity.)

    (b) Prove that if $I \subset \mathbf{R}$ is an open interval and $g : I \to \mathbf{R}$ is differentiable then $g$ is convex if and only if $g'$ is nondecreasing on $I$, and $g$ is strictly convex if and only if $g'$ is strictly increasing on $I$. Deduce that if $g$ is twice differentiable then $g$ is convex if and only if $g''$ is nonnegative on $I$, and $g$ is strictly convex if $g''$ is strictly positive on $I$.

    (c) Deduce that if $f$ is twice continuously differentiable on $S$ then $f$ is convex if and only if its Hessian matrix is positive semidefinite everywhere on $S$, and $f$ is strictly convex if its Hessian matrix is positive definite everywhere on $S$. (Hint: Apply part (b) to the function $g$ defined by $g(t) = f(x + td)$ for small real $t$, points $x$ in $S$, and directions $d$ in $\mathbf{E}$.)

(d) Find a strictly convex function $f : (-1, 1) \to \mathbf{R}$ with $f''(0) = 0$.

(e) Prove that a continuous function $h : \operatorname{cl} S \to \mathbf{R}$ is convex if and only if its restriction to $S$ is convex. What about strictly convex functions?

13. **(Local convexity)** Suppose the function $f : \mathbf{R}^n \to \mathbf{R}$ is twice continuously differentiable near 0 and $\nabla^2 f(0)$ is positive definite. Prove $f|_{\delta B}$ is convex for some real $\delta > 0$.

14. **(Examples of convex functions)** As we shall see in Section 4.2, most natural convex functions occur in pairs. The table in Section 3.3 lists many examples on $\mathbf{R}$. Use Exercise 12 to prove each function $f$ and $f^*$ in the table is convex.

15. **(Examples of convex functions)** Prove the following functions of $x \in \mathbf{R}$ are convex:

(a) $\log\left(\dfrac{\sinh ax}{\sinh x}\right)$ for $a \geq 1$.

(b) $\log\left(\dfrac{e^{ax} - 1}{e^x - 1}\right)$ for $a \geq 1$.

16. * **(Bregman distances [48])** For a function $\phi : \mathbf{E} \to (\infty, +\infty]$ that is strictly convex and differentiable on $\operatorname{int}(\operatorname{dom}\phi)$, define the *Bregman distance* $d_\phi : \operatorname{dom}\phi \times \operatorname{int}(\operatorname{dom}\phi) \to \mathbf{R}$ by

$$d_\phi(x, y) = \phi(x) - \phi(y) - \phi'(y)(x - y).$$

(a) Prove $d_\phi(x, y) \geq 0$, with equality if and only if $x = y$.

(b) Compute $d_\phi$ when $\phi(t) = t^2/2$ and when $\phi$ is the function $p$ defined in Exercise 27.

(c) Suppose $\phi$ is three times differentiable. Prove $d_\phi$ is convex if and only if $-1/\phi''$ is convex on $\operatorname{int}(\operatorname{dom}\phi)$.

(d) Extend the results above to the function

$$D_\phi : (\operatorname{dom}\phi)^n \times (\operatorname{int}(\operatorname{dom}\phi))^n \to \mathbf{R}$$

defined by $D_\phi(x, y) = \sum_i d_\phi(x_i, y_i)$.

17. * **(Convex functions on $\mathbf{R}^2$)** Prove the following functions of $x \in \mathbf{R}^2$ are convex:

(a)
$$\begin{cases} (x_1 - x_2)(\log x_1 - \log x_2) & \text{if } x \in \mathbf{R}^2_{++} \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

(Hint: See Exercise 16.)

(b)

$$\begin{cases} \dfrac{x_1^2}{x_2} & \text{if } x_2 > 0 \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

18. * Prove the function

$$f(x) = \begin{cases} -(x_1 x_2 \ldots x_n)^{1/n} & \text{if } x \in \mathbf{R}^n_+ \\ +\infty & \text{otherwise} \end{cases}$$

is convex.

19. **(Domain of subdifferential)** If the function $f : \mathbf{R}^2 \to (\infty, +\infty]$ is defined by

$$f(x_1, x_2) = \begin{cases} \max\{1 - \sqrt{x_1}, |x_2|\} & \text{if } x_1 \geq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

prove that $f$ is convex but that $\operatorname{dom} \partial f$ is not convex.

20. * **(Monotonicity of gradients)** Suppose that the set $S \subset \mathbf{R}^n$ is open and convex and that the function $f : S \to \mathbf{R}$ is differentiable. Prove $f$ is convex if and only if

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0 \ \text{ for all } x, y \in S,$$

and $f$ is strictly convex if and only if the above inequality holds strictly whenever $x \neq y$. (You may use Exercise 12.)

21. ** **(The log barrier)** Use Exercise 20 (Monotonicity of gradients), Exercise 10 in Section 2.1 and Exercise 8 in Section 1.2 to prove that the function $f : \mathbf{S}^n_{++} \to \mathbf{R}$ defined by $f(X) = -\log \det X$ is strictly convex. Deduce the uniqueness of the minimum volume ellipsoid in Section 2.3, Exercise 8, and the matrix completion in Section 2.1, Exercise 12.

22. Prove the function (2.2.5) is convex on $\mathbf{R}^n$ by calculating its Hessian.

23. * If the function $f : \mathbf{E} \to (\infty, +\infty]$ is essentially strictly convex, prove all distinct points $x$ and $y$ in $\mathbf{E}$ satisfy $\partial f(x) \cap \partial f(y) = \emptyset$. Deduce that $f$ has at most one minimizer.

24. **(Minimizers of essentially smooth functions)** Prove that any minimizer of an essentially smooth function $f$ must lie in core $(\operatorname{dom} f)$.

25. ** **(Convex matrix functions)** Consider a matrix $C$ in $\mathbf{S}^n_+$.

(a) For matrices $X$ in $\mathbf{S}^n_{++}$ and $D$ in $\mathbf{S}^n$, use a power series expansion to prove

$$\frac{d^2}{dt^2}\mathrm{tr}\,(C(X+tD)^{-1})\Big|_{t=0} \geq 0.$$

(b) Deduce $X \in \mathbf{S}^n_{++} \mapsto \mathrm{tr}\,(CX^{-1})$ is convex.

(c) Prove similarly the function $X \in \mathbf{S}^n \mapsto \mathrm{tr}\,(CX^2)$ and the function $X \in \mathbf{S}^n_+ \mapsto -\mathrm{tr}\,(CX^{1/2})$ are convex.

26. ** **(Log-convexity)** Given a convex set $C \subset \mathbf{E}$, we say that a function $f : C \to \mathbf{R}_{++}$ is *log-convex* if $\log f(\cdot)$ is convex.

(a) Prove any log-convex function is convex, using Section 1.1, Exercise 9 (Composing convex functions).

(b) If a polynomial $p : \mathbf{R} \to \mathbf{R}$ has all real roots, prove $1/p$ is log-convex on any interval on which $p$ is strictly positive.

(c) One version of *Hölder's inequality* states, for real $p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$ and functions $u, v : \mathbf{R}_+ \to \mathbf{R}$,

$$\int uv \leq \Big(\int |u|^p\Big)^{1/p}\Big(\int |v|^q\Big)^{1/q}$$

when the right hand side is well-defined. Use this to prove the *gamma function* $\Gamma : \mathbf{R} \to \mathbf{R}$ given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

is log-convex.

27. ** **(Maximum entropy [36])** Define a convex function $p : \mathbf{R} \to (-\infty, +\infty]$ by

$$p(u) = \begin{cases} u\log u - u & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ +\infty & \text{if } u < 0 \end{cases}$$

and a convex function $f : \mathbf{R}^n \to (-\infty, +\infty]$ by

$$f(x) = \sum_{i=1}^n p(x_i).$$

Suppose $\hat{x}$ lies in the interior of $\mathbf{R}^n_+$.

(a) Prove $f$ is strictly convex on $\mathbf{R}^n_+$ with compact level sets.

(b) Prove $f'(x; \hat{x} - x) = -\infty$ for any point $x$ on the boundary of $\mathbf{R}^n_+$.

(c) Suppose the map $G : \mathbf{R}^n \to \mathbf{R}^m$ is linear with $G\hat{x} = b$. Prove for any vector $c$ in $\mathbf{R}^n$ that the problem

$$\inf\{f(x) + \langle c, x\rangle \mid Gx = b,\ x \in \mathbf{R}^n\}$$

has a unique optimal solution $\bar{x}$, lying in $\mathbf{R}^n_{++}$.

(d) Use Corollary 2.1.3 (First order conditions for linear constraints) to prove that some vector $\lambda$ in $\mathbf{R}^m$ satisfies $\nabla f(\bar{x}) = G^*\lambda - c$, and deduce $\bar{x}_i = \exp(G^*\lambda - c)_i$.

28. ** (***DAD*** **problems [36])** Consider the following example of Exercise 27 (Maximum entropy). Suppose the $k \times k$ matrix $A$ has each entry $a_{ij}$ nonnegative. We say $A$ *has doubly stochastic pattern* if there is a doubly stochastic matrix with exactly the same zero entries as $A$. Define a set $Z = \{(i,j) | a_{ij} > 0\}$, and let $\mathbf{R}^Z$ denote the set of vectors with components indexed by $Z$ and $\mathbf{R}^Z_+$ denote those vectors in $\mathbf{R}^Z$ with all nonnegative components. Consider the problem

$$\begin{array}{lrcl} \inf & \sum_{(i,j)\in Z}(p(x_{ij}) - x_{ij}\log a_{ij}) & & \\ \text{subject to} & \sum_{i:(i,j)\in Z} x_{ij} & = & 1 \quad \text{for } j = 1,2,\ldots,k \\ & \sum_{j:(i,j)\in Z} x_{ij} & = & 1 \quad \text{for } i = 1,2,\ldots,k \\ & x & \in & \mathbf{R}^Z. \end{array}$$

(a) Suppose $A$ has doubly stochastic pattern. Prove there is a point $\hat{x}$ in the interior of $\mathbf{R}^Z_+$ which is feasible for the problem above. Deduce that the problem has a unique optimal solution $\bar{x}$, and, for some vectors $\lambda$ and $\mu$ in $\mathbf{R}^k$, $\bar{x}$ satisfies

$$\bar{x}_{ij} = a_{ij}\exp(\lambda_i + \mu_j) \quad \text{for } (i,j) \in Z.$$

(b) Deduce that $A$ has doubly stochastic pattern if and only if there are diagonal matrices $D_1$ and $D_2$ with strictly positive diagonal entries and $D_1 A D_2$ doubly stochastic.

29. ** **(Relativizing the Max formula)** If $f : \mathbf{E} \to (\infty, +\infty]$ is a convex function then for points $\bar{x}$ in $\mathrm{ri}\,(\mathrm{dom}\, f)$ and directions $d$ in $\mathbf{E}$, prove the subdifferential $\partial f(\bar{x})$ is nonempty and

$$f'(\bar{x}; d) = \sup\{\langle \phi, d\rangle \mid \phi \in \partial f(\bar{x})\},$$

with attainment when finite.

## 3.2 The Value Function

In this section we describe another approach to the Karush–Kuhn–Tucker conditions (2.3.8) in the convex case using the existence of subgradients we established in the previous section. We consider an *(inequality-constrained) convex program*

$$\inf\{f(x) \mid g_i(x) \leq 0 \ \text{ for } i = 1, 2, \ldots, m, \ x \in \mathbf{E}\}, \tag{3.2.1}$$

where the functions $f, g_1, g_2, \ldots, g_m : \mathbf{E} \to (\infty, +\infty]$ are convex and satisfy $\emptyset \neq \operatorname{dom} f \subset \cap_i \operatorname{dom} g_i$. Denoting the vector with components $g_i(x)$ by $g(x)$, the function $L : \mathbf{E} \times \mathbf{R}^m_+ \to (\infty, +\infty]$ defined by

$$L(x; \lambda) = f(x) + \lambda^T g(x), \tag{3.2.2}$$

is called the *Lagrangian.* A *feasible solution* is a point $x$ in $\operatorname{dom} f$ satisfying the constraints.

We should emphasize that the term "Lagrange multiplier" has different meanings in different contexts. In the present context we say a vector $\bar{\lambda} \in \mathbf{R}^m_+$ is a *Lagrange multiplier vector* for a feasible solution $\bar{x}$ if $\bar{x}$ minimizes the function $L(\,\cdot\,; \bar{\lambda})$ over $\mathbf{E}$ and $\bar{\lambda}$ satisfies the complementary slackness conditions: $\bar{\lambda}_i = 0$ whenever $g_i(\bar{x}) < 0$.

We can often use the following principle to solve simple optimization problems.

**Proposition 3.2.3 (Lagrangian sufficient conditions)** *If the point $\bar{x}$ is feasible for the convex program (3.2.1) and there is a Lagrange multiplier vector, then $\bar{x}$ is optimal.*

The proof is immediate, and in fact does not rely on convexity.

The Karush–Kuhn–Tucker conditions (2.3.8) are a converse to the above result when the functions $f, g_1, g_2, \ldots, g_m$ are convex and differentiable. We next follow a very different, and surprising, route to this result, circumventing differentiability. We *perturb* the problem (3.2.1), and analyze the resulting *(optimal) value function* $v : \mathbf{R}^m \to [-\infty, +\infty]$, defined by the equation

$$v(b) = \inf\{f(x) \mid g(x) \leq b\}. \tag{3.2.4}$$

We show that Lagrange multiplier vectors $\bar{\lambda}$ correspond to subgradients of $v$ (Exercise 9).

Our old definition of convexity for functions does not naturally extend to functions $h : \mathbf{E} \to [-\infty, +\infty]$ (due to the possible occurrence of $\infty - \infty$). To generalize the definition we introduce the idea of the *epigraph* of $h$:

$$\operatorname{epi}(h) = \{(y, r) \in \mathbf{E} \times \mathbf{R} \mid h(y) \leq r\}, \tag{3.2.5}$$

and we say $h$ is a *convex function* if $\text{epi}\,(h)$ is a convex set. An exercise shows in this case that the domain

$$\text{dom}\,(h) = \{y \mid h(y) < +\infty\}$$

is convex, and further that the value function $v$ defined by equation (3.2.4) is convex. We say $h$ is *proper* if $\text{dom}\, h$ is nonempty and $h$ never takes the value $-\infty$: if we wish to demonstrate the existence of subgradients for $v$ using the results in the previous section then we need to exclude $-\infty$.

**Lemma 3.2.6** *If the function $h : \mathbf{E} \to [-\infty, +\infty]$ is convex and some point $\hat{y}$ in* $\text{core}\,(\text{dom}\, h)$ *satisfies $h(\hat{y}) > -\infty$, then $h$ never takes the value $-\infty$.*

**Proof.** Suppose some point $y$ in $\mathbf{E}$ satisfies $h(y) = -\infty$. Since $\hat{y}$ lies in $\text{core}\,(\text{dom}\, h)$, there is a real $t > 0$ with $\hat{y} + t(\hat{y} - y)$ in $\text{dom}\,(h)$, and hence a real $r$ with $(\hat{y} + t(\hat{y} - y), r)$ in $\text{epi}\,(h)$. Now for any real $s$, $(y, s)$ lies in $\text{epi}\,(h)$, so we know

$$\left(\hat{y}, \frac{r + ts}{1 + t}\right) = \frac{1}{1+t}(\hat{y} + t(\hat{y} - y), r) + \frac{t}{1+t}(y, s) \in \text{epi}\,(h),$$

Letting $s \to -\infty$ gives a contradiction. □

In Section 2.3 we saw that the Karush–Kuhn–Tucker conditions needed a regularity condition. In this approach we will apply a different condition, known as the *Slater constraint qualification*, for the problem (3.2.1):

$$\text{There exists } \hat{x} \text{ in } \text{dom}\,(f) \text{ with } g_i(\hat{x}) < 0 \text{ for } i = 1, 2, \ldots, m. \qquad (3.2.7)$$

**Theorem 3.2.8 (Lagrangian necessary conditions)** *Suppose that the point $\bar{x}$ in* $\text{dom}\,(f)$ *is optimal for the convex program (3.2.1) and that the Slater condition (3.2.7) holds. Then there is a Lagrange multiplier vector for $\bar{x}$.*

**Proof.** Defining the value function $v$ by equation (3.2.4), certainly $v(0) > -\infty$, and the Slater condition shows $0 \in \text{core}\,(\text{dom}\, v)$, so in particular Lemma 3.2.6 shows that $v$ never takes the value $-\infty$. (An incidental consequence, from Section 4.1, is the continuity of $v$ at 0.) We now deduce the existence of a subgradient $-\bar{\lambda}$ of $v$ at 0, by the Max formula (3.1.8).

Any vector $b$ in $\mathbf{R}^m_+$ obviously satisfies $g(\bar{x}) \le b$, whence the inequality

$$f(\bar{x}) = v(0) \le v(b) + \bar{\lambda}^T b \le f(\bar{x}) + \bar{\lambda}^T b.$$

Hence, $\bar{\lambda}$ lies in $\mathbf{R}^m_+$. Furthermore, any point $x$ in $\text{dom}\, f$ clearly satisfies

$$f(x) \ge v(g(x)) \ge v(0) - \bar{\lambda}^T g(x) = f(\bar{x}) - \bar{\lambda}^T g(x).$$

The case $x = \bar{x}$, using the inequalities $\bar{\lambda} \geq 0$ and $g(\bar{x}) \leq 0$, shows $\bar{\lambda}^T g(\bar{x}) = 0$, which yields the complementary slackness conditions. Finally, all points $x$ in $\operatorname{dom} f$ must satisfy $f(x) + \bar{\lambda}^T g(x) \geq f(\bar{x}) = f(\bar{x}) + \bar{\lambda}^T g(\bar{x})$. □

In particular, if in the above result $\bar{x}$ lies in $\operatorname{core}(\operatorname{dom} f)$ and the functions $f, g_1, g_2, \ldots, g_m$ are differentiable at $\bar{x}$ then

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0,$$

so we recapture the Karush–Kuhn–Tucker conditions (2.3.8). In fact, in this case it is easy to see that the Slater condition is equivalent to the Mangasarian–Fromovitz constraint qualification (Assumption 2.3.7).

## Exercises and Commentary

Versions of the Lagrangian necessary conditions above appeared in [182] and [110]; for a survey see [158]. The approach here is analogous to [81]. The Slater condition first appeared in [173].

1. Prove the Lagrangian sufficient conditions (3.2.3).
2. Use the Lagrangian sufficient conditions (3.2.3) to solve the following problems.

   (a)
   $$\begin{array}{lrcl} \inf & x_1^2 + x_2^2 - 6x_1 - 2x_2 + 10 & & \\ \text{subject to} & 2x_1 + x_2 - 2 & \leq & 0 \\ & x_2 - 1 & \leq & 0 \\ & x & \in & \mathbf{R}^2. \end{array}$$

   (b)
   $$\begin{array}{lrcl} \inf & -2x_1 + x_2 & & \\ \text{subject to} & x_1^2 - x_2 & \leq & 0 \\ & x_2 - 4 & \leq & 0 \\ & x & \in & \mathbf{R}^2. \end{array}$$

   (c)
   $$\begin{array}{lrcl} \inf & x_1 + \dfrac{2}{x_2} & & \\ \text{subject to} & -x_2 + \dfrac{1}{2} & \leq & 0 \\ & -x_1 + x_2^2 & \leq & 0 \\ & x & \in & \{(x_1, x_2) \mid x_2 > 0\}. \end{array}$$

3. Given strictly positive reals $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_n$ and $b$, use the Lagrangian sufficient conditions to solve the problem

$$\inf \left\{ \sum_{i=1}^{n} \frac{c_i}{x_i} \;\middle|\; \sum_{i=1}^{n} a_i x_i \leq b,\ x \in \mathbf{R}^n_{++} \right\}.$$

4. For a matrix $A$ in $\mathbf{S}^n_{++}$ and a real $b > 0$, use the Lagrangian sufficient conditions to solve the problem

$$\inf\{-\log\det X \mid \operatorname{tr} AX \le b,\ X \in \mathbf{S}^n_{++}\}.$$

You may use the fact that the objective function is convex with derivative $-X^{-1}$ (see Section 3.1, Exercise 21 (The log barrier)).

5. * **(Mixed constraints)** Consider the convex program (3.2.1) with some additional linear constraints $\langle a^j, x\rangle = d_j$ for vectors $a^j$ in $\mathbf{E}$ and reals $d_j$. By rewriting each equality as two inequalities (or otherwise), prove a version of the Lagrangian sufficient conditions for this problem.

6. **(Extended convex functions)**

   (a) Give an example of a convex function that takes the values 0 and $-\infty$.

   (b) Prove the value function $v$ defined by equation (3.2.4) is convex.

   (c) Prove that a function $h : \mathbf{E} \to [-\infty, +\infty]$ is convex if and only if it satisfies the inequality

   $$h(\lambda x + (1-\lambda)y) \le \lambda h(x) + (1-\lambda)h(y)$$

   for any points $x$ and $y$ in $\operatorname{dom} h$ (or $\mathbf{E}$ if $h$ is proper) and any real $\lambda$ in $(0, 1)$.

   (d) Prove that if the function $h : \mathbf{E} \to [-\infty, +\infty]$ is convex then $\operatorname{dom}(h)$ is convex.

7. **(Nonexistence of multiplier)** For the function $f : \mathbf{R} \to (\infty, +\infty]$ defined by $f(x) = -\sqrt{x}$ for $x$ in $\mathbf{R}_+$ and $+\infty$ otherwise, show there is no Lagrange multiplier at the optimal solution of $\inf\{f(x) \,|\, x \le 0\}$.

8. **(Duffin's duality gap)** Consider the following problem (for real $b$):

$$\inf\{e^{x_2} \mid \|x\| - x_1 \le b,\ x \in \mathbf{R}^2\}. \qquad (3.2.9)$$

   (a) Sketch the feasible region for $b > 0$ and for $b = 0$.

   (b) Plot the value function $v$.

   (c) Show that when $b = 0$ there is no Lagrange multiplier for any feasible solution. Explain why the Lagrangian necessary conditions (3.2.8) do not apply.

   (d) Repeat the above exercises with the objective function $e^{x_2}$ replaced by $x_2$.

9. ** **(Karush–Kuhn–Tucker vectors [167])** Consider the convex program (3.2.1). Suppose the value function $v$ given by equation (3.2.4) is finite at 0. We say the vector $\bar{\lambda}$ in $\mathbf{R}^m_+$ is a *Karush–Kuhn–Tucker vector* if it satisfies $v(0) = \inf\{L(x; \bar{\lambda}) \mid x \in \mathbf{E}\}$.

   (a) Prove that the set of Karush–Kuhn–Tucker vectors is $-\partial v(0)$.

   (b) Suppose the point $\bar{x}$ is an optimal solution of problem (3.2.1). Prove that the set of Karush–Kuhn–Tucker vectors coincides with the set of Lagrange multiplier vectors for $\bar{x}$.

   (c) Prove the Slater condition ensures the existence of a Karush–Kuhn–Tucker vector.

   (d) Suppose $\bar{\lambda}$ is a Karush–Kuhn–Tucker vector. Prove a feasible point $\bar{x}$ is optimal for problem (3.2.1) if and only if $\bar{\lambda}$ is a Lagrange multiplier vector for $\bar{x}$.

10. Prove the equivalence of the Slater and Mangasarian–Fromovitz conditions asserted at the end of the section.

11. **(Normals to epigraphs)** For a function $f : \mathbf{E} \to (\infty, +\infty]$ and a point $\bar{x}$ in core (dom $f$), calculate the normal cone $N_{\mathrm{epi}\, f}(\bar{x}, f(\bar{x}))$.

12. * **(Normals to level sets)** Suppose the function $f : \mathbf{E} \to (\infty, +\infty]$ is convex. If the point $\bar{x}$ lies in core (dom $f$) and is not a minimizer for $f$, prove that the normal cone at $\bar{x}$ to the level set

$$C = \{x \in \mathbf{E} \mid f(x) \le f(\bar{x})\}$$

   is given by $N_C(\bar{x}) = \mathbf{R}_+ \partial f(\bar{x})$. Is the assumption $\bar{x} \in$ core (dom $f$) and $f(\bar{x}) > \inf f$ necessary?

13. * **(Subdifferential of max-function)** Consider convex functions

$$g_1, g_2, \ldots, g_m : \mathbf{E} \to (\infty, +\infty],$$

   and define a function $g(x) = \max_i g_i(x)$ for all points $x$ in $\mathbf{E}$. For a fixed point $\bar{x}$ in $\mathbf{E}$, define the index set $I = \{i \mid g_i(\bar{x}) = g(\bar{x})\}$ and let

$$C = \bigcup \Big\{ \partial\Big( \sum_{i \in I} \lambda_i g_i \Big)(\bar{x}) \;\Big|\; \lambda \in \mathbf{R}^I_+, \ \sum_{i \in I} \lambda_i = 1 \Big\}.$$

   (a) Prove $C \subset \partial g(\bar{x})$.

   (b) Suppose $0 \in \partial g(\bar{x})$. By considering the convex program

$$\inf_{t \in \mathbf{R},\ x \in \mathbf{E}} \{t \mid g_i(x) - t \le 0 \ \text{ for } i = 1, 2, \ldots, m\},$$

   prove $0 \in C$.

(c) Deduce $\partial g(\bar{x}) = C$.

14. ** **(Minimum volume ellipsoid)** Denote the standard basis of $\mathbf{R}^n$ by $\{e^1, e^2, \ldots, e^n\}$ and consider the minimum volume ellipsoid problem (see Section 2.3, Exercise 8)

$$\begin{array}{lrcl} \inf & -\log\det X & & \\ \text{subject to} & \|Xe^i\|^2 - 1 & \leq & 0 \quad \text{for } i = 1, 2, \ldots, n \\ & X & \in & \mathbf{S}^n_{++}. \end{array}$$

Use the Lagrangian sufficient conditions (3.2.3) to prove $X = I$ is the unique optimal solution. (Hint: Use Section 3.1, Exercise 21 (The log barrier).) Deduce the following special case of *Hadamard's inequality*: Any matrix $(x^1 \; x^2 \; \ldots \; x^n)$ in $\mathbf{S}^n_{++}$ satisfies

$$\det(x^1 \; x^2 \; \ldots \; x^n) \leq \|x^1\|\|x^2\| \ldots \|x^n\|.$$

## 3.3 The Fenchel Conjugate

In the next few sections we sketch a little of the elegant and concise theory of Fenchel conjugation, and we use it to gain a deeper understanding of the Lagrangian necessary conditions for convex programs (3.2.8). The *Fenchel conjugate* of a function $h : \mathbf{E} \to [-\infty, +\infty]$ is the function $h^* : \mathbf{E} \to [-\infty, +\infty]$ defined by

$$h^*(\phi) = \sup_{x \in \mathbf{E}} \{\langle \phi, x \rangle - h(x)\}.$$

The function $h^*$ is convex and if the domain of $h$ is nonempty then $h^*$ never takes the value $-\infty$. Clearly the conjugacy operation is *order-reversing*: for functions $f, g : \mathbf{E} \to [-\infty, +\infty]$, the inequality $f \geq g$ implies $f^* \leq g^*$.

Conjugate functions are ubiquitous in optimization. For example, we have already seen the conjugate of the exponential, defined by

$$\exp^*(t) = \begin{cases} t \log t - t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ +\infty & \text{if } t < 0 \end{cases}$$

(see Section 3.1, Exercise 27). A rather more subtle example is the function $g : \mathbf{E} \to (\infty, +\infty]$ defined, for points $a^0, a^1, \ldots, a^m$ in $\mathbf{E}$, by

$$g(z) = \inf_{x \in \mathbf{R}^{m+1}} \Big\{ \sum_i \exp^*(x_i) \,\Big|\, \sum_i x_i = 1, \ \sum_i x_i a^i = z \Big\}. \tag{3.3.1}$$

The conjugate is the function we used in Section 2.2 to prove various theorems of the alternative:

$$g^*(y) = 1 + \log \Big( \sum_i \exp \langle a^i, y \rangle \Big) \tag{3.3.2}$$

(see Exercise 7).

As we shall see later (Section 4.2), many important convex functions $h$ equal their *biconjugates* $h^{**}$. Such functions thus occur as natural pairs, $h$ and $h^*$. Table 3.1 shows some elegant examples on $\mathbf{R}$, and Table 3.2 describes some simple transformations of these examples.

The following result summarizes the properties of two particularly important convex functions.

**Proposition 3.3.3 (Log barriers)** *The functions* $\mathrm{lb} : \mathbf{R}^n \to (\infty, +\infty]$ *and* $\mathrm{ld} : \mathbf{S}^n \to (\infty, +\infty]$ *defined by*

$$\mathrm{lb}(x) = \begin{cases} -\sum_{i=1}^n \log x_i & \textit{if } x \in \mathbf{R}^n_{++} \\ +\infty & \textit{otherwise} \end{cases}$$

| $f(x) = g^*(x)$ | $\operatorname{dom} f$ | $g(y) = f^*(y)$ | $\operatorname{dom} g$ |
|---|---|---|---|
| $0$ | $\mathbf{R}$ | $0$ | $\{0\}$ |
| $0$ | $\mathbf{R}_+$ | $0$ | $-\mathbf{R}_+$ |
| $0$ | $[-1,1]$ | $\|y\|$ | $\mathbf{R}$ |
| $0$ | $[0,1]$ | $y^+$ | $\mathbf{R}$ |
| $\|x\|^p/p, \quad p>1$ | $\mathbf{R}$ | $\|y\|^q/q \quad (\frac{1}{p}+\frac{1}{q}=1)$ | $\mathbf{R}$ |
| $\|x\|^p/p, \quad p>1$ | $\mathbf{R}_+$ | $\|y^+\|^q/q \quad (\frac{1}{p}+\frac{1}{q}=1)$ | $\mathbf{R}$ |
| $-x^p/p, \quad 0<p<1$ | $\mathbf{R}_+$ | $-(-y)^q/q \quad (\frac{1}{p}+\frac{1}{q}=1)$ | $-\mathbf{R}_{++}$ |
| $\sqrt{1+x^2}$ | $\mathbf{R}$ | $-\sqrt{1-y^2}$ | $[-1,1]$ |
| $-\log x$ | $\mathbf{R}_{++}$ | $-1-\log(-y)$ | $-\mathbf{R}_{++}$ |
| $\cosh x$ | $\mathbf{R}$ | $y\sinh^{-1}(y)-\sqrt{1+y^2}$ | $\mathbf{R}$ |
| $-\log(\cos x)$ | $(-\frac{\pi}{2},\frac{\pi}{2})$ | $y\tan^{-1}(y)-\frac{1}{2}\log(1+y^2)$ | $\mathbf{R}$ |
| $\log(\cosh x)$ | $\mathbf{R}$ | $y\tanh^{-1}(y)+\frac{1}{2}\log(1-y^2)$ | $(-1,1)$ |
| $e^x$ | $\mathbf{R}$ | $\begin{cases} y\log y - y & (y>0) \\ 0 & (y=0) \end{cases}$ | $\mathbf{R}_+$ |
| $\log(1+e^x)$ | $\mathbf{R}$ | $\begin{cases} y\log y+(1-y)\log(1-y) \\ \qquad (y\in(0,1)) \\ 0 \qquad (y=0,1) \end{cases}$ | $[0,1]$ |
| $-\log(1-e^x)$ | $\mathbf{R}$ | $\begin{cases} y\log y-(1+y)\log(1+y) \\ \qquad (y>0) \\ 0 \qquad (y=0) \end{cases}$ | $\mathbf{R}_+$ |

Table 3.1: Conjugate pairs of convex functions on **R**.

| $f = g^*$ | $g = f^*$ |
|---|---|
| $f(x)$ | $g(y)$ |
| $h(ax) \quad (a \neq 0)$ | $h^*(y/a)$ |
| $h(x+b)$ | $h^*(y) - by$ |
| $ah(x) \quad (a > 0)$ | $ah^*(y/a)$ |

Table 3.2: Transformed conjugates.

*and*

$$\mathrm{ld}\,(X) = \begin{cases} -\log\det X & \textit{if } X \in \mathbf{S}^n_{++} \\ +\infty & \textit{otherwise} \end{cases}$$

*are essentially smooth, and strictly convex on their domains. They satisfy the conjugacy relations*

$$\begin{aligned} \mathrm{lb}^*(x) &= \mathrm{lb}\,(-x) - n \quad \textit{for all } x \in \mathbf{R}^n, \textit{ and} \\ \mathrm{ld}^*(X) &= \mathrm{ld}\,(-X) - n \quad \textit{for all } X \in \mathbf{S}^n. \end{aligned}$$

*The perturbed functions* $\mathrm{lb} + \langle c, \cdot \rangle$ *and* $\mathrm{ld} + \langle C, \cdot \rangle$ *have compact level sets for any vector* $c \in \mathbf{R}^n_{++}$ *and matrix* $C \in \mathbf{S}^n_{++}$*, respectively.*

(See Section 3.1, Exercise 21 (The log barrier), and Section 1.2, Exercise 14 (Level sets of perturbed log barriers); the conjugacy formulas are simple calculations.) Notice the simple relationships $\mathrm{lb} = \mathrm{ld} \circ \mathrm{Diag}$ and $\mathrm{ld} = \mathrm{lb} \circ \lambda$ between these two functions.

The next elementary but important result relates conjugation with the subgradient. The proof is an exercise.

**Proposition 3.3.4 (Fenchel–Young inequality)** *Any points* $\phi$ *in* $\mathbf{E}$ *and* $x$ *in the domain of a function* $h : \mathbf{E} \to (\infty, +\infty]$ *satisfy the inequality*

$$h(x) + h^*(\phi) \geq \langle \phi, x \rangle.$$

*Equality holds if and only if* $\phi \in \partial h(x)$*.*

In Section 3.2 we analyzed the standard inequality-constrained convex program by studying its optimal value under perturbations. A similar approach works for another model for convex programming, particularly suited to problems with linear constraints. An interesting byproduct is a convex analogue of the chain rule for differentiable functions,

$$\nabla(f + g \circ A)(x) = \nabla f(x) + A^* \nabla g(Ax)$$

(for a linear map $A$). When $A$ is the identity map we obtain a *sum rule.*

In this section we fix a Euclidean space $\mathbf{Y}$. We denote the set of points where a function $g : \mathbf{Y} \to [-\infty, +\infty]$ is finite and continuous by $\operatorname{cont} g$.

**Theorem 3.3.5 (Fenchel duality and convex calculus)** *For given functions $f : \mathbf{E} \to (\infty, +\infty]$ and $g : \mathbf{Y} \to (\infty, +\infty]$ and a linear map $A : \mathbf{E} \to \mathbf{Y}$, let $p, d \in [-\infty, +\infty]$ be primal and dual values defined, respectively, by the* **Fenchel problems**

$$p = \inf_{x \in \mathbf{E}} \{f(x) + g(Ax)\} \tag{3.3.6}$$

$$d = \sup_{\phi \in \mathbf{Y}} \{-f^*(A^*\phi) - g^*(-\phi)\}. \tag{3.3.7}$$

*These values satisfy the* **weak duality** *inequality $p \geq d$. If, furthermore, $f$ and $g$ are convex and satisfy the condition*

$$0 \in \operatorname{core}(\operatorname{dom} g - A\operatorname{dom} f) \tag{3.3.8}$$

*or the stronger condition*

$$A\operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset \tag{3.3.9}$$

*then the values are equal ($p = d$), and the supremum in the dual problem (3.3.7) is attained if finite.*

*At any point $x$ in $\mathbf{E}$, the calculus rule*

$$\partial(f + g \circ A)(x) \supset \partial f(x) + A^* \partial g(Ax) \tag{3.3.10}$$

*holds, with equality if $f$ and $g$ are convex and either condition (3.3.8) or (3.3.9) holds.*

**Proof.** The weak duality inequality follows immediately from the Fenchel–Young inequality (3.3.4). To prove equality we define an optimal value function $h : \mathbf{Y} \to [-\infty, +\infty]$ by

$$h(u) = \inf_{x \in \mathbf{E}} \{f(x) + g(Ax + u)\}.$$

It is easy to check $h$ is convex and $\operatorname{dom} h = \operatorname{dom} g - A\operatorname{dom} f$. If $p$ is $-\infty$ there is nothing to prove, while if condition (3.3.8) holds and $p$ is finite then Lemma 3.2.6 and the Max formula (3.1.8) show there is a subgradient $-\phi \in \partial h(0)$. Hence we deduce, for all $u$ in $\mathbf{Y}$ and $x$ in $\mathbf{E}$, the inequalities

$$\begin{aligned} h(0) &\le h(u) + \langle \phi, u \rangle \\ &\le f(x) + g(Ax+u) + \langle \phi, u \rangle \\ &= \{f(x) - \langle A^*\phi, x \rangle\} + \{g(Ax+u) - \langle -\phi, Ax+u \rangle\}. \end{aligned}$$

Taking the infimum over all points $u$, and then over all points $x$, gives the inequalities

$$h(0) \le -f^*(A^*\phi) - g^*(-\phi) \le d \le p = h(0).$$

Thus $\phi$ attains the supremum in problem (3.3.7), and $p = d$. An easy exercise shows that condition (3.3.9) implies condition (3.3.8). The proof of the calculus rule in the second part of the theorem is a simple consequence of the first part (Exercise 9). □

The case of the Fenchel theorem above, when the function $g$ is simply the indicator function of a point, gives the following particularly elegant and useful corollary.

**Corollary 3.3.11 (Fenchel duality for linear constraints)** *Given any function $f : \mathbf{E} \to (\infty, +\infty]$, any linear map $A : \mathbf{E} \to \mathbf{Y}$, and any element $b$ of $\mathbf{Y}$, the weak duality inequality*

$$\inf_{x \in \mathbf{E}} \{f(x) \mid Ax = b\} \ge \sup_{\phi \in \mathbf{Y}} \{\langle b, \phi \rangle - f^*(A^*\phi)\}$$

*holds. If $f$ is convex and $b$ belongs to* $\mathrm{core}\,(A\mathrm{dom}\, f)$ *then equality holds, and the supremum is attained when finite.*

A pretty application of the Fenchel duality circle of ideas is the calculation of polar cones. The *(negative) polar cone* of the set $K \subset \mathbf{E}$ is the convex cone

$$K^- = \{\phi \in \mathbf{E} \mid \langle \phi, x \rangle \le 0 \ \text{ for all } x \in K\},$$

and the cone $K^{--}$ is called the *bipolar*. A particularly important example of the polar cone is the normal cone to a convex set $C \subset \mathbf{E}$ at a point $x$ in $C$, since $N_C(x) = (C - x)^-$.

We use the following two examples extensively; the proofs are simple exercises.

**Proposition 3.3.12 (Self-dual cones)**

$$(\mathbf{R}^n_+)^- = -\mathbf{R}^n_+ \quad \textit{and} \quad (\mathbf{S}^n_+)^- = -\mathbf{S}^n_+.$$

The next result shows how the calculus rules above can be used to derive geometric consequences.

**Corollary 3.3.13 (Krein–Rutman polar cone calculus)** *Any cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$ and linear map $A : \mathbf{E} \to \mathbf{Y}$ satisfy*

$$(K \cap A^{-1}H)^- \supset A^*H^- + K^-.$$

*Equality holds if $H$ and $K$ are convex and satisfy $H - AK = Y$ (or in particular $AK \cap \operatorname{int} H \neq \emptyset$).*

**Proof.** Rephrasing the definition of the polar cone shows that for any cone $K \subset \mathbf{E}$, the polar cone $K^-$ is just $\partial\delta_K(0)$. The result now follows by the Fenchel theorem above. □

The polarity operation arises naturally from Fenchel conjugation, since for any cone $K \subset \mathbf{E}$ we have $\delta_{K^-} = \delta_K^*$, whence $\delta_{K^{--}} = \delta_K^{**}$. The next result, which is an elementary application of the Basic separation theorem (2.1.6), leads naturally into the development of the next chapter by identifying $K^{--}$ as the closed convex cone generated by $K$.

**Theorem 3.3.14 (Bipolar cone)** *The bipolar cone of any nonempty set $K \subset \mathbf{E}$ is given by $K^{--} = \operatorname{cl}(\operatorname{conv}(\mathbf{R}_+K))$.*

For example, we deduce immediately that the normal cone $N_C(x)$ to a convex set $C$ at a point $x$ in $C$, and the *(convex) tangent cone* to $C$ at $x$ defined by $T_C(x) = \operatorname{cl}\mathbf{R}_+(C - x)$, are polars of each other.

Exercise 20 outlines how to use these two results about cones to characterize *pointed* cones (those closed convex cones $K$ satisfying $K \cap -K = \{0\}$).

**Theorem 3.3.15 (Pointed cones)** *If $K \subset \mathbf{E}$ is a closed convex cone, then $K$ is pointed if and only if there is an element $y$ of $\mathbf{E}$ for which the set*

$$C = \{x \in K \mid \langle x, y \rangle = 1\}$$

*is compact and generates $K$ (that is, $K = \mathbf{R}_+C$).*

## Exercises and Commentary

The conjugation operation has been closely associated with the names of Legendre, Moreau, and Rockafellar, as well as Fenchel; see [167, 70]. Fenchel's original work is [76]. A good reference for properties of convex cones is [151]; see also [20]. The log barriers of Proposition 3.3.3 play a key role in interior point methods for linear and semidefinite programming—see, for example, [148]. The self-duality of the positive semidefinite cone is due to Fejer [99]. Hahn–Banach extension (Exercise 13(e)) is a key technique in functional analysis; see, for example, [98]. Exercise 21 (Order subgradients) is aimed at multicriteria optimization; a good reference is [176]. Our approach may be found, for example, in [20]. The last three functions $g$ in Table 3.1 are respectively known as the *Boltzmann–Shannon*, *Fermi–Dirac*, and *Bose–Einstein* entropies.

1. For each of the functions $f$ in Table 3.1, check the calculation of $f^*$ and check $f = f^{**}$.

2. **(Quadratics)** For all matrices $A$ in $\mathbf{S}^n_{++}$, prove the function $x \in \mathbf{R}^n \mapsto x^T Ax/2$ is convex and calculate its conjugate. Use the order-reversing property of the conjugacy operation to prove

$$A \succeq B \;\Leftrightarrow\; B^{-1} \succeq A^{-1} \quad \text{for } A \text{ and } B \text{ in } \mathbf{S}^n_{++}.$$

3. Verify the conjugates of the log barriers lb and ld claimed in Proposition 3.3.3.

4. * **(Self-conjugacy)** Consider functions $f : \mathbf{E} \to (\infty, +\infty]$.
   (a) Prove $f = f^*$ if and only if $f(x) = \|x\|^2/2$ for all points $x$ in $\mathbf{E}$.
   (b) Find two distinct functions $f$ satisfying $f(-x) = f^*(x)$ for all points $x$ in $\mathbf{E}$.

5. * **(Support functions)** The conjugate of the indicator function of a nonempty set $C \subset \mathbf{E}$, namely $\delta^*_C : \mathbf{E} \to (\infty, +\infty]$, is called the *support function* of $C$. Calculate it for the following sets:
   (a) the halfspace $\{x \mid \langle a, x\rangle \le b\}$ for $0 \ne a \in \mathbf{E}$ and $b \in \mathbf{R}$
   (b) the unit ball $B$
   (c) $\{x \in \mathbf{R}^n_+ \mid \|x\| \le 1\}$
   (d) the *polytope* $\mathrm{conv}\,\{a^1, a^2, \ldots, a^m\}$ for given points $a^1, a^2, \ldots, a^m$ in $\mathbf{E}$
   (e) a cone $K$
   (f) the epigraph of a convex function $f : \mathbf{E} \to (\infty, +\infty]$
   (g) the subdifferential $\partial f(\bar{x})$, where the function $f : \mathbf{E} \to (\infty, +\infty]$ is convex and the point $\bar{x}$ lies in $\mathrm{core}\,(\mathrm{dom}\, f)$
   (h) $\{Y \in \mathbf{S}^n_+ \mid \mathrm{tr}\, Y = 1\}$

6. Calculate the conjugate and biconjugate of the function

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1^2}{2x_2} + x_2 \log x_2 - x_2 & \text{if } x_2 > 0 \\ 0 & \text{if } x_1 = x_2 = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

7. ** **(Maximum entropy example)**

   (a) Prove the function $g$ defined by (3.3.1) is convex.

   (b) For any point $y$ in $\mathbf{R}^{m+1}$, prove

$$g^*(y) = \sup_{x \in \mathbf{R}^{m+1}} \Big\{ \sum_i (x_i \langle a^i, y \rangle - \exp^*(x_i)) \,\Big|\, \sum_i x_i = 1 \Big\}.$$

   (c) Apply Exercise 27 in Section 3.1 to deduce the conjugacy formula (3.3.2).

   (d) Compute the conjugate of the function of $x \in \mathbf{R}^{m+1}$,

$$\begin{cases} \sum_i \exp^*(x_i) & \text{if } \sum_i x_i = 1 \\ +\infty & \text{otherwise.} \end{cases}$$

8. Prove the Fenchel–Young inequality.

9. * **(Fenchel duality and convex calculus)** Fill in the details for the proof of Theorem 3.3.5 as follows.

   (a) Prove the weak duality inequality.

   (b) Prove the inclusion (3.3.10).

   Now assume $f$ and $g$ are convex.

   (c) Prove the function $h$ defined in the proof is convex with domain $\operatorname{dom} g - A \operatorname{dom} f$.

   (d) Prove the implication (3.3.9) $\Rightarrow$ (3.3.8).

   Finally, assume in addition that condition (3.3.8) holds.

   (e) Suppose $\phi \in \partial(f + g \circ A)(\bar{x})$. Use the first part of the theorem and the fact that $\bar{x}$ is an optimal solution of the problem

$$\inf_{x \in \mathbf{E}} \{(f(x) - \langle \phi, x \rangle) + g(Ax)\}$$

   to deduce equality in part (b).

   (f) Prove points $\bar{x} \in \mathbf{E}$ and $\bar{\phi} \in \mathbf{Y}$ are optimal for problems (3.3.6) and (3.3.7), respectively, if and only if they satisfy the conditions $A^*\bar{\phi} \in \partial f(\bar{x})$ and $-\bar{\phi} \in \partial g(A\bar{x})$.

10. **(Normals to an intersection)** If the point $x$ lies in two convex subsets $C$ and $D$ of $\mathbf{E}$ satisfying $0 \in \operatorname{core}(C - D)$ (or in particular $C \cap \operatorname{int} D \neq \emptyset$), use Section 3.1, Exercise 4 (Subgradients and normal cones) to prove

$$N_{C \cap D}(x) = N_C(x) + N_D(x).$$

11. * **(Failure of convex calculus)**

 (a) Find convex functions $f, g : \mathbf{R} \to (\infty, +\infty]$ with

$$\partial f(0) + \partial g(0) \neq \partial(f + g)(0).$$

 (Hint: Section 3.1, Exercise 5.)

 (b) Find a convex function $g : \mathbf{R}^2 \to (\infty, +\infty]$ and a linear map $A : \mathbf{R} \to \mathbf{R}^2$ with $A^* \partial g(0) \neq \partial(g \circ A)(0)$.

12. * **(Infimal convolution)** If the functions $f, g : \mathbf{E} \to (-\infty, +\infty]$ are convex, we define the *infimal convolution* $f \odot g : \mathbf{E} \to [-\infty, +\infty]$ by

$$(f \odot g)(y) = \inf_x \{f(x) + g(y - x)\}.$$

 (a) Prove $f \odot g$ is convex. (On the other hand, if $g$ is concave prove so is $f \odot g$.)

 (b) Prove $(f \odot g)^* = f^* + g^*$.

 (c) If $\operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset$, prove $(f + g)^* = f^* \odot g^*$.

 (d) Given a nonempty set $C \subset \mathbf{E}$, define the *distance function* by

$$d_C(x) = \inf_{y \in C} \|x - y\|.$$

 (i) Prove $d_C^2$ is a difference of convex functions, by observing

$$(d_C(x))^2 = \frac{\|x\|^2}{2} - \left(\frac{\|\cdot\|^2}{2} + \delta_C\right)^*(x).$$

 Now suppose $C$ is convex.

 (ii) Prove $d_C$ is convex and $d_C^* = \delta_B + \delta_C^*$.

 (iii) For $x$ in $C$ prove $\partial d_C(x) = B \cap N_C(x)$.

 (iv) If $C$ is closed and $x \notin C$, prove

$$\nabla d_C(x) = d_C(x)^{-1}(x - P_C(x)),$$

 where $P_C(x)$ is the nearest point to $x$ in $C$.

 (v) If $C$ is closed, prove

$$\nabla \frac{d_C^2}{2}(x) = x - P_C(x)$$

 for all points $x$.

(e) Define the *Lambert W-function* $W : \mathbf{R}_+ \to \mathbf{R}_+$ as the inverse of $y \in \mathbf{R}_+ \mapsto ye^y$. Prove the conjugate of the function

$$x \in \mathbf{R} \mapsto \exp^*(x) + \frac{x^2}{2}$$

is the function

$$y \in \mathbf{R} \mapsto W(e^y) + \frac{(W(e^y))^2}{2}.$$

13. * **(Applications of Fenchel duality)**

(a) **(Sandwich theorem)** Let the functions $f : \mathbf{E} \to (\infty, +\infty]$ and $g : \mathbf{Y} \to (\infty, +\infty]$ be convex and the map $A : \mathbf{E} \to \mathbf{Y}$ be linear. Suppose $f \geq -g \circ A$ and $0 \in \operatorname{core}(\operatorname{dom} g - A\operatorname{dom} f)$ (or $A\operatorname{dom} f \cap \operatorname{cont} g \neq \emptyset$). Prove there is an affine function $\alpha : \mathbf{E} \to \mathbf{R}$ satisfying $f \geq \alpha \geq -g \circ A$.

(b) Interpret the Sandwich theorem geometrically in the case when $A$ is the identity.

(c) **(Pshenichnii–Rockafellar conditions [159])** If the convex set $C$ in $\mathbf{E}$ satisfies the condition $C \cap \operatorname{cont} f \neq \emptyset$ (or the condition $\operatorname{int} C \cap \operatorname{dom} f \neq \emptyset$), and if $f$ is bounded below on $C$, use part (a) to prove there is an affine function $\alpha \leq f$ with $\inf_C f = \inf_C \alpha$. Deduce that a point $\bar{x}$ minimizes $f$ on $C$ if and only if it satisfies $0 \in \partial f(\bar{x}) + N_C(\bar{x})$.

(d) Apply part (c) to the following two cases:

(i) $C$ a single point $\{x^0\} \subset \mathbf{E}$

(ii) $C$ a polyhedron $\{x \mid Ax \leq b\}$, where $b \in \mathbf{R}^n = \mathbf{Y}$

(e) **(Hahn–Banach extension)** If the function $f : \mathbf{E} \to \mathbf{R}$ is everywhere finite and sublinear, and for some linear subspace $L$ of $\mathbf{E}$ the function $h : L \to \mathbf{R}$ is linear and *dominated* by $f$ (in other words $f \geq h$ on $L$), prove there is a linear function $\alpha : \mathbf{E} \to \mathbf{R}$, dominated by $f$, which agrees with $h$ on $L$.

14. Fill in the details of the proof of the Krein–Rutman calculus (3.3.13).

15. * **(Bipolar theorem)** For any nonempty set $K \subset \mathbf{E}$, prove the set $\operatorname{cl}(\operatorname{conv}(\mathbf{R}_+ K))$ is the smallest closed convex cone containing $K$. Deduce Theorem 3.3.14 (Bipolar cones).

16. * **(Sums of closed cones)**

(a) Prove that any cones $H, K \subset \mathbf{E}$ satisfy $(H + K)^- = H^- \cap K^-$.

(b) Deduce that if $H$ and $K$ are closed convex cones then they satisfy $(H \cap K)^- = \mathrm{cl}\,(H^- + K^-)$, and prove that the closure can be omitted under the condition $K \cap \mathrm{int}\, H \neq \emptyset$.

In $\mathbf{R}^3$, define sets

$$\begin{aligned} H &= \{x \mid x_1^2 + x_2^2 \leq x_3^2,\ x_3 \leq 0\} \text{ and} \\ K &= \{x \mid x_2 = -x_3\}. \end{aligned}$$

(c) Prove $H$ and $K$ are closed convex cones.

(d) Calculate the polar cones $H^-$, $K^-$, and $(H \cap K)^-$.

(e) Prove $(1,1,1) \in (H \cap K)^- \setminus (H^- + K^-)$, and deduce that the sum of two closed convex cones is not necessarily closed.

17. * **(Subdifferential of a max-function)** With the notation of Section 3.2, Exercise 13, suppose

$$\mathrm{dom}\, g_j \cap \bigcap_{i \in I \setminus \{j\}} \mathrm{cont}\, g_i \neq \emptyset$$

for some index $j$ in $I$. Prove

$$\partial(\max_i g_i)(\bar{x}) = \mathrm{conv} \bigcup_{i \in I} \partial g_i(\bar{x}).$$

18. * **(Order convexity)** Given a Euclidean space $\mathbf{Y}$ and a closed convex cone $S \subset \mathbf{Y}$, we write $u \leq_S v$ for points $u$ and $v$ in $\mathbf{Y}$ if $v - u$ lies in $S$.

(a) Identify the partial order $\leq_S$ in the following cases:

(i) $S = \{0\}$

(ii) $S = \mathbf{Y}$

(iii) $\mathbf{Y} = \mathbf{R}^n$ and $S = \mathbf{R}^n_+$

Given a convex set $C \subset \mathbf{E}$, we say a function $F : C \to \mathbf{Y}$ is *$S$-convex* if it satisfies

$$F(\lambda x + \mu z) \leq_S \lambda F(x) + \mu F(z)$$

for all points $x$ and $z$ in $\mathbf{E}$ and nonnegative reals $\lambda$ and $\mu$ satisfying $\lambda + \mu = 1$. If, furthermore, $C$ is a cone and this inequality holds for *all* $\lambda$ and $\mu$ in $\mathbf{R}_+$ then we say $F$ is *$S$-sublinear.*

(b) Identify $S$-convexity in the cases listed in part (a).

(c) Prove $F$ is $S$-convex if and only if the function $\langle \phi, F(\cdot) \rangle$ is convex for all elements $\phi$ of $-S^-$.

(d) Prove the following functions are $\mathbf{S}^n_+$-convex:

(i) $X \in \mathbf{S}^n \mapsto X^2$

(ii) $X \in \mathbf{S}^n_{++} \mapsto X^{-1}$

(iii) $X \in \mathbf{S}^n_+ \mapsto -X^{1/2}$

Hint: Use Exercise 25 in Section 3.1.

(e) Prove the function $X \in \mathbf{S}^2 \mapsto X^4$ is not $\mathbf{S}^2_+$-convex. Hint: Consider the matrices

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}.$$

19. **(Order convexity of inversion)** For any matrix $A$ in $\mathbf{S}^n_{++}$, define a function $q_A : \mathbf{R}^n \to \mathbf{R}$ by $q_A(x) = x^T Ax/2$.

(a) Prove $q_A^* = q_{A^{-1}}$.

(b) For any other matrix $B$ in $\mathbf{S}^n_{++}$, prove $2(q_A \odot q_B) \leq q_{(A+B)/2}$. (See Exercise 12.)

(c) Deduce $(A^{-1} + B^{-1})/2 \succeq ((A+B)/2)^{-1}$.

20. ** **(Pointed cones and bases)** Consider a closed convex cone $K$ in $\mathbf{E}$. A *base* for $K$ is a convex set $C$ with $0 \notin \mathrm{cl}\, C$ and $K = \mathbf{R}_+ C$. Using Exercise 16, prove the following properties are equivalent by showing the implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a).$$

(a) $K$ is pointed.

(b) $\mathrm{cl}\,(K^- - K^-) = \mathbf{E}$.

(c) $K^- - K^- = \mathbf{E}$.

(d) $K^-$ has nonempty interior. (Here you may use the fact that $K^-$ has nonempty relative interior—see Section 1.1, Exercise 13.)

(e) There exists a vector $y$ in $\mathbf{E}$ and real $\epsilon > 0$ with $\langle y, x \rangle \geq \epsilon \|x\|$ for all points $x$ in $K$.

(f) $K$ has a bounded base.

21. ** **(Order-subgradients)** This exercise uses the terminology of Exercise 18, and we assume the cone $S \subset \mathbf{Y}$ is pointed: $S \cap -S = \{0\}$. An element $y$ of $\mathbf{Y}$ is the $S$-*infimum* of a set $D \subset \mathbf{Y}$ (written $y = \inf_S D$) if the conditions

(i) $D \subset y + S$ and

(ii) $D \subset z + S$ for some $z$ in $Y$ implies $y \in z + S$

both hold.

(a) Verify that this notion corresponds to the usual infimum when $\mathbf{Y} = \mathbf{R}$ and $S = \mathbf{R}_+$.

(b) Prove every subset of $\mathbf{Y}$ has at most one $S$-infimum.

(c) Prove *decreasing* sequences in $S$ converge:

$$x_0 \geq_S x_1 \geq_S x_2 \ldots \geq_S 0$$

implies $\lim_n x_n$ exists and equals $\inf_S(x_n)$. (Hint: Prove first that $S \cap (x_0 - S)$ is compact using Section 1.1, Exercise 6 (Recession cones).)

An *$S$-subgradient* of $F$ at a point $x$ in $C$ is a linear map $T : \mathbf{E} \to \mathbf{Y}$ satisfying

$$T(z - x) \leq_S F(z) - F(x) \text{ for all } z \text{ in } C.$$

The set of $S$-subgradients is denoted $\partial_S F(x)$. Suppose now $x \in \operatorname{core} C$. Generalize the arguments of Section 3.1 in the following steps.

(d) For any direction $h$ in $\mathbf{E}$, prove

$$\nabla_S F(x; h) = \inf_S \{t^{-1}(F(x + th) - F(x)) \mid t > 0,\ x + th \in C\}$$

exists and, as a function of $h$, is $S$-sublinear.

(e) For any $S$-subgradient $T \in \partial_S F(x)$ and direction $h \in \mathbf{E}$, prove $Th \leq_S \nabla_S F(x; h)$.

(f) Given $h$ in $\mathbf{E}$, prove there exists $T$ in $\partial_S F(x)$ satisfying $Th = \nabla_S F(x; h)$. Deduce the max formula

$$\nabla_S F(x; h) = \max\{Th \mid T \in \partial_S F(x)\}$$

and, in particular, that $\partial_S F(x)$ is nonempty. (You should interpret the "max" in the formula.)

(g) The function $F$ is *Gâteaux differentiable* at $x$ (with derivative the linear map $\nabla F(x) : \mathbf{E} \to \mathbf{Y}$) if

$$\lim_{t \to 0} t^{-1}(F(x + th) - F(x)) = (\nabla F(x))h$$

holds for all $h$ in $\mathbf{E}$. Prove this is the case if and only if $\partial_S F(x)$ is a singleton.

Now fix an element $\phi$ of $-\operatorname{int}(S^-)$.

(h) Prove $\langle \phi, F(\cdot) \rangle'(x; h) = \langle \phi, \nabla_S F(x; h) \rangle$.

(i) Prove $F$ is Gâteaux differentiable at $x$ if and only if $\langle \phi, F(\cdot) \rangle$ is likewise.

22. ** **(Linearly constrained examples)** Prove Corollary 3.3.11 (Fenchel duality for linear constraints). Deduce duality theorems for the following problems.

(a) **Separable problems**

$$\inf \Big\{ \sum_{i=1}^{n} p(x_i) \;\Big|\; Ax = b \Big\},$$

where the map $A : \mathbf{R}^n \to \mathbf{R}^m$ is linear, $b \in \mathbf{R}^m$, and the function $p : \mathbf{R} \to (\infty, +\infty]$ is convex, defined as follows:

(i) **(Nearest points in polyhedrons)** $p(t) = t^2/2$ with domain $\mathbf{R}_+$.

(ii) **(Analytic center)** $p(t) = -\log t$ with domain $\mathbf{R}_{++}$.

(iii) **(Maximum entropy)** $p = \exp^*$.

What happens if the objective function is replaced by $\sum_i p_i(x_i)$?

(b) The **BFGS update** problem in Section 2.1, Exercise 13.

(c) The **DAD problem** in Section 3.1, Exercise 28.

(d) Example (3.3.1).

23. * **(Linear inequalities)** What does Corollary 3.3.11 (Fenchel duality for linear constraints) become if we replace the constraint $Ax = b$ by $Ax \in b + K$ where $K \subset \mathbf{Y}$ is a convex cone? Write down the dual problem for Section 3.2, Exercise 2, part (a), solve it, and verify the duality theorem.

24. **(Symmetric Fenchel duality)** For functions $f, g : \mathbf{E} \to [-\infty, +\infty]$, define the *concave conjugate* $g_* : \mathbf{E} \to [-\infty, +\infty]$ by

$$g_*(\phi) = \inf_{x \in \mathbf{E}} \{\langle \phi, x \rangle - g(x)\}.$$

Prove

$$\inf(f - g) \geq \sup(g_* - f^*),$$

with equality if $f$ is convex, $g$ is concave, and

$$0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom}(-g)).$$

25. ** **(Divergence bounds [135])**

(a) Prove the function

$$t \in \mathbf{R} \mapsto 2(2+t)(\exp^* t + 1) - 3(t-1)^2$$

is convex and is minimized when $t = 1$.

(b) For $v$ in $\mathbf{R}_{++}$ and $u$ in $\mathbf{R}_+$, deduce the inequality

$$3(u-v)^2 \le 2(u+2v)\Big(u \log\left(\frac{u}{v}\right) - u + v\Big).$$

Now suppose the vector $p$ in $\mathbf{R}^n_{++}$ satisfies $\sum_1^n p_i = 1$.

(c) If the vector $q \in \mathbf{R}^n_{++}$ satisfies $\sum_1^n q_i = 1$, use the Cauchy–Schwarz inequality to prove the inequality

$$\Big(\sum_1^n |p_i - q_i|\Big)^2 \le 3 \sum_1^n \frac{(p_i - q_i)^2}{p_i + 2q_i},$$

and deduce the inequality

$$\sum_1^n p_i \log\left(\frac{p_i}{q_i}\right) \ge \frac{1}{2}\Big(\sum_1^n |p_i - q_i|\Big)^2.$$

(d) Hence show the inequality

$$\log n + \sum_1^n p_i \log p_i \ge \frac{1}{2}\Big(\sum_1^n \Big|p_i - \frac{1}{n}\Big|\Big)^2.$$

(e) Use convexity to prove the inequality

$$\sum_1^n p_i \log p_i \le \log \sum_1^n p_i^2.$$

(f) Deduce the bound

$$\log n + \sum_1^n p_i \log p_i \le \frac{\max p_i}{\min p_i} - 1.$$

# Chapter 4

# Convex Analysis

## 4.1 Continuity of Convex Functions

We have already seen that linear functions are always continuous. More generally, a remarkable feature of convex functions on $\mathbf{E}$ is that they must be continuous on the interior of their domains. Part of the surprise is that an *algebraic/geometric* assumption (convexity) leads to a *topological* conclusion (continuity). It is this powerful fact that guarantees the usefulness of regularity conditions like $A\,\mathrm{dom}\,f \cap \mathrm{cont}\,g \neq \emptyset$ (3.3.9), which we studied in the previous section.

Clearly an arbitrary function $f$ is bounded above on some neighbourhood of any point in $\mathrm{cont}\,f$. For convex functions the converse is also true, and in a rather strong sense, needing the following definition. For a real $L \geq 0$, we say that a function $f : \mathbf{E} \to (\infty, +\infty]$ is *Lipschitz (with constant $L$)* on a subset $C$ of $\mathrm{dom}\,f$ if $|f(x) - f(y)| \leq L\|x - y\|$ for any points $x$ and $y$ in $C$. If $f$ is Lipschitz on a neighbourhood of a point $z$ then we say that $f$ is *locally Lipschitz around $z$*. If $\mathbf{Y}$ is another Euclidean space we make analogous definitions for functions $F : \mathbf{E} \to \mathbf{Y}$, with $\|F(x) - F(y)\|$ replacing $|f(x) - f(y)|$.

**Theorem 4.1.1 (Local boundedness)** *Let $f : \mathbf{E} \to (\infty, +\infty]$ be a convex function. Then $f$ is locally Lipschitz around a point $z$ in its domain if and only if it is bounded above on a neighbourhood of $z$.*

**Proof.** One direction is clear, so let us without loss of generality take $z = 0$, $f(0) = 0$, and suppose $f \leq 1$ on $2B$; we shall deduce $f$ is Lipschitz on $B$.

Notice first the bound $f \geq -1$ on $2B$, since convexity implies $f(-x) \geq -f(x)$ on $2B$. Now for any distinct points $x$ and $y$ in $B$, define $\alpha = \|y - x\|$ and fix a point $w = y + \alpha^{-1}(y - x)$, which lies in $2B$. By convexity we obtain

$$f(y) - f(x) \le \frac{1}{1+\alpha} f(x) + \frac{\alpha}{1+\alpha} f(w) - f(x) \le \frac{2\alpha}{1+\alpha} \le 2\|y - x\|,$$

and the result now follows, since $x$ and $y$ may be interchanged. □

This result makes it easy to identify the set of points at which a convex function on $\mathbf{E}$ is continuous. First we prove a key lemma.

**Lemma 4.1.2** *Let $\Delta$ be the* **simplex** $\{x \in \mathbf{R}^n_+ \mid \sum x_i \le 1\}$. *If the function $g : \Delta \to \mathbf{R}$ is convex then it is continuous on* $\mathrm{int}\,\Delta$.

**Proof.** By the above result, we just need to show $g$ is bounded above on $\Delta$. But any point $x$ in $\Delta$ satisfies

$$\begin{aligned} g(x) &= g\Big(\sum_1^n x_i e^i + (1 - \textstyle\sum x_i)0\Big) \le \sum_1^n x_i g(e^i) + (1 - \textstyle\sum x_i) g(0) \\ &\le \max\{g(e^1), g(e^2), \ldots, g(e^n), g(0)\} \end{aligned}$$

(where $\{e^1, e^2, \ldots, e^n\}$ is the standard basis in $\mathbf{R}^n$). □

**Theorem 4.1.3 (Convexity and continuity)** *Let $f : \mathbf{E} \to (\infty, +\infty]$ be a convex function. Then $f$ is continuous (in fact locally Lipschitz) on the interior of its domain.*

**Proof.** We lose no generality if we restrict ourselves to the case $\mathbf{E} = \mathbf{R}^n$. For any point $x$ in $\mathrm{int}\,(\mathrm{dom}\, f)$ we can choose a neighbourhood of $x$ in $\mathrm{dom}\, f$ that is a scaled down, translated copy of the simplex (since the simplex is bounded with nonempty interior). The proof of the preceding lemma now shows $f$ is bounded above on a neighbourhood of $x$, and the result follows by Theorem 4.1.1 (Local boundedness). □

Since it is easy to see that if the convex function $f$ is locally Lipschitz around a point $\bar{x}$ in $\mathrm{int}\,(\mathrm{dom}\, f)$ with constant $L$ then $\partial f(\bar{x}) \subset LB$, we can also conclude that $\partial f(\bar{x})$ is a nonempty compact convex set. Furthermore, this result allows us to conclude quickly that "all norms on $\mathbf{E}$ are equivalent" (see Exercise 2).

We have seen that for a convex function $f$, the two sets $\mathrm{cont}\, f$ and $\mathrm{int}\,(\mathrm{dom}\, f)$ are identical. By contrast, our algebraic approach to the existence of subgradients involved $\mathrm{core}\,(\mathrm{dom}\, f)$. It transpires that this is the same set. To see this we introduce the idea of the *gauge function* $\gamma_C : \mathbf{E} \to (\infty, +\infty]$ associated with a nonempty set $C$ in $\mathbf{E}$:

$$\gamma_C(x) = \inf\{\lambda \in \mathbf{R}_+ \mid x \in \lambda C\}.$$

It is easy to check $\gamma_C$ is sublinear (and in particular convex) when $C$ is convex. Notice $\gamma_B = \|\cdot\|$.

**Theorem 4.1.4 (Core and interior)** *The core and the interior of any convex set in* $\mathbf{E}$ *are identical and convex.*

**Proof.** Any convex set $C \subset \mathbf{E}$ clearly satisfies $\operatorname{int} C \subset \operatorname{core} C$. If we suppose, without loss of generality, $0 \in \operatorname{core} C$, then $\gamma_C$ is everywhere finite, and hence continuous by the previous result. We claim

$$\operatorname{int} C = \{x \mid \gamma_C(x) < 1\}.$$

To see this, observe that the right hand side is contained in $C$, and is open by continuity, and hence is contained in $\operatorname{int} C$. The reverse inclusion is easy, and we deduce $\operatorname{int} C$ is convex. Finally, since $\gamma_C(0) = 0$, we see $0 \in \operatorname{int} C$, which completes the proof. □

The conjugate of the gauge function $\gamma_C$ is the indicator function of a set $C^\circ \subset \mathbf{E}$ defined by

$$C^\circ = \{\phi \in \mathbf{E} \mid \langle \phi, x \rangle \le 1 \text{ for all } x \in C\}.$$

We call $C^\circ$ the *polar set* for $C$. Clearly it is a closed convex set containing 0, and when $C$ is a cone it coincides with the polar cone $C^-$. The following result therefore generalizes the Bipolar cone theorem (3.3.14).

**Theorem 4.1.5 (Bipolar set)** *The bipolar set of any subset* $C$ *of* $\mathbf{E}$ *is given by*

$$C^{\circ\circ} = \operatorname{cl}(\operatorname{conv}(C \cup \{0\})).$$

The ideas of polarity and separating hyperplanes are intimately related. The separation-based proof of the above result (Exercise 5) is a good example, as is the next theorem, whose proof is outlined in Exercise 6.

**Theorem 4.1.6 (Supporting hyperplane)** *Suppose that the convex set* $C \subset \mathbf{E}$ *has nonempty interior and that the point* $\bar{x}$ *lies on the boundary of* $C$*. Then there is a* **supporting hyperplane** *to* $C$ *at* $\bar{x}$*: there is a nonzero element* $a$ *of* $\mathbf{E}$ *satisfying* $\langle a, x \rangle \ge \langle a, \bar{x} \rangle$ *for all points* $x$ *in* $C$*.*

(The set $\{x \in \mathbf{E} \mid \langle a, x - \bar{x} \rangle = 0\}$ is the supporting hyperplane.)

To end this section we use this result to prove a remarkable theorem of Minkowski describing an extremal representation of finite-dimensional compact convex sets. An *extreme point* of a convex set $C \subset \mathbf{E}$ is a point $x$ in $C$ whose complement $C \setminus \{x\}$ is convex. We denote the set of extreme points by $\operatorname{ext} C$. We start with another exercise.

**Lemma 4.1.7** *Given a supporting hyperplane* $H$ *of a convex set* $C \subset \mathbf{E}$*, any extreme point of* $C \cap H$ *is also an extreme point of* $C$*.*

Our proof of Minkowski's theorem depends on two facts: first, any convex set that spans $\mathbf{E}$ and contains the origin has nonempty interior (see Section 1.1, Exercise 13(b)); second, we can define the *dimension* of a set $C \subset \mathbf{E}$ (written $\dim C$) as the dimension of span $(C - x)$ for *any* point $x$ in $C$ (see Section 1.1, Exercise 12 (Affine sets)).

**Theorem 4.1.8 (Minkowski)** *Any compact convex set $C \subset \mathbf{E}$ is the convex hull of its extreme points.*

**Proof.** Our proof is by induction on $\dim C$; clearly the result holds when $\dim C = 0$. Assume the result holds for all sets of dimension less than $\dim C$. We will deduce it for the set $C$.

By translating $C$ and redefining $\mathbf{E}$, we can assume $0 \in C$ and $\operatorname{span} C = \mathbf{E}$. Thus $C$ has nonempty interior.

Given any point $x$ in $\operatorname{bd} C$, the Supporting hyperplane theorem (4.1.6) shows $C$ has a supporting hyperplane $H$ at $x$. By the induction hypothesis applied to the set $C \cap H$ we deduce, using Lemma 4.1.7,

$$x \in \operatorname{conv}(\operatorname{ext}(C \cap H)) \subset \operatorname{conv}(\operatorname{ext} C).$$

Thus we have proved $\operatorname{bd} C \subset \operatorname{conv}(\operatorname{ext} C)$, so $\operatorname{conv}(\operatorname{bd} C) \subset \operatorname{conv}(\operatorname{ext} C)$. But since $C$ is compact it is easy to see $\operatorname{conv}(\operatorname{bd} C) = C$, and the result now follows. □

## Exercises and Commentary

An easy introduction to convex analysis in finite dimensions is [181]. The approach we adopt here (and in the exercises) extends easily to infinite dimensions; see [98, 131, 153]. The Lipschitz condition was introduced in [129]. Minkowski's theorem first appeared in [141, 142]. The Open mapping theorem (Exercise 9) is another fundamental tool of functional analysis [98]. For recent references on Pareto minimization (Exercise 12), see [44].

1. * **(Points of continuity)** Suppose the function $f : \mathbf{E} \to (\infty, +\infty]$ is convex.

   (a) Use the Local boundedness theorem (4.1.1) to prove that $f$ is continuous and finite at $x$ if and only if it minorizes a function $g : \mathbf{E} \to (\infty, +\infty]$ which is continuous and finite at $x$.

   (b) Suppose $f$ is continuous at some point $y$ in $\operatorname{dom} f$. Use part (a) to prove directly that $f$ is continuous at any point $z$ in $\operatorname{core}(\operatorname{dom} f)$. (Hint: Pick a point $u$ in $\operatorname{dom} f$ such that $z = \delta y + (1 - \delta)u$ for some real $\delta \in (0, 1)$; now observe that the function

   $$x \in \mathbf{E} \mapsto \delta^{-1}(f(\delta x + (1 - \delta)u) - (1 - \delta)f(u))$$

minorizes $f$.)

(c) Prove that $f$ is continuous at a point $x$ in $\operatorname{dom} f$ if and only if

$$(x, f(x)+\epsilon) \in \operatorname{int}(\operatorname{epi} f)$$

for some (all) real $\epsilon > 0$.

(d) Assuming $0 \in \operatorname{cont} f$, prove $f^*$ has bounded level sets. Deduce that the function $X \in \mathbf{S}^n \mapsto \langle C, X\rangle + \operatorname{ld}(X)$ has compact level sets for any matrix $C$ in $\mathbf{S}^n_{++}$.

(e) Assuming $x \in \operatorname{cont} f$, prove $\partial f(x)$ is a nonempty compact convex set.

2. **(Equivalent norms)** A *norm* is a sublinear function $|||\cdot||| : \mathbf{E} \to \mathbf{R}_+$ that satisfies $|||x||| = |||-x||| > 0$ for all nonzero points $x$ in $\mathbf{E}$. By considering the function $|||\cdot|||$ on the standard unit ball $B$, prove any norm $|||\cdot|||$ is *equivalent* to the Euclidean norm $\|\cdot\|$: that is, there are constants $K \geq k > 0$ with $k\|x\| \leq |||x||| \leq K\|x\|$ for all $x$.

3. **(Examples of polars)** Calculate the polars of the following sets:

(a) $\operatorname{conv}\big(B \cup \{(1,1),(-1,-1)\}\big) \subset \mathbf{R}^2$.

(b) $\left\{(x,y) \in \mathbf{R}^2 \;\middle|\; y \geq b + \dfrac{x^2}{2}\right\}$ $(b \in \mathbf{R})$.

4. **(Polar sets and cones)** Suppose the set $C \subset \mathbf{E}$ is closed, convex, and contains 0. Prove the convex cones in $\mathbf{E} \times \mathbf{R}$

$$\operatorname{cl} \mathbf{R}_+(C \times \{1\}) \;\text{ and }\; \operatorname{cl} \mathbf{R}_+(C^\circ \times \{-1\})$$

are mutually polar.

5. $^*$ **(Polar sets)** Suppose $C$ is a nonempty subset of $\mathbf{E}$.

(a) Prove $\gamma_C^* = \delta_{C^\circ}$.

(b) Prove $C^\circ$ is a closed convex set containing 0.

(c) Prove $C \subset C^{\circ\circ}$.

(d) If $C$ is a cone, prove $C^\circ = C^-$.

(e) For a subset $D$ of $\mathbf{E}$, prove $C \subset D$ implies $D^\circ \subset C^\circ$.

(f) Prove $C$ is bounded if and only if $0 \in \operatorname{int} C^\circ$.

(g) For any closed halfspace $H \subset \mathbf{E}$ containing 0, prove $H^{\circ\circ} = H$.

(h) Prove Theorem 4.1.5 (Bipolar set).

6. * **(Polar sets and strict separation)** Fix a nonempty set $C$ in $\mathbf{E}$.

   (a) For points $x$ in $\operatorname{int} C$ and $\phi$ in $C^\circ$, prove $\langle \phi, x \rangle < 1$.

   (b) Assume further that $C$ is a convex set. Prove $\gamma_C$ is sublinear.

   (c) Assume in addition $0 \in \operatorname{core} C$. Deduce

   $$\operatorname{cl} C = \{x \mid \gamma_C(x) \le 1\}.$$

   (d) Finally, suppose in addition that $D \subset \mathbf{E}$ is a convex set disjoint from the interior of $C$. By considering the Fenchel problem $\inf\{\delta_D + \gamma_C\}$, prove there is a closed halfspace containing $D$ but disjoint from the interior of $C$.

7. * **(Polar calculus [23])** Suppose $C$ and $D$ are subsets of $\mathbf{E}$.

   (a) Prove $(C \cup D)^\circ = C^\circ \cap D^\circ$.

   (b) If $C$ and $D$ are convex, prove

   $$\operatorname{conv}(C \cup D) = \bigcup_{\lambda \in [0,1]} (\lambda C + (1-\lambda) D).$$

   (c) If $C$ is a convex cone and the convex set $D$ contains 0, prove

   $$C + D \subset \operatorname{cl} \operatorname{conv}(C \cup D).$$

   Now suppose the closed convex sets $K$ and $H$ of $\mathbf{E}$ both contain 0.

   (d) Prove $(K \cap H)^\circ = \operatorname{cl} \operatorname{conv}(K^\circ \cup H^\circ)$.

   (e) If furthermore $K$ is a cone, prove $(K \cap H)^\circ = \operatorname{cl}(K^\circ + H^\circ)$.

8. ** **(Polar calculus [23])** Suppose $P$ is a cone in $\mathbf{E}$ and $C$ is a nonempty subset of a Euclidean space $\mathbf{Y}$.

   (a) Prove $(P \times C)^\circ = P^\circ \times C^\circ$.

   (b) If furthermore $C$ is compact and convex (possibly not containing 0), and $K$ is a cone in $\mathbf{E} \times \mathbf{Y}$, prove

   $$(K \cap (P \times C))^\circ = (K \cap (P \times C^{\circ\circ}))^\circ.$$

   (c) If furthermore $K$ and $P$ are closed and convex, use Exercise 7 to prove

   $$(K \cap (P \times C))^\circ = \operatorname{cl}(K^\circ + (P^\circ \times C^\circ)).$$

   (d) Find a counterexample to part (c) when $C$ is unbounded.

9. * **(Open mapping theorem)** Suppose the linear map $A : \mathbf{E} \to \mathbf{Y}$ is surjective.

   (a) Prove any set $C \subset \mathbf{E}$ satisfies $A\,\mathrm{core}\, C \subset \mathrm{core}\, AC$.

   (b) Deduce $A$ is an *open map*: that is, the image of any open set is open.

   (c) Prove another condition ensuring condition (3.3.8) in the Fenchel theorem is that there is a point $\hat{x}$ in $\mathrm{int}\,(\mathrm{dom}\, f)$ with $A\hat{x}$ in $\mathrm{dom}\, g$ and $A$ is surjective. Prove similarly that a sufficient condition for Fenchel duality with linear constraints (Corollary 3.3.11) to hold is $A$ surjective and $b \in A(\mathrm{int}\,(\mathrm{dom}\, f))$.

   (d) Deduce that any cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$, and any surjective linear map $A : \mathbf{E} \to \mathbf{Y}$ satisfy $(K \cap A^{-1}H)^- = A^*H^- + K^-$, providing $H \cap A(\mathrm{int}\, K) \neq \emptyset$.

10. * **(Conical absorption)**

    (a) If the set $A \subset \mathbf{E}$ is convex, the set $C \subset \mathbf{E}$ is bounded, and $\mathbf{R}_+ A = \mathbf{E}$, prove there exists a real $\delta > 0$ such that $\delta C \subset A$.

    Now define two sets in $\mathbf{S}^2_+$ by

$$A = \left\{ \begin{bmatrix} y & x \\ x & z \end{bmatrix} \in \mathbf{S}^2_+ \;\middle|\; |x| \leq y^{2/3} \right\}, \quad \text{and}$$
$$C = \{X \in \mathbf{S}^2_+ \mid \mathrm{tr}\, X \leq 1\}.$$

    (b) Prove that both $A$ and $C$ are closed, convex, and contain 0, and that $C$ is bounded.

    (c) Prove $\mathbf{R}_+ A = \mathbf{S}^2_+ = \mathbf{R}_+ C$.

    (d) Prove there is no real $\delta > 0$ such that $\delta C \subset A$.

11. **(Hölder's inequality)** This question develops an alternative approach to the theory of the $p$-norm $\|\cdot\|_p$ defined in Section 2.3, Exercise 6.

    (a) Prove $p^{-1}\|x\|_p^p$ is a convex function, and deduce the set

$$B_p = \{x \mid \|x\|_p \leq 1\}$$

    is convex.

    (b) Prove the gauge function $\gamma_{B_p}(\cdot)$ is exactly $\|\cdot\|_p$, and deduce $\|\cdot\|_p$ is convex.

    (c) Use the Fenchel–Young inequality (3.3.4) to prove that any vectors $x$ and $\phi$ in $\mathbf{R}^n$ satisfy the inequality

$$p^{-1}\|x\|_p^p + q^{-1}\|\phi\|_q^q \geq \langle \phi, x \rangle.$$

(d) Assuming $\|u\|_p = \|v\|_q = 1$, deduce $\langle u, v\rangle \le 1$, and hence prove that any vectors $x$ and $\phi$ in $\mathbf{R}^n$ satisfy the inequality

$$\langle \phi, x\rangle \le \|\phi\|_q \|x\|_p.$$

(e) Calculate $B_p^\circ$.

12. * **(Pareto minimization)** We use the notation of Section 3.3, Exercise 18 (Order convexity), and we assume the cone $S$ is pointed and has nonempty interior. Given a set $D \subset \mathbf{Y}$, we say a point $y$ in $D$ is a *Pareto minimum of $D$ (with respect to $S$)* if

$$(y - D) \cap S = \{0\},$$

and a *weak minimum* if

$$(y - D) \cap \operatorname{int} S = \emptyset.$$

(a) Prove $y$ is a Pareto (respectively weak) minimum of $D$ if and only if it is a Pareto (respectively weak) minimum of $D + S$.

(b) The map $X \in \mathbf{S}^n_+ \mapsto X^{1/2}$ is $\mathbf{S}^n_+$-order-preserving (Section 1.2, Exercise 5). Use this fact to prove, for any matrix $Z$ in $\mathbf{S}^n_+$, the unique Pareto minimum of the set

$$\{X \in \mathbf{S}^n \mid X^2 \succeq Z^2\}$$

with respect to $\mathbf{S}^n_+$ is $Z$.

For a convex set $C \subset \mathbf{E}$ and an $S$-convex function $F : C \to \mathbf{Y}$, we say a point $\bar{x}$ in $C$ is a *Pareto* (respectively, *weak*) *minimum* of the *vector optimization problem*

$$\inf\{F(x) \mid x \in C\} \tag{4.1.9}$$

if $F(\bar{x})$ is a Pareto (respectively weak) minimum of $F(C)$.

(c) Prove $F(C) + S$ is convex.

(d) **(Scalarization)** Suppose $\bar{x}$ is a weak minimum of the problem (4.1.9). By separating $(F(\bar{x}) - F(C) - S)$ and $\operatorname{int} S$ (using Exercise 6), prove there is a nonzero element $\phi$ of $-S^-$ such that $\bar{x}$ solves the *scalarized* convex optimization problem

$$\inf\{\langle \phi, F(x)\rangle \mid x \in C\}.$$

Conversely, show any solution of this problem is a weak minimum of (4.1.9).

13. **(Existence of extreme points)** Prove any nonempty compact convex set $C \subset \mathbf{E}$ has an extreme point, without using Minkowski's theorem, by considering the furthest point in $C$ from the origin.

14. Prove Lemma 4.1.7.

15. For any compact convex set $C \subset \mathbf{E}$, prove $C = \mathrm{conv}\,(\mathrm{bd}\, C)$.

16. * **(A converse of Minkowski's theorem)** Suppose $D$ is a subset of a compact convex set $C \subset \mathbf{E}$ satisfying $\mathrm{cl}\,(\mathrm{conv}\, D) = C$. Prove $\mathrm{ext}\, C \subset \mathrm{cl}\, D$.

17. * **(Extreme points)** Consider a compact convex set $C \subset \mathbf{E}$.
    (a) If $\dim \mathbf{E} \leq 2$, prove the set $\mathrm{ext}\, C$ is closed.
    (b) If $\mathbf{E}$ is $\mathbf{R}^3$ and $C$ is the convex hull of the set
    $$\{(x, y, 0) \mid x^2 + y^2 = 1\} \cup \{(1, 0, 1), (1, 0, -1)\},$$
    prove $\mathrm{ext}\, C$ is not closed.

18. * **(Exposed points)** A point $x$ in a convex set $C \subset \mathbf{E}$ is called *exposed* if there is an element $\phi$ of $\mathbf{E}$ such that $\langle \phi, x \rangle > \langle \phi, z \rangle$ for all points $z \neq x$ in $C$.
    (a) Prove any exposed point is an extreme point.
    (b) Find a set in $\mathbf{R}^2$ with an extreme point which is not exposed.

19. ** **(Tangency conditions)** Let $\mathbf{Y}$ be a Euclidean space. Fix a convex set $C$ in $\mathbf{E}$ and a point $x$ in $C$.
    (a) Show $x \in \mathrm{core}\, C$ if and only if $T_C(x) = \mathbf{E}$. (You may use Exercise 20(a).)
    (b) For a linear map $A : \mathbf{E} \to \mathbf{Y}$, prove $AT_C(x) \subset T_{AC}(Ax)$.
    (c) For another convex set $D$ in $\mathbf{Y}$ and a point $y$ in $D$, prove
    $$\begin{aligned} N_{C \times D}(x, y) &= N_C(x) \times N_D(y) \ \text{ and} \\ T_{C \times D}(x, y) &= T_C(x) \times T_D(y). \end{aligned}$$
    (d) Suppose the point $x$ also lies in the convex set $G \subset \mathbf{E}$. Prove $T_C(x) - T_G(x) \subset T_{C-G}(0)$, and deduce
    $$0 \in \mathrm{core}\,(C - G) \iff T_C(x) - T_G(x) = \mathbf{E}.$$
    (e) Show that the condition (3.3.8) in the Fenchel theorem can be replaced by the condition
    $$T_{\mathrm{dom}\, g}(Ax) - AT_{\mathrm{dom}\, f}(x) = \mathbf{Y}$$
    for an arbitrary point $x$ in $\mathrm{dom}\, f \cap A^{-1}\mathrm{dom}\, g$.

20. ** **(Properties of the relative interior)** (We use Exercise 9 (Open mapping theorem), as well as Section 1.1, Exercise 13.)

   (a) Let $D$ be a nonempty convex set in $\mathbf{E}$. Prove $D$ is a linear subspace if and only if $\mathrm{cl}\, D$ is a linear subspace. (Hint: $\mathrm{ri}\, D \neq \emptyset$.)

   (b) For a point $x$ in a convex set $C \subset \mathbf{E}$, prove the following properties are equivalent:

      (i) $x \in \mathrm{ri}\, C$.
      (ii) The tangent cone $\mathrm{cl}\, \mathbf{R}_+(C - x)$ is a linear subspace.
      (iii) The normal cone $N_C(x)$ is a linear subspace.
      (iv) $y \in N_C(x) \Rightarrow -y \in N_C(x)$.

   (c) For a convex set $C \subset \mathbf{E}$ and a linear map $A : \mathbf{E} \to \mathbf{Y}$, prove $A \mathrm{ri}\, C \supset \mathrm{ri}\, AC$, and deduce

$$A \mathrm{ri}\, C = \mathrm{ri}\, AC.$$

   (d) Suppose $U$ and $V$ are convex sets in $\mathbf{E}$. Deduce

$$\mathrm{ri}\,(U - V) = \mathrm{ri}\, U - \mathrm{ri}\, V.$$

   (e) Apply Section 3.1, Exercise 29 (Relativizing the Max formula) to conclude that the condition (3.3.8) in the Fenchel theorem (3.3.5) can be replaced by

$$\mathrm{ri}\,(\mathrm{dom}\, g) \cap A \mathrm{ri}\,(\mathrm{dom}\, f) \neq \emptyset.$$

   (f) Suppose the function $f : \mathbf{E} \to (\infty, +\infty]$ is bounded below on the convex set $C \subset \mathbf{E}$, and $\mathrm{ri}\, C \cap \mathrm{ri}\,(\mathrm{dom}\, f) \neq \emptyset$. Prove there is an affine function $\alpha \leq f$ with $\inf_C f = \inf_C \alpha$.

21. ** **(Essential smoothness)** For any convex function $f$ and any point $x \in \mathrm{bd}(\mathrm{dom}\, f)$, prove $\partial f(x)$ is either empty or unbounded. Deduce that a function is essentially smooth if and only if its subdifferential is always singleton or empty.

22. ** **(Birkhoff's theorem [15])** We use the notation of Section 1.2.

   (a) Prove $\mathbf{P}^n = \{(z_{ij}) \in \mathbf{\Gamma}^n \mid z_{ij} = 0 \text{ or } 1 \text{ for all } i, j\}$.

   (b) Prove $\mathbf{P}^n \subset \mathrm{ext}\,(\mathbf{\Gamma}^n)$.

   (c) Suppose $(z_{ij}) \in \mathbf{\Gamma}^n \setminus \mathbf{P}^n$. Prove there exist sequences of distinct indices $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$ such that

$$0 < z_{i_r j_r}, z_{i_{r+1} j_r} < 1 \quad (r = 1, 2, \ldots, m)$$

(where $i_{m+1} = i_1$). For these sequences, show the matrix $(z'_{ij})$ defined by

$$z'_{ij} - z_{ij} = \begin{cases} \epsilon & \text{if } (i,j) = (i_r, j_r) \text{ for some } r \\ -\epsilon & \text{if } (i,j) = (i_{r+1}, j_r) \text{ for some } r \\ 0 & \text{otherwise} \end{cases}$$

is doubly stochastic for all small real $\epsilon$. Deduce $(z_{ij}) \notin \text{ext}\,(\mathbf{\Gamma}^n)$.

(d) Deduce $\text{ext}\,(\mathbf{\Gamma}^n) = \mathbf{P}^n$. Hence prove Birkhoff's theorem (1.2.5).

(e) Use Carathéodory's theorem (Section 2.2, Exercise 5) to bound the number of permutation matrices needed to represent a doubly stochastic matrix in Birkhoff's theorem.

## 4.2 Fenchel Biconjugation

We have seen that many important convex functions $h : \mathbf{E} \to (\infty, +\infty]$ agree identically with their biconjugates $h^{**}$. Table 3.1 in Section 3.3 lists many one-dimensional examples, and the Bipolar cone theorem (3.3.14) shows $\delta_K = \delta_K^{**}$ for any closed convex cone $K$. In this section we isolate exactly the circumstances when $h = h^{**}$.

We can easily check that $h^{**}$ is a minorant of $h$ (that is, $h^{**} \leq h$ pointwise). Our specific aim in this section is to find conditions on a point $x$ in $\mathbf{E}$ guaranteeing $h^{**}(x) = h(x)$. This becomes the key relationship for the study of duality in optimization. As we see in this section, the conditions we need are both geometric and topological. This is neither particularly surprising or stringent. Since any conjugate function must have a closed convex epigraph, we cannot expect a function to agree with its biconjugate unless the function itself has a closed convex epigraph. On the other hand, this restriction is not particularly strong since, as we saw in the previous section, convex functions automatically have strong continuity properties.

We say the function $h : \mathbf{E} \to [-\infty, +\infty]$ is *closed* if its epigraph is a closed set. We say $h$ is *lower semicontinuous* at a point $x$ in $\mathbf{E}$ if

$$\liminf h(x^r) \;\Big(= \lim_{s\to\infty} \inf_{r\geq s} h(x^r)\Big) \geq h(x)$$

for any sequence $x^r \to x$. A function $h : \mathbf{E} \to [-\infty, +\infty]$ is *lower semicontinuous* if it is lower semicontinuous at every point in $\mathbf{E}$; this is in fact equivalent to $h$ being closed, which in turn holds if and only if $h$ has closed level sets. Any two functions $h$ and $g$ satisfying $h \leq g$ (in which case we call $h$ a *minorant* of $g$) must satisfy $h^* \geq g^*$, and hence $h^{**} \leq g^{**}$.

**Theorem 4.2.1 (Fenchel biconjugation)** *The three properties below are equivalent for any function* $h : \mathbf{E} \to (-\infty, +\infty]$:

(*i*) *$h$ is closed and convex.*

(*ii*) $h = h^{**}$.

(*iii*) *For all points $x$ in* $\mathbf{E}$,

$$h(x) = \sup\{\alpha(x) \mid \alpha \text{ an affine minorant of } h\}.$$

*Hence the conjugacy operation induces a bijection between proper closed convex functions.*

**Proof.** We can assume $h$ is proper. Since conjugate functions are always closed and convex we know property (ii) implies property (i). Also, any affine minorant $\alpha$ of $h$ satisfies $\alpha = \alpha^{**} \leq h^{**} \leq h$, and hence property (iii) implies (ii). It remains to show (i) implies (iii).

Fix a point $x^0$ in $\mathbf{E}$. Assume first $x^0 \in \mathrm{cl}\,(\mathrm{dom}\, h)$, and fix any real $r < h(x^0)$. Since $h$ is closed, the set $\{x \mid h(x) > r\}$ is open, so there is an open convex neighbourhood $U$ of $x^0$ with $h(x) > r$ on $U$. Now note that the set $\mathrm{dom}\, h \cap \mathrm{cont}\, \delta_U$ is nonempty, so we can apply the Fenchel theorem (3.3.5) to deduce that some element $\phi$ of $\mathbf{E}$ satisfies

$$r \leq \inf_x \{h(x) + \delta_U(x)\} = \{-h^*(\phi) - \delta_U^*(-\phi)\}. \qquad (4.2.2)$$

Now define an affine function $\alpha(\cdot) = \langle \phi, \cdot \rangle + \delta_U^*(-\phi) + r$. Inequality (4.2.2) shows that $\alpha$ minorizes $h$, and by definition we know $\alpha(x^0) \geq r$. Since $r$ was arbitrary, (iii) follows at the point $x = x^0$.

Suppose on the other hand $x^0$ does not lie in $\mathrm{cl}\,(\mathrm{dom}\, h)$. By the Basic separation theorem (2.1.6) there is a real $b$ and a nonzero element $a$ of $\mathbf{E}$ satisfying

$$\langle a, x^0 \rangle > b \geq \langle a, x \rangle \ \text{ for all points } x \text{ in } \mathrm{dom}\, h.$$

The argument in the preceding paragraph shows there is an affine minorant $\alpha$ of $h$. But now the affine function $\alpha(\cdot) + k(\langle a, \cdot \rangle - b)$ is a minorant of $h$ for all $k = 1, 2, \ldots$. Evaluating these functions at $x = x^0$ proves property (iii) at $x^0$. The final remark follows easily. □

We immediately deduce that a closed convex function $h : \mathbf{E} \to [-\infty, +\infty]$ equals its biconjugate if and only if it is proper or identically $+\infty$ or $-\infty$.

Restricting the conjugacy bijection to finite sublinear functions gives the following result.

**Corollary 4.2.3 (Support functions)** *Fenchel conjugacy induces a bijection between everywhere-finite sublinear functions and nonempty compact convex sets in* $\mathbf{E}$:

(a) *If the set $C \subset \mathbf{E}$ is compact, convex and nonempty then the* **support function** $\delta_C^*$ *is everywhere finite and sublinear.*

(b) *If the function $h : \mathbf{E} \to \mathbf{R}$ is sublinear then $h^* = \delta_C$, where the set*

$$C = \{\phi \in \mathbf{E} \mid \langle \phi, d \rangle \leq h(d) \text{ for all } d \in \mathbf{E}\}$$

*is nonempty, compact, and convex.*

**Proof.** See Exercise 9. □

Conjugacy offers a convenient way to recognize when a convex function has bounded level sets.

**Theorem 4.2.4 (Moreau–Rockafellar)** *A closed convex proper function on* $\mathbf{E}$ *has bounded level sets if and only if its conjugate is continuous at 0.*

**Proof.** By Proposition 1.1.5, a convex function $f : \mathbf{E} \to (\infty, +\infty]$ has bounded level sets if and only if it satisfies the growth condition

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > 0.$$

Since $f$ is closed we can check that this is equivalent to the existence of a minorant of the form $\epsilon\|\cdot\| + k \leq f(\cdot)$ for some constants $\epsilon > 0$ and $k$. Taking conjugates, this is in turn equivalent to $f^*$ being bounded above near 0, and the result then follows by Theorem 4.1.1 (Local boundedness). □

Strict convexity is also easy to recognize via conjugacy, using the following result (see Exercise 19 for the proof).

**Theorem 4.2.5 (Strict-smooth duality)** *A proper closed convex function on* $\mathbf{E}$ *is essentially strictly convex if and only if its conjugate is essentially smooth.*

What can we say about $h^{**}$ when the function $h : \mathbf{E} \to [-\infty, +\infty]$ is not necessarily closed? To answer this question we introduce the idea of the *closure* of $h$, denoted $\mathrm{cl}\, h$, defined by

$$\mathrm{epi}\,(\mathrm{cl}\, h) = \mathrm{cl}\,(\mathrm{epi}\, h). \tag{4.2.6}$$

It is easy to verify that $\mathrm{cl}\, h$ is then well-defined. The definition immediately implies $\mathrm{cl}\, h$ is the largest closed function minorizing $h$. Clearly if $h$ is convex, so is $\mathrm{cl}\, h$. We leave the proof of the next simple result as an exercise.

**Proposition 4.2.7 (Lower semicontinuity and closure)** *If a function* $f : \mathbf{E} \to [-\infty, +\infty]$ *is convex then it is lower semicontinuous at a point* $x$ *where it is finite if and only if* $f(x) = (\mathrm{cl}\, f)(x)$. *In this case* $f$ *is proper.*

We can now answer the question we posed at the beginning of the section.

**Theorem 4.2.8** *Suppose the function* $h : \mathbf{E} \to [-\infty, +\infty]$ *is convex.*

(a) *If* $h^{**}$ *is somewhere finite then* $h^{**} = \mathrm{cl}\, h$.

(b) *For any point* $x$ *where* $h$ *is finite,* $h(x) = h^{**}(x)$ *if and only if* $h$ *is lower semicontinuous at* $x$.

**Proof.** Observe first that since $h^{**}$ is closed and minorizes $h$, we know $h^{**} \leq \operatorname{cl} h \leq h$. If $h^{**}$ is somewhere finite then $h^{**}$ (and hence $\operatorname{cl} h$) is never $-\infty$ by applying Proposition 4.2.7 (Lower semicontinuity and closure) to $h^{**}$. On the other hand, if $h$ is finite and lower semicontinuous at $x$ then Proposition 4.2.7 shows $\operatorname{cl} h(x)$ is finite, and applying the proposition again to $\operatorname{cl} h$ shows once more that $\operatorname{cl} h$ is never $-\infty$. In either case, the Fenchel biconjugation theorem implies $\operatorname{cl} h = (\operatorname{cl} h)^{**} \leq h^{**} \leq \operatorname{cl} h$, so $\operatorname{cl} h = h^{**}$. Part (a) is now immediate, while part (b) follows by using Proposition 4.2.7 once more. $\square$

Any proper convex function $h$ with an affine minorant has its biconjugate $h^{**}$ somewhere finite. (In fact, because $\mathbf{E}$ is finite-dimensional, $h^{**}$ is somewhere finite if and only if $h$ is proper—see Exercise 25.)

## Exercises and Commentary

Our approach in this section again extends easily to infinite dimensions; see for example [70]. Our definition of a closed function is a little different to that in [167], although they coincide for proper functions. The original version of von Neumann's minimax theorem (Exercise 16) had both the sets $C$ and $D$ simplices. The proof was by Brouwer's fixed point theorem (8.1.3). The Fisher information function introduced in Exercise 24 is useful in signal reconstruction [35]. The inequality in Exercise 20 (Logarithmic homogeneity) is important for interior point methods [148, Prop. 2.4.1].

1. Prove that any function $h : \mathbf{E} \to [-\infty, +\infty]$ satisfies $h^{**} \leq h$.

2. **(Lower semicontinuity and closedness)** For any given function $h : \mathbf{E} \to [-\infty, +\infty]$, prove the following properties are equivalent:

   (a) $h$ is lower semicontinuous.

   (b) $h$ has closed level sets.

   (c) $h$ is closed.

   Prove that such a function has a global minimizer on any nonempty, compact set.

3. **(Pointwise maxima)** If the functions $f_\gamma : \mathbf{E} \to [-\infty, +\infty]$ are all convex (respectively closed) then prove the function defined by $f(x) = \sup_\gamma f_\gamma(x)$ is convex (respectively closed). Deduce that for any function $h : \mathbf{E} \to [-\infty, +\infty]$, the conjugate function $h^*$ is closed and convex.

4. Verify directly that any affine function equals its biconjugate.

5. * **(Midpoint convexity)**

   (a) A function $f : \mathbf{E} \to (\infty, +\infty]$ is *midpoint convex* if it satisfies

   $$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} \quad \text{for all } x \text{ and } y \text{ in } \mathbf{E}.$$

   Prove a closed function is convex if and only if it is midpoint convex.

   (b) Use the inequality

   $$2(X^2+Y^2) \succeq (X+Y)^2 \quad \text{for all } X \text{ and } Y \text{ in } \mathbf{S}^n$$

   to prove the function $Z \in \mathbf{S}^n_+ \mapsto -Z^{1/2}$ is $\mathbf{S}^n_+$-convex (see Section 3.3, Exercise 18 (Order convexity)).

6. Is the Fenchel biconjugation theorem (4.2.1) valid for arbitrary functions $h : \mathbf{E} \to [-\infty, +\infty]$?

7. **(Inverse of subdifferential)** For a function $h : \mathbf{E} \to (\infty, +\infty]$, if points $x$ and $\phi$ in $\mathbf{E}$ satisfy $\phi \in \partial h(x)$, prove $x \in \partial h^*(\phi)$. Prove the converse if $h$ is closed and convex.

8. * **(Closed subdifferential)** If a function $h : \mathbf{E} \to (\infty, +\infty]$ is closed, prove the multifunction $\partial h$ is *closed*: that is,

   $$\phi_r \in \partial h(x_r),\ x_r \to x,\ \phi_r \to \phi \ \Rightarrow \phi \in \partial h(x).$$

   Deduce that if $h$ is essentially smooth and a sequence of points $x_r$ in $\operatorname{int}(\operatorname{dom} h)$ approaches a point in $\operatorname{bd}(\operatorname{dom} h)$ then $\|\nabla h(x_r)\| \to \infty$.

9. * **(Support functions)**

   (a) Prove that if the set $C \subset \mathbf{E}$ is nonempty then $\delta^*_C$ is a closed sublinear function and $\delta^{**}_C = \delta_{\text{cl conv}C}$. Prove that if $C$ is also bounded then $\delta^*_C$ is everywhere finite.

   (b) Prove that any sets $C, D \subset \mathbf{E}$ satisfy

   $$\begin{aligned} \delta^*_{C+D} &= \delta^*_C + \delta^*_D \quad \text{and} \\ \delta^*_{\text{conv}(C\cup D)} &= \max(\delta^*_C, \delta^*_D). \end{aligned}$$

   (c) Suppose the function $h : \mathbf{E} \to (-\infty, +\infty]$ is positively homogeneous, and define a closed convex set

   $$C = \{\phi \in \mathbf{E} \mid \langle \phi, d\rangle \le h(d)\ \forall d\}.$$

   Prove $h^* = \delta_C$. Prove that if $h$ is in fact sublinear and everywhere finite then $C$ is nonempty and compact.

(d) Deduce Corollary 4.2.3 (Support functions).

10. * **(Almost homogeneous functions [19])** Prove that a function $f : \mathbf{E} \to \mathbf{R}$ has a representation

$$f(x) = \max_{i \in I}\{\langle a^i, x\rangle - b_i\} \quad (x \in \mathbf{E})$$

for a compact set $\{(a^i, b_i) \mid i \in I\} \subset \mathbf{E} \times \mathbf{R}$ if and only if $f$ is convex and satisfies $\sup_{\mathbf{E}} |f - g| < \infty$ for some sublinear function $g$.

11. * Complete the details of the proof of the Moreau–Rockafellar theorem (4.2.4).

12. **(Compact bases for cones)** Consider a closed convex cone $K$. Using the Moreau–Rockafellar theorem (4.2.4), show that a point $x$ lies in $\operatorname{int} K$ if and only if the set $\{\phi \in K^- \mid \langle \phi, x\rangle \geq -1\}$ is bounded. If the set $\{\phi \in K^- \mid \langle \phi, x\rangle = -1\}$ is nonempty and bounded, prove $x \in \operatorname{int} K$.

13. For any function $h : \mathbf{E} \to [-\infty, +\infty]$, prove the set $\operatorname{cl}(\operatorname{epi} h)$ is the epigraph of some function.

14. * **(Lower semicontinuity and closure)** For any convex function $h : \mathbf{E} \to [-\infty, +\infty]$ and any point $x^0$ in $\mathbf{E}$, prove

$$(\operatorname{cl} h)(x^0) = \lim_{\delta \downarrow 0} \inf_{\|x - x^0\| \leq \delta} h(x).$$

Deduce Proposition 4.2.7.

15. For any point $x$ in $\mathbf{E}$ and any function $h : \mathbf{E} \to (-\infty, +\infty]$ with a subgradient at $x$, prove $h$ is lower semicontinuous at $x$.

16. * **(Von Neumann's minimax theorem [185])** Suppose $\mathbf{Y}$ is a Euclidean space. Suppose that the sets $C \subset \mathbf{E}$ and $D \subset \mathbf{Y}$ are nonempty and convex with $D$ closed and that the map $A : \mathbf{E} \to \mathbf{Y}$ is linear.

(a) By considering the Fenchel problem

$$\inf_{x \in \mathbf{E}}\{\delta_C(x) + \delta_D^*(Ax)\}$$

prove

$$\inf_{x \in C} \sup_{y \in D}\langle y, Ax\rangle = \max_{y \in D} \inf_{x \in C}\langle y, Ax\rangle$$

(where the max is attained if finite), under the assumption

$$0 \in \operatorname{core}(\operatorname{dom} \delta_D^* - AC). \qquad (4.2.9)$$

(b) Prove property (4.2.9) holds in either of the two cases

(i) $D$ is bounded, or

(ii) $A$ is surjective and 0 lies in $\operatorname{int} C$. (Hint: Use the Open mapping theorem, Section 4.1, Exercise 9.)

(c) Suppose both $C$ and $D$ are compact. Prove

$$\min_{x \in C} \max_{y \in D} \langle y, Ax \rangle = \max_{y \in D} \min_{x \in C} \langle y, Ax \rangle.$$

17. **(Recovering primal solutions)** Assume all the conditions for the Fenchel theorem (3.3.5) hold, and that in addition the functions $f$ and $g$ are closed.

(a) Prove that if the point $\bar{\phi} \in \mathbf{Y}$ is an optimal dual solution then the point $\bar{x} \in \mathbf{E}$ is optimal for the primal problem if and only if it satisfies the two conditions $\bar{x} \in \partial f^*(A^*\bar{\phi})$ and $A\bar{x} \in \partial g^*(-\bar{\phi})$.

(b) Deduce that if $f^*$ is differentiable at the point $A^*\bar{\phi}$ then the only possible primal optimal solution is $\bar{x} = \nabla f^*(A^*\bar{\phi})$.

(c) ** Apply this result to the problems in Section 3.3, Exercise 22.

18. Calculate the support function $\delta_C^*$ of the set $C = \{x \in \mathbf{R}^2 \mid x_2 \geq x_1^2\}$. Prove the "contour" $\{y \mid \delta_C^*(y) = 1\}$ is not closed.

19. * **(Strict-smooth duality)** Consider a proper closed convex function $f : \mathbf{E} \to (\infty, +\infty]$.

(a) If $f$ has Gâteaux derivative $y$ at a point $x$ in $\mathbf{E}$, prove the inequality

$$f^*(z) > f^*(y) + \langle x, z - y \rangle$$

for elements $z$ of $\mathbf{E}$ distinct from $y$.

(b) If $f$ is essentially smooth, prove that $f^*$ is essentially strictly convex.

(c) Deduce the Strict-smooth duality theorem (4.2.5) using Exercise 23 in Section 3.1.

20. * **(Logarithmic homogeneity)** If the function $f : \mathbf{E} \to (\infty, +\infty]$ is closed, convex, and proper, then for any real $\nu > 0$ prove the inequality

$$f(x) + f^*(\phi) + \nu \log \langle x, -\phi \rangle \geq \nu \log \nu - \nu \quad \text{for all } x, \phi \in \mathbf{E}$$

holds (where we interpret $\log \alpha = -\infty$ when $\alpha \leq 0$) if and only $f$ satisfies the condition

$$f(tx) = f(x) - \nu \log t \quad \text{for all } x \in \mathbf{E},\ t \in \mathbf{R}_{++}.$$

Hint: Consider first the case $\nu = 1$, and use the inequality

$$\alpha \leq -1 - \log(-\alpha).$$

21. * **(Cofiniteness)** Consider a function $h : \mathbf{E} \to (\infty, +\infty]$ and the following properties:

    (i) $h(\cdot) - \langle \phi, \cdot \rangle$ has bounded level sets for all $\phi$ in $\mathbf{E}$.

    (ii) $\lim_{\|x\| \to \infty} \|x\|^{-1} h(x) = +\infty$.

    (iii) $h^*$ is everywhere finite.

    Complete the following steps.

    (a) Prove properties (i) and (ii) are equivalent.

    (b) If $h$ is closed, convex and proper, use the Moreau–Rockafellar theorem (4.2.4) to prove properties (i) and (iii) are equivalent.

22. ** **(Computing closures)**

    (a) Prove any closed convex function $g : \mathbf{R} \to (\infty, +\infty]$ is continuous on its domain.

    (b) Consider a convex function $f : \mathbf{E} \to (\infty, +\infty]$. For any points $x$ in $\mathbf{E}$ and $y$ in $\operatorname{int}(\operatorname{dom} f)$, prove

    $$f^{**}(x) = \lim_{t \uparrow 1} f(y + t(x - y)).$$

    Hint: Use part (a) and the Accessibility lemma (Section 1.1, Exercise 11).

23. ** **(Recession functions)** This exercise uses Section 1.1, Exercise 6 (Recession cones). The *recession function* of a closed convex function $f : \mathbf{E} \to (\infty, +\infty]$ is defined by

    $$0^+ f(d) = \sup_{t \in \mathbf{R}_{++}} \frac{f(x + td) - f(x)}{t} \quad \text{for } d \text{ in } \mathbf{E},$$

    where $x$ is any point in $\operatorname{dom} f$.

    (a) Prove $0^+ f$ is closed and sublinear.

    (b) Prove $\operatorname{epi}(0^+ f) = 0^+(\operatorname{epi} f)$, and deduce that $0^+ f$ is independent of the choice of the point $x$.

    (c) For any real $\alpha > \inf f$, prove

    $$0^+\{y \in \mathbf{E} \mid f(y) \leq \alpha\} = \{d \in \mathbf{E} \mid 0^+ f(d) \leq 0\}.$$

24. ** **(Fisher information function)** Let $f : \mathbf{R} \to (\infty, +\infty]$ be a given function, and define a function $g : \mathbf{R}^2 \to (\infty, +\infty]$ by

$$g(x,y) = \begin{cases} yf\left(\dfrac{x}{y}\right) & \text{if } y > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

 (a) Prove $g$ is convex if and only if $f$ is convex.

 (b) Suppose $f$ is essentially strictly convex. For $y$ and $v$ in $\mathbf{R}_{++}$ and $x$ and $u$ in $\mathbf{R}$, prove

$$g(x,y) + g(u,v) = g(x+y, u+v) \;\Leftrightarrow\; \frac{x}{y} = \frac{u}{v}.$$

 (c) Calculate $g^*$.

 (d) Suppose $f$ is closed, convex, and finite at 0. Using Exercises 22 and 23, prove

$$g^{**}(x,y) = \begin{cases} yf\left(\dfrac{x}{y}\right) & \text{if } y > 0 \\ 0^+ f(x) & \text{if } y = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

 (e) If $f(x) = x^2/2$ for all $x$ in $\mathbf{R}$, calculate $g$.

 (f) Define a set $C = \{(x,y) \in \mathbf{R}^2 \mid x^2 \le y \le x\}$ and a function

$$h(x,y) = \begin{cases} \dfrac{x^3}{y^2} & \text{if } (x,y) \in C \setminus \{0\} \\ 0 & \text{if } (x,y) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

 Prove $h$ is closed and convex but is not continuous relative to its (compact) domain $C$. Construct another such example with $\sup_C h$ finite.

25. ** **(Finiteness of biconjugate)** Consider a convex function $h : \mathbf{E} \to [-\infty, +\infty]$.

 (a) If $h$ is proper and has an affine minorant, prove $h^{**}$ is somewhere finite.

 (b) If $h^{**}$ is somewhere finite, prove $h$ is proper.

 (c) Use the fact that any proper convex function has a subgradient (Section 3.1, Exercise 29) to deduce that $h^{**}$ is somewhere finite if and only if $h$ is proper.

 (d) Deduce $h^{**} = \operatorname{cl} h$ for any convex function $h : E \to (\infty, +\infty]$.

26. ** **(Self-dual cones [8])** Consider a function $h : \mathbf{E} \to [-\infty, \infty)$ for which $-h$ is closed and sublinear, and suppose there is a point $\hat{x} \in \mathbf{E}$ satisfying $h(\hat{x}) > 0$. Define the *concave polar* of $h$ as the function $h_\circ : \mathbf{E} \to [-\infty, \infty)$ given by

$$h_\circ(y) = \inf\{\langle x, y\rangle \mid h(x) \geq 1\}.$$

(a) Prove $-h_\circ$ is closed and sublinear, and, for real $\lambda > 0$, we have $\lambda(\lambda h)_\circ = h_\circ$.

(b) Prove the closed convex cone

$$K_h = \{(x, t) \in \mathbf{E} \times \mathbf{R} \mid |t| \leq h(x)\}$$

has polar $(K_h)^- = -K_{h_\circ}$.

(c) Suppose the vector $\alpha \in \mathbf{R}^n_{++}$ satisfies $\sum_i \alpha_i = 1$, and define a function $h^\alpha : \mathbf{R}^n \to [-\infty, +\infty)$ by

$$h^\alpha(x) = \begin{cases} \prod_i x_i^{\alpha_i} & \text{if } x \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

Prove $h^\alpha_\circ = h^\alpha / h^\alpha(\alpha)$, and deduce the cone

$$P_\alpha = K_{(h^\alpha(\alpha))^{-1/2} h^\alpha}$$

is *self-dual*: $P_\alpha^- = -P_\alpha$.

(d) Prove the cones

$$\begin{aligned} Q_2 &= \{(x, t, z) \in \mathbf{R}^3 \mid t^2 \leq 2xz,\ x, z \geq 0\} \text{ and} \\ Q_3 &= \{(x, t, z) \in \mathbf{R}^3 \mid 2|t|^3 \leq \sqrt{27}xz^2,\ x, z \geq 0\} \end{aligned}$$

are self-dual.

(e) Prove $Q_2$ is *isometric* to $\mathbf{S}^2_+$; in other words, there is a linear map $A : \mathbf{R}^3 \to \mathbf{S}^2_+$ preserving the norm and satisfying $AQ_2 = \mathbf{S}^2_+$.

27. ** **(Conical open mapping [8])** Define two closed convex cones in $\mathbf{R}^3$:

$$\begin{aligned} Q &= \{(x, y, z) \in \mathbf{R}^3 \mid y^2 \leq 2xz,\ x, z \geq 0\}. \text{ and} \\ S &= \{(w, x, y) \in \mathbf{R}^3 \mid 2|x|^3 \leq \sqrt{27}wy^2,\ w, y \geq 0\}. \end{aligned}$$

These cones are self-dual by Exercise 26. Now define convex cones in $\mathbf{R}^4$ by

$$C = (0 \times Q) + (S \times 0) \quad \text{and} \quad D = 0 \times \mathbf{R}^3.$$

(a) Prove $C \cap D = \{0\} \times Q$.

(b) Prove $-C^- = (\mathbf{R} \times Q) \cap (S \times \mathbf{R})$.

(c) Define the projection $P : \mathbf{R}^4 \to \mathbf{R}^3$ by $P(w, x, y, z) = (x, y, z)$. Prove $P(C^-) = -Q$, or equivalently,

$$C^- + D^- = (C \cap D)^-.$$

(d) Deduce the normal cone formula

$$N_{C \cap D}(x) = N_C(x) + N_D(x) \quad \text{for all } x \text{ in } C \cap D$$

and, by taking polars, the tangent cone formula

$$T_{C \cap D}(x) = T_C(x) \cap T_D(x) \quad \text{for all } x \text{ in } C \cap D.$$

(e) Prove $C^-$ is a closed convex pointed cone with nonempty interior and $D^-$ is a line, and yet there is no constant $\epsilon > 0$ satisfying

$$(C^- + D^-) \cap \epsilon B \subset (C^- \cap B) + (D^- \cap B).$$

(Hint: Prove equivalently there is no $\epsilon > 0$ satisfying

$$P(C^-) \cap \epsilon B \subset P(C^- \cap B)$$

by considering the path $\{(t^2, t^3, t) \mid t \geq 0\}$ in $Q$.) Compare this with the situation when $C$ and $D$ are subspaces, using the Open mapping theorem (Section 4.1, Exercise 9).

(f) Consider the path

$$u(t) = \left(\frac{2}{\sqrt{27}}, t^2, t^3, 0\right) \quad \text{if } t \geq 0.$$

Prove $d_C(u(t)) = 0$ and $d_D(u(t)) = 2/\sqrt{27}$ for all $t \geq 0$, and yet

$$d_{C \cap D}(u(t)) \to +\infty \quad \text{as } t \to +\infty.$$

(Hint: Use the isometry in Exercise 26.)

28. ** **(Expected surprise [18])** An event occurs once every $n$ days, with probability $p_i$ on day $i$ for $i = 1, 2, \ldots, n$. We seek a distribution maximizing the average surprise caused by the event. Define the "surprise" as minus the logarithm of the probability that the event occurs on day $i$ given that it has not occurred so far. Using Bayes conditional probability rule, our problem is

$$\inf \left\{ S(p) \,\middle|\, \sum_1^n p_i = 1 \right\},$$

where we define the function $S : \mathbf{R}^n \to (\infty, +\infty]$ by

$$S(p) = \sum_{i=1}^{n} h\Big(p_i, \sum_{j=i}^{n} p_j\Big),$$

and the function $h : \mathbf{R}^2 \to (\infty, +\infty]$ by

$$h(x,y) = \begin{cases} x \log\left(\dfrac{x}{y}\right) & \text{if } x, y > 0 \\ 0 & \text{if } x \geq 0,\ y = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

(a) Prove $h$ is closed and convex using Exercise 24 (Fisher information function).

(b) Hence prove $S$ is closed and convex.

(c) Prove the problem has an optimal solution.

(d) By imitating Section 3.1, Exercise 27 (Maximum entropy), show the solution $\bar{p}$ is unique and is expressed recursively by

$$\bar{p}_1 = \mu_1, \quad \bar{p}_k = \mu_k\Big(1 - \sum_{1}^{k-1} \bar{p}_j\Big) \quad \text{for } k = 2, 3, \ldots, n,$$

where the numbers $\mu_k$ are defined by the recursion

$$\mu_n = 1, \quad \mu_{k-1} = \mu_k e^{-\mu_k} \quad \text{for } k = 2, 3, \ldots, n.$$

(e) Deduce that the components of $\bar{p}$ form an increasing sequence and that $\bar{p}_{n-j}$ is independent of $j$.

(f) Prove $\bar{p}_1 \sim 1/n$ for large $n$.

## 4.3 Lagrangian Duality

The duality between a convex function $h$ and its Fenchel conjugate $h^*$ which we outlined earlier is an elegant piece of theory. The real significance, however, lies in its power to describe duality theory for convex programs, one of the most far-reaching ideas in the study of optimization.

We return to the convex program that we studied in Section 3.2:

$$\inf\{f(x) \mid g(x) \le 0,\ x \in \mathbf{E}\}. \tag{4.3.1}$$

Here the function $f$ and the components $g_1, g_2, \ldots, g_m : \mathbf{E} \to (\infty, +\infty]$ are convex, and satisfy $\emptyset \neq \operatorname{dom} f \subset \cap_1^m \operatorname{dom} g_i$. As before, the Lagrangian function $L : \mathbf{E} \times \mathbf{R}_+^m \to (\infty, +\infty]$ is defined by $L(x; \lambda) = f(x) + \lambda^T g(x)$.

Notice that the Lagrangian encapsulates all the information of the *primal problem* (4.3.1): clearly

$$\sup_{\lambda \in \mathbf{R}_+^m} L(x; \lambda) = \begin{cases} f(x) & \text{if } x \text{ is feasible} \\ +\infty & \text{otherwise,} \end{cases}$$

so if we denote the optimal value of (4.3.1) by $p \in [-\infty, +\infty]$, we could rewrite the problem in the following form:

$$p = \inf_{x \in \mathbf{E}} \sup_{\lambda \in \mathbf{R}_+^m} L(x; \lambda). \tag{4.3.2}$$

This makes it rather natural to consider an associated problem

$$d = \sup_{\lambda \in \mathbf{R}_+^m} \inf_{x \in \mathbf{E}} L(x; \lambda) \tag{4.3.3}$$

where $d \in [-\infty, +\infty]$ is called the *dual value*. Thus the *dual problem* consists of maximizing over vectors $\lambda$ in $\mathbf{R}_+^m$ the *dual function* $\Phi(\lambda) = \inf_x L(x; \lambda)$. This dual problem is perfectly well-defined without any assumptions on the functions $f$ and $g$. It is an easy exercise to show the "weak duality inequality" $p \ge d$. Notice $\Phi$ is concave.

It can happen that the primal value $p$ is strictly larger than the dual value $d$ (Exercise 5). In this case we say there is a *duality gap*. We next investigate conditions ensuring there is no duality gap. As in Section 3.2, the chief tool in our analysis is the primal value function $v : \mathbf{R}^m \to [-\infty, +\infty]$, defined by

$$v(b) = \inf\{f(x) \mid g(x) \le b\}. \tag{4.3.4}$$

Below we summarize the relationships among these various ideas and pieces of notation.

**Proposition 4.3.5 (Dual optimal value)**

(*a*) *The primal optimal value $p$ is $v(0)$.*

(*b*) *The conjugate of the value function satisfies*

$$v^*(-\lambda) = \begin{cases} -\Phi(\lambda) & \textit{if } \lambda \geq 0 \\ +\infty & \textit{otherwise.} \end{cases}$$

(*c*) *The dual optimal value $d$ is $v^{**}(0)$.*

**Proof.** Part (a) is just the definition of $p$. Part (b) follows from the identities

$$\begin{aligned} v^*(-\lambda) &= \sup\{-\lambda^T b - v(b) \mid b \in \mathbf{R}^m\} \\ &= \sup\{-\lambda^T b - f(x) \mid g(x) + z = b,\ x \in \operatorname{dom} f,\ b \in \mathbf{R}^m,\ z \in \mathbf{R}^m_+\} \\ &= \sup\{-\lambda^T(g(x) + z) - f(x) \mid x \in \operatorname{dom} f,\ z \in \mathbf{R}^m_+\} \\ &= -\inf\{f(x) + \lambda^T g(x) \mid x \in \operatorname{dom} f\} + \sup\{-\lambda^T z \mid z \in \mathbf{R}^m_+\} \\ &= \begin{cases} -\Phi(\lambda) & \text{if } \lambda \geq 0 \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, we observe

$$d = \sup_{\lambda \in \mathbf{R}^m_+} \Phi(\lambda) = -\inf_{\lambda \in \mathbf{R}^m_+} -\Phi(\lambda) = -\inf_{\lambda \in \mathbf{R}^m_+} v^*(-\lambda) = v^{**}(0),$$

so part (c) follows. □

Notice the above result does not use convexity.

The reason for our interest in the relationship between a convex function and its biconjugate should now be clear, in light of parts (a) and (c) above.

**Corollary 4.3.6 (Zero duality gap)** *Suppose the value of the primal problem (4.3.1) is finite. Then the primal and dual values are equal if and only if the value function $v$ is lower semicontinuous at 0. In this case the set of optimal dual solutions is $-\partial v(0)$.*

**Proof.** By the previous result, there is no duality gap exactly when the value function satisfies $v(0) = v^{**}(0)$, so Theorem 4.2.8 proves the first assertion. By part (b) of the previous result, dual optimal solutions $\lambda$ are characterized by the property $0 \in \partial v^*(-\lambda)$ or equivalently $v^*(-\lambda) + v^{**}(0) = 0$. But we know $v(0) = v^{**}(0)$, so this property is equivalent to the condition $-\lambda \in \partial v(0)$. □

This result sheds new light on our proof of the Lagrangian necessary conditions (3.2.8); the proof in fact demonstrates the existence of a dual optimal solution. We consider below two distinct approaches to proving the absence of a duality gap. The first uses the Slater condition, as in Theorem 3.2.8, to force attainment in the dual problem. The second (dual) approach uses compactness to force attainment in the primal problem.

**Theorem 4.3.7 (Dual attainment)** *If the Slater condition holds for the primal problem (4.3.1) then the primal and dual values are equal, and the dual value is attained if finite.*

**Proof.** If $p$ is $-\infty$ there is nothing to prove, since we know $p \geq d$. If on the other hand $p$ is finite then, as in the proof of the Lagrangian necessary conditions (3.2.8), the Slater condition forces $\partial v(0) \neq \emptyset$. Hence $v$ is finite and lower semicontinuous at 0 (Section 4.2, Exercise 15), and the result follows by Corollary 4.3.6 (Zero duality gap). □

An indirect way of stating the Slater condition is that there is a point $\hat{x}$ in $\mathbf{E}$ for which the set $\{\lambda \in \mathbf{R}^m_+ \mid L(\hat{x};\lambda) \geq \alpha\}$ is compact for all real $\alpha$. The second approach uses a "dual" condition to ensure the value function is closed.

**Theorem 4.3.8 (Primal attainment)** *Suppose that the functions*

$$f, g_1, g_2, \ldots, g_m : \mathbf{E} \to (\infty, +\infty]$$

*are closed and that for some real $\hat{\lambda}_0 \geq 0$ and some vector $\hat{\lambda}$ in $\mathbf{R}^m_+$, the function $\hat{\lambda}_0 f + \hat{\lambda}^T g$ has compact level sets. Then the value function $v$ defined by equation (4.3.4) is closed, and the infimum in this equation is attained when finite. Consequently, if the functions $f, g_1, g_2, \ldots, g_m$ are, in addition, convex and the dual value for the problem (4.3.1) is not $-\infty$, then the primal and dual values $p$ and $d$ are equal, and the primal value is attained when finite.*

**Proof.** If the points $(b^r, s_r)$ lie in $\operatorname{epi} v$ for $r = 1, 2, \ldots$ and approach the point $(b, s)$ then for each integer $r$ there is a point $x^r$ in $\mathbf{E}$ satisfying $f(x^r) \leq s_r + r^{-1}$ and $g(x^r) \leq b^r$. Hence we deduce

$$(\hat{\lambda}_0 f + \hat{\lambda}^T g)(x^r) \leq \hat{\lambda}_0 (s_r + r^{-1}) + \hat{\lambda}^T b^r \to \hat{\lambda}_0 s + \hat{\lambda}^T b.$$

By the compact level set assumption, the sequence $(x^r)$ has a subsequence converging to some point $\bar{x}$, and since all the functions are closed, we know $f(\bar{x}) \leq s$ and $g(\bar{x}) \leq b$. We deduce $v(b) \leq s$, so $(b, s)$ lies in $\operatorname{epi} v$ as we required. When $v(b)$ is finite, the same argument with $(b^r, s_r)$ replaced by $(b, v(b))$ for each $r$ shows the infimum is attained.

If the functions $f, g_1, g_2, \ldots, g_m$ are convex then we know (from Section 3.2) $v$ is convex. If $d$ is $+\infty$ then again from the inequality $p \geq d$, there is nothing to prove. If $d$ $(= v^{**}(0))$ is finite then Theorem 4.2.8 shows $v^{**} = \mathrm{cl}\, v$, and the above argument shows $\mathrm{cl}\, v = v$. Hence $p = v(0) = v^{**}(0) = d$, and the result follows. $\square$

Notice that if either the objective function $f$ or any one of the constraint functions $g_1, g_2, \ldots, g_m$ has compact level sets then the compact level set condition in the above result holds.

## Exercises and Commentary

An attractive elementary account of finite-dimensional convex duality theory appears in [152]. A good reference for this kind of development in infinite dimensions is [98]. When the value function $v$ is lower semicontinuous at 0 we say the problem (4.3.1) is *normal*; see [167]. If $\partial v(0) \neq \emptyset$ (or $v(0) = -\infty$) the problem is called *stable*; see, for example, [6]). For a straightforward account of interior point methods and the penalized linear program in Exercise 4 (Examples of duals) see [187, p. 40]. For more on the minimax theory in Exercise 14 see, for example, [60].

1. **(Weak duality)** Prove that the primal and dual values $p$ and $d$ defined by equations (4.3.2) and (4.3.3) satisfy $p \geq d$.

2. Calculate the Lagrangian dual of the problem in Section 3.2, Exercise 3.

3. **(Slater and compactness)** Prove the Slater condition holds for problem (4.3.1) if and only if there is a point $\hat{x}$ in $\mathbf{E}$ for which the level sets

$$\{\lambda \in \mathbf{R}^m_+ \mid -L(\hat{x}; \lambda) \leq \alpha\}$$

are compact for all real $\alpha$.

4. **(Examples of duals)** Calculate the Lagrangian dual problem for the following problems (for given vectors $a^1, a^2, \ldots, a^m$, and $c$ in $\mathbf{R}^n$).

   (a) The *linear program*

$$\inf_{x \in \mathbf{R}^n} \{\langle c, x\rangle \mid \langle a^i, x\rangle \leq b_i \ \text{ for } i = 1, 2, \ldots, m\}.$$

   (b) Another linear program

$$\inf_{x \in \mathbf{R}^n} \{\langle c, x\rangle + \delta_{\mathbf{R}^n_+}(x) \mid \langle a^i, x\rangle \leq b_i \ \text{ for } i = 1, 2, \ldots, m\}.$$

   (c) The *quadratic program* (for $C \in \mathbf{S}^n_{++}$)

$$\inf_{x \in \mathbf{R}^n} \left\{ \frac{x^T C x}{2} \;\middle|\; \langle a^i, x\rangle \leq b_i \ \text{ for } i = 1, 2, \ldots, m \right\}.$$

(d) The separable problem

$$\inf_{x \in \mathbf{R}^n} \left\{ \sum_{j=1}^{n} p(x_j) \;\middle|\; \langle a^i, x \rangle \le b_i \ \text{ for } i = 1, 2, \ldots, m \right\}$$

for a given function $p : \mathbf{R} \to (\infty, +\infty]$.

(e) The *penalized linear program*

$$\inf_{x \in \mathbf{R}^n} \{ \langle c, x \rangle + \epsilon \mathrm{lb}\,(x) \mid \langle a^i, x \rangle \le b_i \ \text{ for } i = 1, 2, \ldots, m \}$$

for real $\epsilon > 0$.

For given matrices $A_1, A_2, \ldots, A_m$, and $C$ in $\mathbf{S}^n$, calculate the dual of the *semidefinite program*

$$\inf_{X \in \mathbf{S}^n_+} \{ \mathrm{tr}\,(CX) + \delta_{\mathbf{S}^n_+}(X) \mid \mathrm{tr}\,(A_i X) \le b_i \ \text{ for } i = 1, 2, \ldots, m \},$$

and the *penalized semidefinite program*

$$\inf_{X \in \mathbf{S}^n_+} \{ \mathrm{tr}\,(CX) + \epsilon \mathrm{ld}\, X \mid \mathrm{tr}\,(A_i X) \le b_i \ \text{ for } i = 1, 2, \ldots, m \}$$

for real $\epsilon > 0$.

5. **(Duffin's duality gap, continued)**

(a) For the problem considered in Section 3.2, Exercise 8, namely

$$\inf_{x \in \mathbf{R}^2} \{ e^{x_2} \mid \|x\| - x_1 \le 0 \},$$

calculate the dual function, and hence find the dual value.

(b) Repeat part (a) with the objective function $e^{x_2}$ replaced by $x_2$.

6. Consider the problem

$$\inf\{ \exp^*(x_1) + \exp^*(x_2) \mid x_1 + 2x_2 - 1 \le 0, \ x \in \mathbf{R}^2 \}.$$

Write down the Lagrangian dual problem, solve the primal and dual problems, and verify that the optimal values are equal.

7. Given a matrix $C$ in $\mathbf{S}^n_{++}$, calculate

$$\inf_{X \in \mathbf{S}^n_{++}} \{ \mathrm{tr}\,(CX) \mid -\log(\det X) \le 0 \}$$

by Lagrangian duality.

8. * **(Mixed constraints)** Explain why an appropriate dual for the problem

$$\inf\{f(x) \mid g(x) \le 0,\ h(x) = 0\}$$

for a function $h : \mathrm{dom}\, f \to \mathbf{R}^k$ is

$$\sup_{\lambda \in \mathbf{R}^m_+,\ \mu \in \mathbf{R}^k} \ \inf_{x \in \mathrm{dom}\, f} \{f(x) + \lambda^T g(x) + \mu^T h(x)\}.$$

9. **(Fenchel and Lagrangian duality)** Let $\mathbf{Y}$ be a Euclidean space. By suitably rewriting the primal Fenchel problem

$$\inf_{x \in \mathbf{E}} \{f(x) + g(Ax)\}$$

for given functions $f : \mathbf{E} \to (\infty, +\infty]$, $g : \mathbf{Y} \to (\infty, +\infty]$, and linear $A : \mathbf{E} \to \mathbf{Y}$, interpret the dual Fenchel problem

$$\sup_{\phi \in \mathbf{Y}} \{-f^*(A^*\phi) - g^*(-\phi)\}$$

as a Lagrangian dual problem.

10. **(Trust region subproblem duality [175])** Given a matrix $A$ in $\mathbf{S}^n$ and a vector $b$ in $\mathbf{R}^n$, consider the *nonconvex* problem

$$\inf\left\{x^T Ax + b^T x \,\middle|\, x^T x - 1 \le 0,\ x \in \mathbf{R}^n\right\}.$$

Complete the following steps to prove there is an optimal dual solution, with no duality gap.

(i) Prove the result when $A$ is positive semidefinite.

(ii) If $A$ is not positive definite, prove the primal optimal value does not change if we replace the inequality in the constraint by an equality.

(iii) By observing for any real $\alpha$ the equality

$$\min\left\{x^T Ax + b^T x \,\middle|\, x^T x = 1\right\} = \\ -\alpha + \min\left\{x^T (A + \alpha I)x + b^T x \,\middle|\, x^T x = 1\right\},$$

prove the general result.

11. ** If there is no duality gap, prove that dual optimal solutions are the same as Karush–Kuhn–Tucker vectors (Section 3.2, Exercise 9).

12. * **(Conjugates of compositions)** Consider the composition $g \circ f$ of a nondecreasing convex function $g : \mathbf{R} \to (\infty, +\infty]$ with a convex function $f : \mathbf{E} \to (\infty, +\infty]$. We interpret $g(+\infty) = +\infty$, and we

assume there is a point $\hat{x}$ in $\mathbf{E}$ satisfying $f(\hat{x}) \in \operatorname{int}(\operatorname{dom} g)$. Use Lagrangian duality to prove the formula, for $\phi$ in $\mathbf{E}$,

$$(g \circ f)^*(\phi) = \inf_{t \in \mathbf{R}_+} \left\{ g^*(t) + t f^*\left(\frac{\phi}{t}\right) \right\},$$

where we interpret

$$0 f^*\left(\frac{\phi}{0}\right) = \delta^*_{\operatorname{dom} f}(\phi).$$

13. ** **(A symmetric pair [28])**

(a) Given real $\gamma_1, \gamma_2, \ldots, \gamma_n > 0$, define $h : \mathbf{R}^n \to (\infty, +\infty]$ by

$$h(x) = \begin{cases} \prod_{i=1}^n x_i^{-\gamma_i} & \text{if } x \in \mathbf{R}^n_{++} \\ +\infty & \text{otherwise.} \end{cases}$$

By writing $g(x) = \exp(\log g(x))$ and using the composition formula in Exercise 12, prove

$$h^*(y) = \begin{cases} -(\gamma + 1) \displaystyle\prod_{i=1}^n \left(\frac{-y_i}{\gamma_i}\right)^{\gamma_i/(\gamma+1)} & \text{if } -y \in \mathbf{R}^n_+ \\ +\infty & \text{otherwise,} \end{cases}$$

where $\gamma = \sum_i \gamma_i$.

(b) Given real $\alpha_1, \alpha_2, \ldots, \alpha_n > 0$, define $\alpha = \sum_i \alpha_i$ and suppose a real $\mu$ satisfies $\mu > \alpha + 1$. Now define a function $f : \mathbf{R}^n \times \mathbf{R} \to (\infty, +\infty]$ by

$$f(x, s) = \begin{cases} \mu^{-1} s^\mu \prod_i x_i^{-\alpha_i} & \text{if } x \in \mathbf{R}^n_{++},\ s \in \mathbf{R}_+ \\ +\infty & \text{otherwise.} \end{cases}$$

Use part (a) to prove

$$f^*(y, t) = \begin{cases} \rho \nu^{-1} t^\nu \prod_i (-y_i)^{-\beta_i} & \text{if } -y \in \mathbf{R}^n_{++},\ t \in \mathbf{R}_+ \\ +\infty & \text{otherwise} \end{cases}$$

for constants

$$\nu = \frac{\mu}{\mu - (\alpha + 1)}, \quad \beta_i = \frac{\alpha_i}{\mu - (\alpha + 1)}, \quad \rho = \prod_i \left(\frac{\alpha_i}{\mu}\right)^{\beta_i}.$$

(c) Deduce $f = f^{**}$, whence $f$ is convex.

(d) Give an alternative proof of the convexity of $f$ by using Section 4.2, Exercise 24(a) (Fisher information function) and induction.

(e) Prove $f$ is strictly convex.

14. ** **(Convex minimax theory)** Suppose that $\mathbf{Y}$ is a Euclidean space, that the sets $C \subset \mathbf{Y}$ and $D \subset \mathbf{E}$ are nonempty, and consider a function $\psi : C \times D \to \mathbf{R}$.

    (a) Prove the inequality
$$\sup_{y \in D} \inf_{x \in C} \psi(x, y) \leq \inf_{x \in C} \sup_{y \in D} \psi(x, y).$$

    (b) We call a point $(\bar{x}, \bar{y})$ in $C \times D$ a *saddlepoint* if it satisfies
$$\psi(\bar{x}, y) \leq \psi(\bar{x}, \bar{y}) \leq \psi(x, \bar{y}) \text{ for all } x \in C,\ y \in D.$$
In this case prove
$$\sup_{y \in D} \inf_{x \in C} \psi(x, y) = \psi(\bar{x}, \bar{y}) = \inf_{x \in C} \sup_{y \in D} \psi(x, y).$$

    (c) Suppose the function $p_y : \mathbf{E} \to (\infty, +\infty]$ defined by
$$p_y(x) = \begin{cases} \psi(x, y) & \text{if } x \in C \\ +\infty & \text{otherwise} \end{cases}$$
is convex, for all $y$ in $D$. Prove the function $h : \mathbf{Y} \to [-\infty, +\infty]$ defined by
$$h(z) = \inf_{x \in C} \sup_{y \in D} \{\psi(x, y) + \langle z, y \rangle\}$$
is convex.

    (d) Suppose the function $q_x : \mathbf{Y} \to (\infty, +\infty]$ defined by
$$q_x(y) = \begin{cases} -\psi(x, y) & \text{if } y \in D \\ +\infty & \text{otherwise} \end{cases}$$
is closed and convex for all points $x$ in $C$. Deduce
$$h^{**}(0) = \sup_{y \in D} \inf_{x \in C} \psi(x, y).$$

    (e) Suppose that for all points $y$ in $D$ the function $p_y$ defined in part (c) is closed and convex, and that for some point $\hat{y}$ in $D$, $p_{\hat{y}}$ has compact level sets. If $h$ is finite at 0, prove it is lower semicontinuous there. If the assumption in part (d) also holds, deduce
$$\sup_{y \in D} \inf_{x \in C} \psi(x, y) = \min_{x \in C} \sup_{y \in D} \psi(x, y).$$

    (f) Suppose the functions $f, g_1, g_2, \ldots, g_s : \mathbf{R}^t \to (\infty, +\infty]$ are closed and convex. Interpret the above results in the following two cases:

(i)
$$\begin{aligned} C &= (\operatorname{dom} f) \cap \Big(\bigcap_{i=1}^{s} \operatorname{dom} g_i\Big) \\ D &= \mathbf{R}^s_+ \\ \psi(u,w) &= f(u) + \textstyle\sum_{i=1}^{s} w_i g_i(u). \end{aligned}$$

(ii)
$$\begin{aligned} C &= \mathbf{R}^s_+ \\ D &= (\operatorname{dom} f) \cap \Big(\bigcap_{i=1}^{s} \operatorname{dom} g_i\Big) \\ \psi(u,w) &= -f(w) - \textstyle\sum_{i=1}^{s} u_i g_i(w). \end{aligned}$$

(g) **(Kakutani [109])** Suppose that the nonempty sets $C \subset \mathbf{Y}$ and $D \subset \mathbf{E}$ are compact and convex, that the function $\psi : C \times D \to \mathbf{R}$ is continuous, that $\psi(x, y)$ is convex in the variable $x$ for all fixed $y$ in $D$, and that $-\psi(x, y)$ is convex in the variable $y$ for all points $x$ in $C$. Deduce $\psi$ has a saddlepoint.

# Chapter 5

# Special Cases

## 5.1 Polyhedral Convex Sets and Functions

In our earlier section on theorems of the alternative (Section 2.2), we observed that finitely generated cones are closed. Remarkably, a finite linear-algebraic assumption leads to a topological conclusion. In this section we pursue the consequences of this type of assumption in convex analysis.

There are two natural ways to impose a finite linear structure on the sets and functions we consider. The first we have already seen: a "polyhedron" (or *polyhedral* set) is a finite intersection of closed halfspaces in $\mathbf{E}$, and we say a function $f : \mathbf{E} \to [-\infty, +\infty]$ is *polyhedral* if its epigraph is polyhedral. On the other hand, a *polytope* is the convex hull of a finite subset of $\mathbf{E}$, and we call a subset of $\mathbf{E}$ *finitely generated* if it is the sum of a polytope and a finitely generated cone (in the sense of formula (2.2.11)). Notice we do not yet know if a cone that is a finitely generated set in this sense is finitely generated in the sense of (2.2.11); we return to this point later in the section. The function $f$ is *finitely generated* if its epigraph is finitely generated. A central result of this section is that polyhedra and finitely generated sets in fact coincide.

We begin with some easy observations collected together in the following two results.

**Proposition 5.1.1 (Polyhedral functions)** *Suppose that the function $f : \mathbf{E} \to [-\infty, +\infty]$ is polyhedral. Then $f$ is closed and convex and can be decomposed in the form*

$$f = \max_{i \in I} g_i + \delta_P, \tag{5.1.2}$$

*where the index set $I$ is finite (and possibly empty), the functions $g_i$ are affine, and the set $P \subset \mathbf{E}$ is polyhedral (and possibly empty). Thus the domain of $f$ is polyhedral and coincides with* $\operatorname{dom} \partial f$ *if $f$ is proper.*

**Proof.** Since any polyhedron is closed and convex, so is $f$, and the decomposition (5.1.2) follows directly from the definition. If $f$ is proper then both the sets $I$ and $P$ are nonempty in this decomposition. At any point $x$ in $P$ $(= \operatorname{dom} f)$ we know $0 \in \partial \delta_P(x)$, and the function $\max_i g_i$ certainly has a subgradient at $x$ since it is everywhere finite. Hence we deduce the condition $\partial f(x) \neq \emptyset$. □

**Proposition 5.1.3 (Finitely generated functions)** *Suppose the function $f : \mathbf{E} \to [-\infty, +\infty]$ is finitely generated. Then $f$ is closed and convex and* $\operatorname{dom} f$ *is finitely generated. Furthermore, $f^*$ is polyhedral.*

**Proof.** Polytopes are compact and convex (by Carathéodory's theorem (Section 2.2, Exercise 5)), and finitely generated cones are closed and convex, so finitely generated sets (and therefore functions) are closed and convex (by Section 1.1, Exercise 5(a)). We leave the remainder of the proof as an exercise. □

An easy exercise shows that a set $P \subset \mathbf{E}$ is polyhedral (respectively, finitely generated) if and only if $\delta_P$ is polyhedral (respectively, finitely generated).

To prove that polyhedra and finitely generated sets in fact coincide, we consider the two extreme special cases: first, compact sets, and second, cones. Observe first that compact, finitely generated sets are just polytopes, directly from the definition.

**Lemma 5.1.4** *Any polyhedron has at most finitely many extreme points.*

**Proof.** Fix a finite set of affine functions $\{g_i \mid i \in I\}$ on $\mathbf{E}$, and consider the polyhedron

$$P = \{x \in \mathbf{E} \mid g_i(x) \leq 0 \text{ for } i \in I\}.$$

For any point $x$ in $P$, the "active set" is $\{i \in I \mid g_i(x) = 0\}$. Suppose two distinct extreme points $x$ and $y$ of $P$ have the same active set. Then for any small real $\epsilon$ the points $x \pm \epsilon(y - x)$ both lie in $P$. But this contradicts the assumption that $x$ is extreme. Hence different extreme points have different active sets, and the result follows. □

This lemma together with Minkowski's theorem (4.1.8) reveals the nature of compact polyhedra.

**Theorem 5.1.5** *Any compact polyhedron is a polytope.*

We next turn to cones.

**Lemma 5.1.6** *Any polyhedral cone is a finitely generated cone (in the sense of (2.2.11)).*

**Proof.** Given a polyhedral cone $P \subset \mathbf{E}$, define a subspace $L = P \cap -P$ and a pointed polyhedral cone $K = P \cap L^{\perp}$. Observe the decomposition $P = K \oplus L$. By the Pointed cone theorem (3.3.15), there is an element $y$ of $\mathbf{E}$ for which the set

$$C = \{x \in K \mid \langle x, y \rangle = 1\}$$

is compact and satisfies $K = \mathbf{R}_+ C$. Since $C$ is polyhedral, the previous result shows it is a polytope. Thus $K$ is finitely generated, whence so is $P$. □

**Theorem 5.1.7 (Polyhedrality)** *A set or function is polyhedral if and only if it is finitely generated.*

**Proof.** For finite sets $\{a_i \mid i \in I\} \subset \mathbf{E}$ and $\{b_i \mid i \in I\} \subset \mathbf{R}$, consider the polyhedron in $\mathbf{E}$ defined by

$$P = \{x \in \mathbf{E} \mid \langle a_i, x \rangle \leq b_i \text{ for } i \in I\}.$$

The polyhedral cone in $\mathbf{E} \times \mathbf{R}$ defined by

$$Q = \{(x, r) \in \mathbf{E} \times \mathbf{R} \mid \langle a_i, x \rangle - b_i r \leq 0 \text{ for } i \in I\}$$

is finitely generated by the previous lemma, so there are finite subsets $\{x_j \mid j \in J\}$ and $\{y_t \mid t \in T\}$ of $\mathbf{E}$ with

$$Q = \Big\{ \sum_{j \in J} \lambda_j (x_j, 1) + \sum_{t \in T} \mu_t (y_t, 0) \,\Big|\, \lambda_j \in \mathbf{R}_+ \text{ for } j \in J,\ \mu_t \in \mathbf{R}_+ \text{ for } t \in T \Big\}.$$

We deduce

$$\begin{aligned} P &= \{x \mid (x, 1) \in Q\} \\ &= \operatorname{conv}\{x_j \mid j \in J\} + \Big\{ \sum_{t \in T} \mu_t y_y \,\Big|\, \mu_t \in \mathbf{R}_+ \text{ for } t \in T \Big\}, \end{aligned}$$

so $P$ is finitely generated. We have thus shown that any polyhedral set (and hence function) is finitely generated.

Conversely, suppose the function $f : \mathbf{E} \to [-\infty, +\infty]$ is finitely generated. Consider first the case when $f$ is proper. By Proposition 5.1.3, $f^*$ is polyhedral, and hence (by the above argument) finitely generated. But $f$ is closed and convex, by Proposition 5.1.3, so the Fenchel biconjugation theorem (4.2.1) implies $f = f^{**}$. By applying Proposition 5.1.3 once again we see $f^{**}$ (and hence $f$) is polyhedral. We leave the improper case as an exercise. □

Notice these two results show our two notions of a finitely generated cone do indeed coincide.

The following collection of exercises shows that many linear-algebraic operations preserve polyhedrality.

**Proposition 5.1.8 (Polyhedral algebra)** *Consider a Euclidean space* $\mathbf{Y}$ *and a linear map* $A : \mathbf{E} \to \mathbf{Y}$.

(a) *If the set* $P \subset \mathbf{E}$ *is polyhedral then so is its image* $AP$.

(b) *If the set* $K \subset \mathbf{Y}$ *is polyhedral then so is its inverse image* $A^{-1}K$.

(c) *The sum and pointwise maximum of finitely many polyhedral functions are polyhedral.*

(d) *If the function* $g : \mathbf{Y} \to [-\infty, +\infty]$ *is polyhedral then so is the composite function* $g \circ A$.

(e) *If the function* $q : \mathbf{E} \times \mathbf{Y} \to [-\infty, +\infty]$ *is polyhedral then so is the function* $h : \mathbf{Y} \to [-\infty, +\infty]$ *defined by* $h(u) = \inf_{x \in \mathbf{E}} q(x, u)$.

**Corollary 5.1.9 (Polyhedral Fenchel duality)** *All the conclusions of the Fenchel duality theorem (3.3.5) remain valid if the regularity condition (3.3.8) is replaced by the assumption that the functions* $f$ *and* $g$ *are polyhedral with* $\mathrm{dom}\, g \cap A \mathrm{dom}\, f$ *nonempty.*

**Proof.** We follow the original proof, simply observing that the value function $h$ defined in the proof is polyhedral by the Polyhedral algebra proposition above. Thus, when the optimal value is finite, $h$ has a subgradient at 0. □

We conclude this section with a result emphasizing the power of Fenchel duality for convex problems with linear constraints.

**Corollary 5.1.10 (Mixed Fenchel duality)** *All the conclusions of the Fenchel duality theorem (3.3.5) remain valid if the regularity condition (3.3.8) is replaced by the assumption that* $\mathrm{dom}\, g \cap A \mathrm{cont}\, f$ *is nonempty and the function* $g$ *is polyhedral.*

**Proof.** Assume without loss of generality the primal optimal value

$$p = \inf_{x \in \mathbf{E}} \{f(x) + g(Ax)\} = \inf_{x \in \mathbf{E},\ r \in \mathbf{R}} \{f(x) + r \mid g(Ax) \le r\}$$

is finite. By assumption there is a feasible point for the problem on the right at which the objective function is continuous, so there is an affine function $\alpha : \mathbf{E} \times \mathbf{R} \to \mathbf{R}$ minorizing the function $(x, r) \mapsto f(x) + r$ such that

$$p = \inf_{x \in \mathbf{E},\ r \in \mathbf{R}} \{\alpha(x, r) \mid g(Ax) \le r\}$$

(see Section 3.3, Exercise 13(c)). Clearly $\alpha$ has the form $\alpha(x, r) = \beta(x) + r$ for some affine minorant $\beta$ of $f$, so

$$p = \inf_{x \in \mathbf{E}} \{\beta(x) + g(Ax)\}.$$

Now we apply polyhedral Fenchel duality (Corollary 5.1.9) to deduce the existence of an element $\phi$ of $\mathbf{Y}$ such that

$$p = -\beta^*(A^*\phi) - g^*(-\phi) \leq -f^*(A^*\phi) - g^*(-\phi) \leq p$$

(using the weak duality inequality), and the duality result follows. The calculus rules follow as before. □

It is interesting to compare this result with the version of Fenchel duality using the Open mapping theorem (Section 4.1, Exercise 9), where the assumption that $g$ is polyhedral is replaced by surjectivity of $A$.

## Exercises and Commentary

Our approach in this section is analogous to [181]. The key idea, Theorem 5.1.7 (Polyhedrality), is due to Minkowski [141] and Weyl [186]. A nice development of geometric programming (see Exercise 13) appears in [152].

1. Prove directly from the definition that any polyhedral function has a decomposition of the form (5.1.2).

2. Fill in the details for the proof of the Finitely generated functions proposition (5.1.3).

3. Use Proposition 4.2.7 (Lower semicontinuity and closure) to show that if a finitely generated function $f$ is not proper then it has the form

$$f(x) = \begin{cases} +\infty & \text{if } x \notin K \\ -\infty & \text{if } x \in K \end{cases}$$

   for some finitely generated set $K$.

4. Prove a set $K \subset \mathbf{E}$ is polyhedral (respectively, finitely generated) if and only if $\delta_K$ is polyhedral (respectively, finitely generated). Do not use the Polyhedrality theorem (5.1.7).

5. Complete the proof of the Polyhedrality theorem (5.1.7) for improper functions using Exercise 3.

6. **(Tangents to polyhedra)** Prove the tangent cone to a polyhedron $P$ at a point $x$ in $P$ is given by $T_P(x) = \mathbf{R}_+(P - x)$.

7. * **(Polyhedral algebra)** Prove Proposition 5.1.8 using the following steps.

   (i) Prove parts (a)–(d).

(ii) In the notation of part (e), consider the natural projection

$$P_{\mathbf{Y}\times\mathbf{R}} : \mathbf{E} \times \mathbf{Y} \times \mathbf{R} \to \mathbf{Y} \times \mathbf{R}.$$

Prove the inclusions

$$P_{\mathbf{Y}\times\mathbf{R}}(\operatorname{epi} q) \subset \operatorname{epi} h \subset \operatorname{cl}(P_{\mathbf{Y}\times\mathbf{R}}(\operatorname{epi} q)).$$

(iii) Deduce part (e).

8. If the function $f : \mathbf{E} \to (\infty, +\infty]$ is polyhedral, prove the subdifferential of $f$ at a point $x$ in $\operatorname{dom} f$ is a nonempty polyhedron and is bounded if and only if $x$ lies in $\operatorname{int}(\operatorname{dom} f)$.

9. **(Polyhedral cones)** For any polyhedral cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$ and any linear map $A : \mathbf{E} \to \mathbf{Y}$, prove the relation

$$(K \cap A^{-1}H)^- = A^*H^- + K^-$$

using convex calculus.

10. Apply the Mixed Fenchel duality corollary (5.1.10) to the problem $\inf\{f(x) \mid Ax \le b\}$, for a linear map $A : \mathbf{E} \to \mathbf{R}^m$ and a point $b$ in $\mathbf{R}^m$.

11. * **(Generalized Fenchel duality)** Consider convex functions

$$h_1, h_2, \ldots, h_m : \mathbf{E} \to (\infty, +\infty]$$

with $\cap_i \operatorname{cont} h_i$ nonempty. By applying the Mixed Fenchel duality corollary (5.1.10) to the problem

$$\inf_{x, x^1, x^2, \ldots, x^m \in \mathbf{E}} \Big\{ \sum_{i=1}^m h_i(x^i) \ \Big|\ x^i = x \text{ for } i = 1, 2, \ldots, m \Big\},$$

prove

$$\inf_{x \in \mathbf{E}} \sum_i h_i(x) = -\inf \Big\{ \sum_i h_i^*(\phi^i) \ \Big|\ \phi^1, \phi^2, \ldots, \phi^m \in \mathbf{E},\ \sum_i \phi^i = 0 \Big\}.$$

12. ** **(Relativizing Mixed Fenchel duality)** In the Mixed Fenchel duality corollary (5.1.10), prove the condition $\operatorname{dom} g \cap A \operatorname{cont} f \ne \emptyset$ can be replaced by $\operatorname{dom} g \cap A \operatorname{ri}(\operatorname{dom} f) \ne \emptyset$.

13. ** **(Geometric programming)** Consider the *constrained geometric program*

$$\inf_{x \in \mathbf{E}} \{h_0(x) \mid h_i(x) \le 1 \text{ for } i = 1, 2, \ldots, m\},$$

where each function $h_i$ is a sum of functions of the form

$$x \in \mathbf{E} \mapsto c \log \Big( \sum_{j=1}^{n} \exp \langle a^j, x \rangle \Big)$$

for real $c > 0$ and elements $a^1, a^2, \ldots, a^n$ of $\mathbf{E}$. Write down the Lagrangian dual problem and simplify it using Exercise 11 and the form of the conjugate of each $h_i$ given by (3.3.1). State a duality theorem.

## 5.2 Functions of Eigenvalues

Fenchel conjugacy gives a concise and beautiful avenue to many eigenvalue inequalities in classical matrix analysis. In this section we outline this approach.

The two cones $\mathbf{R}^n_+$ and $\mathbf{S}^n_+$ appear repeatedly in applications, as do their corresponding logarithmic barriers lb and ld, which we defined in Section 3.3. We can relate the vector and matrix examples, using the notation of Section 1.2, through the identities

$$\delta_{\mathbf{S}^n_+} = \delta_{\mathbf{R}^n_+} \circ \lambda \quad \text{and} \quad \mathrm{ld} = \mathrm{lb} \circ \lambda. \tag{5.2.1}$$

We see in this section that these identities fall into a broader pattern.

Recall the function $[\cdot] : \mathbf{R}^n \to \mathbf{R}^n$ rearranges components into nonincreasing order. We say a function $f$ on $\mathbf{R}^n$ is *symmetric* if $f(x) = f([x])$ for all vectors $x$ in $\mathbf{R}^n$; in other words, permuting components does not change the function value. We call a symmetric function of the eigenvalues of a symmetric matrix a *spectral function.* The following formula is crucial.

**Theorem 5.2.2 (Spectral conjugacy)** *If $f : \mathbf{R}^n \to [-\infty, +\infty]$ is a symmetric function, it satisfies the formula*

$$(f \circ \lambda)^* = f^* \circ \lambda.$$

**Proof.** By Fan's inequality (1.2.2) any matrix $Y$ in $\mathbf{S}^n$ satisfies the inequalities

$$\begin{aligned}
(f \circ \lambda)^*(Y) &= \sup_{X \in \mathbf{S}^n} \{\mathrm{tr}\,(XY) - f(\lambda(X))\} \\
&\le \sup_X \{\lambda(X)^T \lambda(Y) - f(\lambda(X))\} \\
&\le \sup_{x \in \mathbf{R}^n} \{x^T \lambda(Y) - f(x)\} \\
&= f^*(\lambda(Y)).
\end{aligned}$$

On the other hand, fixing a spectral decomposition $Y = U^T(\mathrm{Diag}\,\lambda(Y))U$ for some matrix $U$ in $\mathbf{O}^n$ leads to the reverse inequality

$$\begin{aligned}
f^*(\lambda(Y)) &= \sup_{x \in \mathbf{R}^n} \{x^T \lambda(Y) - f(x)\} \\
&= \sup_x \{\mathrm{tr}\,((\mathrm{Diag}\,x)UYU^T) - f(x)\} \\
&= \sup_x \{\mathrm{tr}\,(U^T(\mathrm{Diag}\,x)UY) - f(\lambda(U^T \mathrm{Diag}\,xU))\} \\
&\le \sup_{X \in \mathbf{S}^n} \{\mathrm{tr}\,(XY) - f(\lambda(X))\} \\
&= (f \circ \lambda)^*(Y),
\end{aligned}$$

which completes the proof. □

This formula, for example, makes it very easy to calculate $\mathrm{ld}^*$ (see the Log barriers proposition (3.3.3)) and to check the self-duality of the cone $\mathbf{S}^n_+$.

Once we can compute conjugates easily, we can also recognize closed convex functions easily using the Fenchel biconjugation theorem (4.2.1).

**Corollary 5.2.3 (Davis)** *Suppose the function $f : \mathbf{R}^n \to (\infty, +\infty]$ is symmetric. Then the "spectral function" $f \circ \lambda$ is closed and convex if and only if $f$ is closed and convex.*

We deduce immediately that the logarithmic barrier ld is closed and convex, as well as the function $X \mapsto \mathrm{tr}\,(X^{-1})$ on $\mathbf{S}^n_{++}$, for example.

Identifying subgradients is also easy using the conjugacy formula and the Fenchel–Young inequality (3.3.4).

**Corollary 5.2.4 (Spectral subgradients)** *If $f : \mathbf{R}^n \to (\infty, +\infty]$ is a symmetric function, then for any two matrices $X$ and $Y$ in $\mathbf{S}^n$, the following properties are equivalent:*

*(i)* $Y \in \partial(f \circ \lambda)(X)$.

*(ii)* *$X$ and $Y$ have a simultaneous ordered spectral decomposition and satisfy $\lambda(Y) \in \partial f(\lambda(X))$.*

*(iii)* *$X = U^T(\mathrm{Diag}\,x)U$ and $Y = U^T(\mathrm{Diag}\,y)U$ for some matrix $U$ in $\mathbf{O}^n$ and vectors $x$ and $y$ in $\mathbf{R}^n$ satisfying $y \in \partial f(x)$.*

**Proof.** Notice the inequalities

$$(f \circ \lambda)(X) + (f \circ \lambda)^*(Y) = f(\lambda(X)) + f^*(\lambda(Y)) \geq \lambda(X)^T\lambda(Y) \geq \mathrm{tr}\,(XY).$$

The condition $Y \in \partial(f \circ \lambda)(X)$ is equivalent to equality between the left and right hand sides (and hence throughout), and the equivalence of properties (i) and (ii) follows using Fan's inequality (1.2.1). For the remainder of the proof, see Exercise 9. □

**Corollary 5.2.5 (Spectral differentiability)** *Suppose that the function $f : \mathbf{R}^n \to (\infty, +\infty]$ is symmetric, closed, and convex. Then $f \circ \lambda$ is differentiable at a matrix $X$ in $\mathbf{S}^n$ if and only if $f$ is differentiable at $\lambda(X)$.*

**Proof.** If $\partial(f \circ \lambda)(X)$ is a singleton, so is $\partial f(\lambda(X))$, by the Spectral subgradients corollary above. Conversely, suppose $\partial f(\lambda(X))$ consists only of the vector $y \in \mathbf{R}^n$. Using Exercise 9(b), we see the components of $y$ are nonincreasing, so by the same corollary, $\partial(f \circ \lambda)(X)$ is the nonempty convex set

$$\{U^T(\mathrm{Diag}\,y)U \mid U \in \mathbf{O}^n,\ U^T\mathrm{Diag}\,(\lambda(X))U = X\}.$$

But every element of this set has the same norm (namely $\|y\|$), so the set must be a singleton. □

Notice that the proof in fact shows that when $f$ is differentiable at $\lambda(X)$ we have the formula

$$\nabla(f \circ \lambda)(X) = U^T(\mathrm{Diag}\,\nabla f(\lambda(X)))U \tag{5.2.6}$$

for any matrix $U$ in $\mathbf{O}^n$ satisfying $U^T(\mathrm{Diag}\,\lambda(X))U = X$.

The pattern of these results is clear: many analytic and geometric properties of the matrix function $f \circ \lambda$ parallel the corresponding properties of the underlying function $f$. The following exercise is another example.

**Corollary 5.2.7** *Suppose the function $f : \mathbf{R}^n \to (\infty, +\infty]$ is symmetric, closed, and convex. Then $f \circ \lambda$ is essentially strictly convex (respectively, essentially smooth) if and only if $f$ is essentially strictly convex (respectively, essentially smooth).*

For example, the logarithmic barrier ld is both essentially smooth and essentially strictly convex.

## Exercises and Commentary

Our approach in this section follows [120]. The Davis theorem (5.2.3) appeared in [58] (without the closure assumption). Many convexity properties of eigenvalues like Exercise 4 (Examples of convex spectral functions) can be found in [99] or [10], for example. Surveys of eigenvalue optimization appear in [128, 127].

1. Prove the identities (5.2.1).

2. Use the Spectral conjugacy theorem (5.2.2) to calculate $\mathrm{ld}^*$ and $\delta^*_{\mathbf{S}^n_+}$.

3. Prove the Davis characterization (Corollary 5.2.3) using the Fenchel biconjugation theorem (4.2.1).

4. **(Examples of convex spectral functions)** Use the Davis characterization (Corollary 5.2.3) to prove the following functions of a matrix $X \in \mathbf{S}^n$ are closed and convex:

   (a) $\mathrm{ld}\,(X)$.

   (b) $\mathrm{tr}\,(X^p)$, for any nonnegative even integer $p$.

   (c)
   $$\begin{cases} -\mathrm{tr}\,(X^{1/2}) & \text{if } X \in \mathbf{S}^n_+ \\ +\infty & \text{otherwise.} \end{cases}$$

(d)
$$\begin{cases} \operatorname{tr}(X^{-p}) & \text{if } X \in \mathbf{S}^n_{++} \\ +\infty & \text{otherwise} \end{cases}$$
for any nonnegative integer $p$.

(e)
$$\begin{cases} \operatorname{tr}(X^{1/2})^{-1} & \text{if } X \in \mathbf{S}^n_{++} \\ +\infty & \text{otherwise.} \end{cases}$$

(f)
$$\begin{cases} -(\det X)^{1/n} & \text{if } X \in \mathbf{S}^n_{+} \\ +\infty & \text{otherwise.} \end{cases}$$

Deduce from the sublinearity of the function in part (f) the property

$$0 \preceq X \preceq Y \Rightarrow 0 \leq \det X \leq \det Y$$

for matrices $X$ and $Y$ in $\mathbf{S}^n$.

5. Calculate the conjugate of each of the functions in Exercise 4.

6. Use formula (5.2.6) to calculate the gradients of the functions in Exercise 4.

7. For a matrix $A$ in $\mathbf{S}^n_{++}$ and a real $b > 0$, use the Lagrangian sufficient conditions (3.2.3) to solve the problem

$$\inf\{f(X) \mid \operatorname{tr}(AX) \leq b,\ X \in \mathbf{S}^n\},$$

where $f$ is one of the functions in Exercise 4.

8. * **(Orthogonal invariance)** A function $h : \mathbf{S}^n \to (\infty, +\infty]$ is *orthogonally invariant* if all matrices $X$ in $\mathbf{S}^n$ and $U$ in $\mathbf{O}^n$ satisfy the relation $h(U^T XU) = h(X)$; in other words, orthogonal similarity transformations do not change the value of $h$.

   (a) Prove $h$ is orthogonally invariant if and only if there is a symmetric function $f : \mathbf{R}^n \to (\infty, +\infty]$ with $h = f \circ \lambda$.

   (b) Prove that an orthogonally invariant function $h$ is closed and convex if and only if $h \circ \mathrm{Diag}$ is closed and convex.

9. * Suppose the function $f : \mathbf{R}^n \to (-\infty, +\infty]$ is symmetric.

   (a) Prove $f^*$ is symmetric.

   (b) If vectors $x$ and $y$ in $\mathbf{R}^n$ satisfy $y \in \partial f(x)$, prove $[y] \in \partial f([x])$ using Proposition 1.2.4.

   (c) Finish the proof of the Spectral subgradients corollary (5.2.4).

   (d) Deduce $\partial(f \circ \lambda)(X) = \emptyset \Leftrightarrow \partial f(\lambda(X)) = \emptyset$.

   (e) Prove Corollary 5.2.7.

10. * **(Fillmore–Williams [78])** Suppose the set $C \subset \mathbf{R}^n$ is *symmetric*: that is, $PC = C$ holds for all permutation matrices $P$. Prove the set

$$\lambda^{-1}(C) = \{X \in \mathbf{S}^n \mid \lambda(X) \in C\}$$

is closed and convex if and only if $C$ is closed and convex.

11. ** **(Semidefinite complementarity)** Suppose matrices $X$ and $Y$ lie in $\mathbf{S}^n_+$.

    (a) If $\operatorname{tr}(XY) = 0$, prove $-Y \in \partial\delta_{\mathbf{S}^n_+}(X)$.

    (b) Hence prove the following properties are equivalent:

      (i) $\operatorname{tr}(XY) = 0$.

      (ii) $XY = 0$.

      (iii) $XY + YX = 0$.

    (c) Using Exercise 5 in Section 1.2, prove for any matrices $U$ and $V$ in $\mathbf{S}^n$

$$(U^2 + V^2)^{1/2} = U + V \quad \Leftrightarrow \quad U, V \succeq 0 \text{ and } \operatorname{tr}(UV) = 0.$$

12. ** **(Eigenvalue sums)** Consider a vector $\mu$ in $\mathbf{R}^n_\geq$.

    (a) Prove the function $\mu^T\lambda(\cdot)$ is sublinear using Section 2.2, Exercise 9 (Schur-convexity).

    (b) Deduce the map $\lambda$ is $(-\mathbf{R}^n_\geq)^-$-sublinear. (See Section 3.3, Exercise 18 (Order convexity).)

    (c) Use Section 3.1, Exercise 10 to prove

$$\partial(\mu^T\lambda)(0) = \lambda^{-1}(\operatorname{conv}(\mathbf{P}^n\mu)).$$

13. ** **(Davis theorem)** Suppose the function $f : \mathbf{R}^n \to [-\infty, +\infty]$ is symmetric (but not necessarily closed). Use Exercise 12 (Eigenvalue sums) and Section 2.2, Exercise 9(d) (Schur-convexity) to prove that $f \circ \lambda$ is convex if and only if $f$ is convex.

14. * **(DC functions)** We call a real function $f$ on a convex set $C \subset \mathbf{E}$ a *DC function* if it can be written as the difference of two real convex functions on $C$.

    (a) Prove the set of DC functions is a vector space.

    (b) If $f$ is a DC function, prove it is locally Lipschitz on $\operatorname{int} C$.

    (c) Prove $\lambda_k$ is a DC function on $\mathbf{S}^n$ for all $k$, and deduce it is locally Lipschitz.

## 5.3 Duality for Linear and Semidefinite Programming

*Linear programming* (LP) is the study of optimization problems involving a linear objective function subject to linear constraints. This simple optimization model has proved enormously powerful in both theory and practice, so we devote this section to deriving linear programming duality theory from our convex-analytic perspective. We contrast this theory with the corresponding results for *semidefinite programming* (SDP), a class of matrix optimization problems analogous to linear programs but involving the positive semidefinite cone.

Linear programs are inherently polyhedral, so our main development follows directly from the polyhedrality section (Section 5.1). But to begin, we sketch an alternative development directly from the Farkas lemma (2.2.7). Given vectors $a^1, a^2, \ldots, a^m$, and $c$ in $\mathbf{R}^n$ and a vector $b$ in $\mathbf{R}^m$, consider the *primal linear program*

$$\left.\begin{array}{lrcl} \inf & & \langle c, x\rangle & \\ \text{subject to} & \langle a^i, x\rangle - b_i & \leq & 0 \ \text{ for } i = 1, 2, \ldots, m \\ & x & \in & \mathbf{R}^n. \end{array}\right\} \tag{5.3.1}$$

Denote the primal optimal value by $p \in [-\infty, +\infty]$. In the Lagrangian duality framework (Section 4.3), the dual problem is

$$\sup\Big\{ -b^T\mu \;\Big|\; \sum_{i=1}^m \mu_i a^i = -c,\ \mu \in \mathbf{R}^m_+ \Big\} \tag{5.3.2}$$

with dual optimal value $d \in [-\infty, +\infty]$. From Section 4.3 we know the weak duality inequality $p \geq d$. If the primal problem (5.3.1) satisfies the Slater condition then the Dual attainment theorem (4.3.7) shows $p = d$ with dual attainment when the values are finite. However, as we shall see, the Slater condition is superfluous here.

Suppose the primal value $p$ is finite. Then it is easy to see that the "homogenized" system of inequalities in $\mathbf{R}^{n+1}$,

$$\left.\begin{array}{rl} \langle a^i, x\rangle - b_i z \leq 0 & \text{for } i = 1, 2, \ldots, m \\ -z \leq 0 & \text{and} \\ \langle -c, x\rangle + pz > 0, & x \in \mathbf{R}^n, \quad z \in \mathbf{R} \end{array}\right\} \tag{5.3.3}$$

has no solution. Applying the Farkas lemma (2.2.7) to this system, we deduce there is a vector $\bar{\mu}$ in $\mathbf{R}^n_+$ and a scalar $\beta$ in $\mathbf{R}_+$ satisfying

$$\sum_{i=1}^m \bar{\mu}_i (a^i, -b_i) + \beta(0, -1) = (-c, p).$$

Thus $\bar{\mu}$ is a feasible solution for the dual problem (5.3.2) with objective value at least $p$. The weak duality inequality now implies $\bar{\mu}$ is optimal and $p = d$. We needed no Slater condition; the assumption of a finite primal optimal value alone implies zero duality gap and dual attainment.

We can be more systematic using our polyhedral theory. Suppose that $\mathbf{Y}$ is a Euclidean space, that the map $A : \mathbf{E} \to \mathbf{Y}$ is linear, and consider cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$. For given elements $c$ of $\mathbf{E}$ and $b$ of $\mathbf{Y}$, consider the primal *abstract linear program*

$$\inf\{\langle c, x\rangle \mid Ax - b \in H,\ x \in K\}. \tag{5.3.4}$$

As usual, denote the optimal value by $p$. We can write this problem in Fenchel form (3.3.6) if we define functions $f$ on $\mathbf{E}$ and $g$ on $\mathbf{Y}$ by $f(x) = \langle c, x\rangle + \delta_K(x)$ and $g(y) = \delta_H(y-b)$. Then the Fenchel dual problem (3.3.7) is

$$\sup\{\langle b, \phi\rangle \mid A^*\phi - c \in K^-,\ \phi \in -H^-\} \tag{5.3.5}$$

with dual optimal value $d$. If we now apply the Fenchel duality theorem (3.3.5) in turn to problem (5.3.4), and then to problem (5.3.5) (using the Bipolar cone theorem (3.3.14)), we obtain the following general result.

**Corollary 5.3.6 (Cone programming duality)** *Suppose the cones $H$ and $K$ in problem (5.3.4) are convex.*

(*a*) *If any of the conditions*

(*i*) $b \in \operatorname{int}(AK - H)$,

(*ii*) $b \in AK - \operatorname{int} H$, *or*

(*iii*) $b \in A(\operatorname{int} K) - H$, *and either $H$ is polyhedral or $A$ is surjective*

*hold then there is no duality gap ($p = d$) and the dual optimal value $d$ is attained if finite.*

(*b*) *Suppose $H$ and $K$ are also closed. If any of the conditions*

(*i*) $-c \in \operatorname{int}(A^*H^- + K^-)$,

(*ii*) $-c \in A^*H^- + \operatorname{int} K^-$, *or*

(*iii*) $-c \in A^*(\operatorname{int} H^-) + K^-$, *and either $K$ is polyhedral or $A^*$ is surjective*

*hold then there is no duality gap and the primal optimal value $p$ is attained if finite.*

In both parts (a) and (b), the sufficiency of condition (iii) follows by applying the Mixed Fenchel duality corollary (5.1.10), or the Open mapping theorem (Section 4.1, Exercise 9). In the fully polyhedral case we obtain the following result.

**Corollary 5.3.7 (Linear programming duality)** *Suppose the cones $H$ and $K$ in the dual pair of problems (5.3.4) and (5.3.5) are polyhedral. If either problem has finite optimal value then there is no duality gap and both problems have optimal solutions.*

**Proof.** We apply the Polyhedral Fenchel duality corollary (5.1.9) to each problem in turn. □

Our earlier result for the linear program (5.3.1) is clearly just a special case of this corollary.

Linear programming has an interesting matrix analogue. Given matrices $A_1, A_2, \ldots, A_m$, and $C$ in $\mathbf{S}^n_+$ and a vector $b$ in $\mathbf{R}^m$, consider the primal *semidefinite program*

$$\left.\begin{array}{lrcl} \inf & \mathrm{tr}\,(CX) & & \\ \text{subject to} & \mathrm{tr}\,(A_iX) & = & b_i \text{ for } i = 1, 2, \ldots, m \\ & X & \in & \mathbf{S}^n_+. \end{array}\right\} \tag{5.3.8}$$

This is a special case of the abstract linear program (5.3.4), so the dual problem is

$$\sup\left\{ b^T\phi \;\middle|\; C - \sum_{i=1}^m \phi_i A_i \in \mathbf{S}^n_+,\ \phi \in \mathbf{R}^m \right\}, \tag{5.3.9}$$

since $(\mathbf{S}^n_+)^- = -\mathbf{S}^n_+$, by the Self-dual cones proposition (3.3.12), and we obtain the following duality theorem from the general result above.

**Corollary 5.3.10 (Semidefinite programming duality)** *If the primal problem (5.3.8) has a positive definite feasible solution, there is no duality gap and the dual optimal value is attained when finite. On the other hand, if there is a vector $\phi$ in $\mathbf{R}^m$ with $C - \sum_i \phi_i A_i$ positive definite then once again there is no duality gap and the primal optimal value is attained when finite.*

Unlike linear programming, we need a condition stronger than mere consistency to guarantee no duality gap. For example, if we consider the primal semidefinite program (5.3.8) with

$$n = 2,\ m = 1,\ C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ \text{and } b = 0,$$

the primal optimal value is 0 (and is attained), whereas the dual problem (5.3.9) is inconsistent.

## Exercises and Commentary

The importance of linear programming duality was first emphasized by Dantzig [57] and that of semidefinite duality by Nesterov and Nemirovskii [148]. A good general reference for linear programming is [53]. A straightforward exposition of the central path (see Exercise 10) may be found in [187]. Semidefinite programming has wide application in control theory [46].

1. Check the form of the dual problem for the linear program (5.3.1).

2. If the optimal value of problem (5.3.1) is finite, prove system (5.3.3) has no solution.

3. **(Linear programming duality gap)** Give an example of a linear program of the form (5.3.1) which is inconsistent ($p = +\infty$) with the dual problem (5.3.2) *also* inconsistent ($d = -\infty$).

4. Check the form of the dual problem for the abstract linear program (5.3.4).

5. Fill in the details of the proof of the Cone programming duality corollary (5.3.6). In particular, when the cones $H$ and $K$ are closed, show how to interpret problem (5.3.4) as the dual of problem (5.3.5).

6. Fill in the details of the proof of the linear programming duality corollary (5.3.7).

7. **(Complementary slackness)** Suppose we know the optimal values of problems (5.3.4) and (5.3.5) are equal and the dual value is attained. Prove a feasible solution $x$ for problem (5.3.4) is optimal if and only if there is a feasible solution $\phi$ for the dual problem (5.3.5) satisfying the conditions

$$\langle Ax - b, \phi \rangle = 0 = \langle x, A^* \phi - c \rangle.$$

8. **(Semidefinite programming duality)** Prove Corollary 5.3.10.

9. **(Semidefinite programming duality gap)** Check the details of the example after Corollary 5.3.10.

10. ** **(Central path)** Consider the dual pair of linear programs (5.3.1) and (5.3.2). Define a linear map $A : \mathbf{R}^n \to \mathbf{R}^m$ by $(Ax)_i = (a^i)^T x$ for each index $i$. Make the following assumptions:

    (i) There is a vector $x$ in $\mathbf{R}^n$ satisfying $b - Ax \in \mathbf{R}^n_{++}$.

    (ii) There is a feasible solution $\mu$ in $\mathbf{R}^m_{++}$ for problem (5.3.2).

(iii) The set $\{a^1, a^2, \ldots, a^m\}$ is linearly independent.

Now consider the "penalized" problem (for real $\epsilon > 0$)

$$\inf_{x \in \mathbf{R}^n} \{c^T x + \epsilon \,\mathrm{lb}\,(b - Ax)\}. \tag{5.3.11}$$

(a) Write this problem as a Fenchel problem (3.3.6), and show the dual problem is

$$\sup\Big\{ -b^T\mu - \epsilon \,\mathrm{lb}\,(\mu) - k(\epsilon) \,\Big|\, \sum_{i=1}^m \mu_i a^i = -c,\ \mu \in \mathbf{R}^m_+ \Big\} \tag{5.3.12}$$

for some function $k : \mathbf{R}_+ \to \mathbf{R}$.

(b) Prove that both problems (5.3.11) and (5.3.12) have optimal solutions, with equal optimal values.

(c) By complementary slackness (Section 3.3, Exercise 9(f)), prove problems (5.3.11) and (5.3.12) have unique optimal solutions $x^\epsilon \in \mathbf{R}^n$ and $\mu^\epsilon \in \mathbf{R}^m$, characterized as the unique solution of the system

$$\begin{aligned} \sum_{i=1}^m \mu_i a^i &= -c \\ \mu_i(b_i - (a^i)^T x) &= \epsilon \ \text{ for each } i \\ b - Ax &\geq 0, \ \text{ and} \\ \mu \in \mathbf{R}^m_+, &\quad x \in \mathbf{R}^n. \end{aligned}$$

(d) Calculate $c^T x^\epsilon + b^T \mu^\epsilon$.

(e) Deduce that, as $\epsilon$ decreases to 0, the feasible solution $x^\epsilon$ approaches optimality in problem (5.3.1) and $\mu^\epsilon$ approaches optimality in problem (5.3.2).

11. ** **(Semidefinite central path)** Imitate the development of Exercise 10 for the semidefinite programs (5.3.8) and (5.3.9).

12. ** **(Relativizing cone programming duality)** Prove other conditions guaranteeing part (a) of Corollary 5.3.6 are

(i) $b \in A(\mathrm{ri}\, K) - \mathrm{ri}\, H$ or

(ii) $b \in A(\mathrm{ri}\, K) - H$ and $H$ polyhedral.

(Hint: Use Section 4.1, Exercise 20, and Section 5.1, Exercise 12.)

# Chapter 7

# Karush–Kuhn–Tucker Theory

## 7.1 An Introduction to Metric Regularity

Our main optimization models so far are inequality-constrained. A little thought shows our techniques are not useful for equality-constrained problems like

$$\inf\{f(x) \mid h(x) = 0\}.$$

In this section we study such problems by linearizing the feasible region $h^{-1}(0)$ using the contingent cone.

Throughout this section we consider an open set $U \subset \mathbf{E}$, a closed set $S \subset U$, a Euclidean space $\mathbf{Y}$, and a continuous map $h : U \to \mathbf{Y}$. The restriction of $h$ to $S$ we denote $h|_S$. The following easy result (Exercise 1) suggests our direction.

**Proposition 7.1.1** *If $h$ is Fréchet differentiable at the point $x \in U$ then*

$$K_{h^{-1}(h(x))}(x) \subset N(\nabla h(x)).$$

Our aim in this section is to find conditions guaranteeing equality in this result.

Our key tool is the next result. It states that if a closed function attains a value close to its infimum at some point then a nearby point minimizes a slightly perturbed function.

**Theorem 7.1.2 (Ekeland variational principle)** *Suppose the function $f : \mathbf{E} \to (\infty, +\infty]$ is closed and the point $x \in \mathbf{E}$ satisfies $f(x) \leq \inf f + \epsilon$ for some real $\epsilon > 0$. Then for any real $\lambda > 0$ there is a point $v \in \mathbf{E}$ satisfying the conditions* (*a*) $\|x - v\| \le \lambda$,

(*b*) $f(v) \le f(x)$, *and*

(*c*) $v$ *is the unique minimizer of the function* $f(\cdot) + (\epsilon/\lambda)\|\cdot - v\|$.

**Proof.** We can assume $f$ is proper, and by assumption it is bounded below. Since the function

$$f(\cdot) + \frac{\epsilon}{\lambda}\|\cdot - x\|$$

therefore has compact level sets, its set of minimizers $M \subset \mathbf{E}$ is nonempty and compact. Choose a minimizer $v$ for $f$ on $M$. Then for points $z \neq v$ in $M$ we know

$$f(v) \le f(z) < f(z) + \frac{\epsilon}{\lambda}\|z - v\|,$$

while for $z$ not in $M$ we have

$$f(v) + \frac{\epsilon}{\lambda}\|v - x\| < f(z) + \frac{\epsilon}{\lambda}\|z - x\|.$$

Part (c) follows by the triangle inequality. Since $v$ lies in $M$ we have

$$f(z) + \frac{\epsilon}{\lambda}\|z - x\| \ge f(v) + \frac{\epsilon}{\lambda}\|v - x\| \text{ for all } z \text{ in } \mathbf{E}.$$

Setting $z = x$ shows the inequalities

$$f(v) + \epsilon \ge \inf f + \epsilon \ge f(x) \ge f(v) + \frac{\epsilon}{\lambda}\|v - x\|.$$

Properties (a) and (b) follow. □

As we shall see, precise calculation of the contingent cone $K_{h^{-1}(h(x))}(x)$ requires us first to bound the distance of a point $z$ to the set $h^{-1}(h(x))$ in terms of the function value $h(z)$. This leads us to the notion of "metric regularity". In this section we present a somewhat simplified version of this idea, which suffices for most of our purposes; we defer a more comprehensive treatment to a later section. We say $h$ is *weakly metrically regular* on $S$ at the point $x$ in $S$ if there is a real constant $k$ such that

$$d_{S \cap h^{-1}(h(x))}(z) \le k\|h(z) - h(x)\| \text{ for all } z \text{ in } S \text{ close to } x.$$

**Lemma 7.1.3** *Suppose* $0 \in S$ *and* $h(0) = 0$. *If* $h$ *is not weakly metrically regular on* $S$ *at zero then there is a sequence* $v_r \to 0$ *in* $S$ *such that* $h(v_r) \neq 0$ *for all* $r$, *and a strictly positive sequence* $\delta_r \downarrow 0$ *such that the function*

$$\|h(\cdot)\| + \delta_r\|\cdot - v_r\|$$

*is minimized on* $S$ *at* $v_r$.

**Proof.** By definition there is a sequence $x_r \to 0$ in $S$ such that

$$d_{S\cap h^{-1}(0)}(x_r) > r\|h(x_r)\| \text{ for all } r. \tag{7.1.4}$$

For each index $r$ we apply the Ekeland principle with

$$f = \|h\| + \delta_S, \quad \epsilon = \|h(x_r)\|, \quad \lambda = \min\{r\epsilon, \sqrt{\epsilon}\}, \quad \text{and } x = x_r$$

to deduce the existence of a point $v_r$ in $S$ such that

(a) $\|x_r - v_r\| \le \min\left\{r\|h(x_r)\|, \sqrt{\|h(x_r)\|}\right\}$ and

(c) $v_r$ minimizes the function

$$\|h(\cdot)\| + \max\left\{r^{-1}, \sqrt{\|h(x_r)\|}\right\} \|\cdot - v_r\|$$

on $S$.

Property (a) shows $v_r \to 0$, while (c) reveals the minimizing property of $v_r$. Finally, inequality (7.1.4) and property (a) prove $h(v_r) \neq 0$. □

We can now present a convenient condition for weak metric regularity.

**Theorem 7.1.5 (Surjectivity and metric regularity)** *If $h$ is strictly differentiable at the point $x$ in $S$ and*

$$\nabla h(x)(T_S(x)) = \mathbf{Y}$$

*then $h$ is weakly metrically regular on $S$ at $x$.*

**Proof.** Notice first $h$ is locally Lipschitz around $x$ (see Theorem 6.2.3). Without loss of generality, suppose $x = 0$ and $h(0) = 0$. If $h$ is not weakly metrically regular on $S$ at zero then by Lemma 7.1.3 there is a sequence $v_r \to 0$ in $S$ such that $h(v_r) \neq 0$ for all $r$, and a real sequence $\delta_r \downarrow 0$ such that the function

$$\|h(\cdot)\| + \delta_r\|\cdot - v_r\|$$

is minimized on $S$ at $v_r$. Denoting the local Lipschitz constant by $L$, we deduce from the sum rule (6.1.6) and the Exact penalization proposition (6.3.2) the condition

$$0 \in \partial_\circ(\|h\|)(v_r) + \delta_r B + L\partial_\circ d_S(v_r).$$

Hence there are elements $u_r$ of $\partial_\circ(\|h\|)(v_r)$ and $w_r$ of $L\partial_\circ d_S(v_r)$ such that $u_r + w_r$ approaches zero.

By choosing a subsequence we can assume

$$\|h(v_r)\|^{-1}h(v_r) \to y \neq 0$$

and an exercise then shows $u_r \to (\nabla h(0))^* y$. Since the Clarke subdifferential is closed at zero (Section 6.2, Exercise 12), we deduce

$$-(\nabla h(0))^* y \in L\partial_\circ d_S(0) \subset N_S(0).$$

However, by assumption there is a nonzero element $p$ of $T_S(0)$ such that $\nabla h(0)p = -y$, so we arrive at the contradiction

$$0 \geq \langle p, -(\nabla h(0))^* y\rangle = \langle \nabla h(0)p, -y\rangle = \|y\|^2 > 0,$$

which completes the proof. □

We can now prove the main result of this section.

**Theorem 7.1.6 (Liusternik)** *If $h$ is strictly differentiable at the point $x$ and $\nabla h(x)$ is surjective then the set $h^{-1}(h(x))$ is tangentially regular at $x$ and*

$$K_{h^{-1}(h(x))}(x) = N(\nabla h(x)).$$

**Proof.** Assume without loss of generality that $x = 0$ and $h(0) = 0$. In light of Proposition 7.1.1, it suffices to prove

$$N(\nabla h(0)) \subset T_{h^{-1}(0)}(0).$$

Fix any element $p$ of $N(\nabla h(0))$ and consider a sequence $x^r \to 0$ in $h^{-1}(0)$ and $t_r \downarrow 0$ in $\mathbf{R}_{++}$. The previous result shows $h$ is weakly metrically regular at zero, so there is a constant $k$ such that

$$d_{h^{-1}(0)}(x^r + t_r p) \leq k\|h(x^r + t_r p)\|$$

holds for all large $r$, and hence there are points $z^r$ in $h^{-1}(0)$ satisfying

$$\|x^r + t_r p - z^r\| \leq k\|h(x^r + t_r p)\|.$$

If we define directions $p^r = t_r^{-1}(z^r - x^r)$ then clearly the points $x^r + t_r p^r$ lie in $h^{-1}(0)$ for large $r$, and since

$$\begin{aligned} \|p - p^r\| &= \frac{\|x^r + t_r p - z^r\|}{t_r} \\ &\leq \frac{k\|h(x^r + t_r p) - h(x^r)\|}{t_r} \\ &\to k\|(\nabla h(0))p\| \\ &= 0, \end{aligned}$$

we deduce $p \in T_{h^{-1}(0)}(0)$. □

## Exercises and Commentary

Liusternik's original study of tangent spaces appeared in [130]. Closely related ideas were pursued by Graves [85] (see [65] for a good survey). The Ekeland principle first appeared in [69], motivated by the study of infinite-dimensional problems where techniques based on compactness might be unavailable. As we see in this section, it is a powerful idea even in finite dimensions; the simplified version we present here was observed in [94]. See also Exercise 14 in Section 9.2. The inversion technique we use (Lemma 7.1.3) is based on the approach in [101]. The recognition of "metric" regularity (a term perhaps best suited to nonsmooth analysis) as a central idea began largely with Robinson; see [162, 163] for example. Many equivalences are discussed in [5, 168].

1. Suppose $h$ is Fréchet differentiable at the point $x \in S$.

   (a) Prove for any set $D \supset h(S)$ the inclusion
   $$\nabla h(x) K_S(x) \subset K_D(h(x)).$$

   (b) If $h$ is constant on $S$, deduce
   $$K_S(x) \subset N(\nabla h(x)).$$

   (c) If $h$ is a real function and $x$ is a local minimizer of $h$ on $S$, prove
   $$-\nabla h(x) \in (K_S(x))^-.$$

2. **(Lipschitz extension)** Suppose the real function $f$ has Lipschitz constant $k$ on the set $C \subset \mathbf{E}$. By considering the infimal convolution of the functions $f + \delta_C$ and $k\|\cdot\|$, prove there is a function $\tilde{f} : \mathbf{E} \to \mathbf{R}$ with Lipschitz constant $k$ that agrees with $f$ on $C$. Prove furthermore that if $f$ and $C$ are convex then $\tilde{f}$ can be assumed convex.

3. * **(Closure and the Ekeland principle)** Given a subset $S$ of $\mathbf{E}$, suppose the conclusion of Ekeland's principle holds for all functions of the form $g + \delta_S$ where the function $g$ is continuous on $S$. Deduce $S$ is closed. (Hint: For any point $x$ in $\operatorname{cl} S$, let $g = \|\cdot - x\|$.)

4. ** Suppose $h$ is strictly differentiable at zero and satisfies
   $$h(0) = 0,\ v_r \to 0,\ \|h(v_r)\|^{-1} h(v_r) \to y, \text{ and } u_r \in \partial_\circ(\|h\|)(v_r).$$
   Prove $u_r \to (\nabla h(0))^* y$. Write out a shorter proof when $h$ is continuously differentiable at zero.

5. ** Interpret Exercise 27 (Conical open mapping) in Section 4.2 in terms of metric regularity.

6. ** **(Transversality)** Suppose the set $V \subset \mathbf{Y}$ is open and the set $R \subset V$ is closed. Suppose furthermore $h$ is strictly differentiable at the point $x$ in $S$ with $h(x)$ in $R$ and

$$\nabla h(x)(T_S(x)) - T_R(h(x)) = \mathbf{Y}. \tag{7.1.7}$$

   (a) Define the function $g : U \times V \to \mathbf{Y}$ by $g(z, y) = h(z) - y$. Prove $g$ is weakly metrically regular on $S \times R$ at the point $(x, h(x))$.

   (b) Deduce the existence of a constant $k'$ such that the inequality

$$d_{(S\times R)\cap g^{-1}(g(x,h(x)))}(z, y) \le k'\|h(z) - y\|$$

   holds for all points $(z, y)$ in $S \times R$ close to $(x, h(x))$.

   (c) Apply Proposition 6.3.2 (Exact penalization) to deduce the existence of a constant $k$ such that the inequality

$$d_{(S\times R)\cap g^{-1}(g(x,h(x)))}(z, y) \le k(\|h(z) - y\| + d_S(z) + d_R(y))$$

   holds for all points $(z, y)$ in $U \times V$ close to $(x, h(x))$.

   (d) Deduce the inequality

$$d_{S\cap h^{-1}(R)}(z) \le k(d_S(z) + d_R(h(z)))$$

   holds for all points $z$ in $U$ close to $x$.

   (e) Imitate the proof of Liusternik's theorem (7.1.6) to deduce the inclusions

$$T_{S\cap h^{-1}(R)}(x) \supset T_S(x) \cap (\nabla h(x))^{-1} T_R(h(x))$$

   and

$$K_{S\cap h^{-1}(R)}(x) \supset K_S(x) \cap (\nabla h(x))^{-1} T_R(h(x)).$$

   (f) Suppose $h$ is the identity map, so

$$T_S(x) - T_R(x) = \mathbf{E}.$$

   If either $R$ or $S$ is tangentially regular at $x$, prove

$$K_{R\cap S}(x) = K_R(x) \cap K_S(x).$$

   (g) **(Guignard)** By taking polars and applying the Krein–Rutman polar cone calculus (3.3.13) and condition (7.1.7) again, deduce

$$N_{S\cap h^{-1}(R)}(x) \subset N_S(x) + (\nabla h(x))^* N_R(h(x)).$$

(h) If $C$ and $D$ are convex subsets of $\mathbf{E}$ satisfying $0 \in \operatorname{core}(C - D)$ (or $\operatorname{ri} C \cap \operatorname{ri} D \neq \emptyset$), and the point $x$ lies in $C \cap D$, use part (e) to prove

$$T_{C \cap D}(x) = T_C(x) \cap T_D(x).$$

7. ** **(Liusternik via inverse functions)** We first fix $\mathbf{E} = \mathbf{R}^n$. The classical inverse function theorem states that if the map $g : U \to \mathbf{R}^n$ is continuously differentiable then at any point $x$ in $U$ at which $\nabla g(x)$ is invertible, $x$ has an open neighbourhood $V$ whose image $g(V)$ is open, and the restricted map $g|_V$ has a continuously differentiable inverse satisfying the condition

$$\nabla \left(g|_V\right)^{-1}(g(x)) = (\nabla g(x))^{-1}.$$

Consider now a continuously differentiable map $h : U \to \mathbf{R}^m$, and a point $x$ in $U$ with $\nabla h(x)$ surjective, and fix a direction $d$ in the null space $N(\nabla h(x))$. Choose any $(n \times (n - m))$ matrix $D$ making the matrix $A = (\nabla h(x), D)$ invertible, define a function $g : U \to \mathbf{R}^n$ by $g(z) = (h(z), Dz)$, and for a small real $\delta > 0$ define a function $p : (-\delta, \delta) \to \mathbf{R}^n$ by

$$p(t) = g^{-1}(g(x) + tAd).$$

(a) Prove $p$ is well-defined providing $\delta$ is small.

(b) Prove the following properties:

(i) $p$ is continuously differentiable.

(ii) $p(0) = x$.

(iii) $p'(0) = d$.

(iv) $h(p(t)) = h(x)$ for all small $t$.

(c) Deduce that a direction $d$ lies in $N(\nabla h(x))$ if and only if there is a function $p : (-\delta, \delta) \to \mathbf{R}^n$ for some $\delta > 0$ in $\mathbf{R}$ satisfying the four conditions in part (b).

(d) Deduce $K_{h^{-1}(h(x))}(x) = N(\nabla h(x))$.

## 7.2 The Karush–Kuhn–Tucker Theorem

The central result of optimization theory describes first order necessary optimality conditions for the general nonlinear problem

$$\inf\{f(x) \mid x \in S\}, \tag{7.2.1}$$

where, given an open set $U \subset \mathbf{E}$, the objective function is $f : U \to \mathbf{R}$ and the feasible region $S$ is described by equality and inequality constraints:

$$S = \{x \in U \mid g_i(x) \leq 0 \text{ for } i = 1, 2, \ldots, m,\ h(x) = 0\}. \tag{7.2.2}$$

The equality constraint map $h : U \to \mathbf{Y}$ (where $\mathbf{Y}$ is a Euclidean space) and the inequality constraint functions $g_i : U \to \mathbf{R}$ (for $i = 1, 2, \ldots, m$) are all continuous. In this section we derive necessary conditions for the point $\bar{x}$ in $S$ to be a local minimizer for the problem (7.2.1).

In outline, the approach takes three steps. We first extend Liusternik's theorem (7.1.6) to describe the contingent cone $K_S(\bar{x})$. Next we calculate this cone's polar cone using the Farkas lemma (2.2.7). Finally, we apply the Contingent necessary condition (6.3.10) to derive the result.

As in our development for the inequality-constrained problem in Section 2.3, we need a regularity condition. Once again, we denote the set of indices of the active inequality constraints by $I(\bar{x}) = \{i \mid g_i(\bar{x}) = 0\}$.

**Assumption 7.2.3 (The Mangasarian–Fromovitz constraint qualification)** *The active constraint functions $g_i$ (for $i$ in $I(\bar{x})$) are Fréchet differentiable at the point $\bar{x}$, the equality constraint map $h$ is strictly differentiable, with a surjective gradient, at $\bar{x}$, and the set*

$$\{p \in N(\nabla h(\bar{x})) \mid \langle \nabla g_i(\bar{x}), p\rangle < 0 \textit{ for } i \textit{ in } I(\bar{x})\} \tag{7.2.4}$$

*is nonempty.*

Notice in particular that the set (7.2.4) is nonempty in the case where the map $h : U \to \mathbf{R}^q$ has components $h_1, h_2, \ldots, h_q$ and the set of gradients

$$\{\nabla h_j(\bar{x}) \mid j = 1, 2, \ldots, q\} \cup \{\nabla g_i(\bar{x}) \mid i \in I(\bar{x})\} \tag{7.2.5}$$

is linearly independent (Exercise 1).

**Theorem 7.2.6** *Suppose the Mangasarian–Fromovitz constraint qualification (7.2.3) holds. Then the contingent cone to the feasible region $S$ defined by equation (7.2.2) is given by*

$$K_S(\bar{x}) = \{p \in N(\nabla h(\bar{x})) \mid \langle \nabla g_i(\bar{x}), p\rangle \leq 0 \textit{ for } i \textit{ in } I(\bar{x})\}. \tag{7.2.7}$$

**Proof.** Denote the set (7.2.4) by $\widetilde{K}$ and the right hand side of formula (7.2.7) by $K$. The inclusion

$$K_S(\bar{x}) \subset K$$

is a straightforward exercise. Furthermore, since $\widetilde{K}$ is nonempty, it is easy to see $K = \operatorname{cl}\widetilde{K}$. If we can show $\widetilde{K} \subset K_S(\bar{x})$ then the result will follow since the contingent cone is always closed.

To see $\widetilde{K} \subset K_S(\bar{x})$, fix an element $p$ of $\widetilde{K}$. Since $p$ lies in $N(\nabla h(\bar{x}))$, Liusternik's theorem (7.1.6) shows $p \in K_{h^{-1}(0)}(\bar{x})$. Hence there are sequences $t_r \downarrow 0$ in $\mathbf{R}_{++}$ and $p^r \to p$ in $\mathbf{E}$ satisfying $h(\bar{x} + t_r p^r) = 0$ for all $r$. Clearly $\bar{x} + t_r p^r \in U$ for all large $r$, and we claim $g_i(\bar{x} + t_r p^r) < 0$. For indices $i$ not in $I(\bar{x})$ this follows by continuity, so we suppose $i \in I(\bar{x})$ and $g_i(\bar{x} + t_r p^r) \geq 0$ for all $r$ in some subsequence $R$ of $\mathbf{N}$. We then obtain the contradiction

$$\begin{aligned} 0 &= \lim_{r\to\infty \text{ in } R} \frac{g_i(\bar{x} + t_r p^r) - g_i(\bar{x}) - \langle \nabla g_i(\bar{x}), t_r p^r \rangle}{t_r \|p^r\|} \\ &\geq -\frac{\langle \nabla g_i(\bar{x}), p \rangle}{\|p\|} \\ &> 0. \end{aligned}$$

The result now follows. □

**Lemma 7.2.8** *Any linear maps $A : \mathbf{E} \to \mathbf{R}^q$ and $G : \mathbf{E} \to \mathbf{Y}$ satisfy*

$$\{x \in N(G) \mid Ax \leq 0\}^- = A^*\mathbf{R}^q_+ + G^*\mathbf{Y}.$$

**Proof.** This is an immediate application of Section 5.1, Exercise 9 (Polyhedral cones). □

**Theorem 7.2.9 (Karush–Kuhn–Tucker conditions)** *Suppose $\bar{x}$ is a local minimizer for problem (7.2.1) and the objective function $f$ is Fréchet differentiable at $\bar{x}$. If the Mangasarian–Fromovitz constraint qualification (7.2.3) holds then there exist multipliers $\lambda_i$ in $\mathbf{R}_+$ (for $i$ in $I(\bar{x})$) and $\mu$ in $\mathbf{Y}$ satisfying*

$$\nabla f(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}) + \nabla h(\bar{x})^*\mu = 0. \tag{7.2.10}$$

**Proof.** The Contingent necessary condition (6.3.10) shows

$$\begin{aligned} -\nabla f(\bar{x}) &\in K_S(\bar{x})^- \\ &= \{p \in N(\nabla h(\bar{x})) \mid \langle \nabla g_i(\bar{x}), p \rangle \leq 0 \text{ for } i \text{ in } I(\bar{x})\}^- \\ &= \sum_{i\in I(\bar{x})} \mathbf{R}_+ \nabla g_i(\bar{x}) + \nabla h(\bar{x})^*\mathbf{Y} \end{aligned}$$

using Theorem 7.2.6 and Lemma 7.2.8. □

## Exercises and Commentary

A survey of the history of these results may be found in [158]. The Mangasarian–Fromovitz condition originated with [133], while the Karush–Kuhn–Tucker conditions first appeared in [111] and [117]. The idea of penalty functions (see Exercise 11 (Quadratic penalties)) is a common technique in optimization. The related notion of a barrier penalty is crucial for interior point methods; examples include the penalized linear and semidefinite programs we considered in Section 4.3, Exercise 4 (Examples of duals).

1. **(Linear independence implies Mangasarian–Fromovitz)** If the set of gradients (7.2.5) is linearly independent, then by considering the equations

$$\begin{aligned}\langle \nabla g_i(\bar{x}), p\rangle &= -1 \ \text{ for } i \text{ in } I(\bar{x})\\ \langle \nabla h_j(\bar{x}), p\rangle &= 0 \ \text{ for } j = 1, 2, \ldots, q,\end{aligned}$$

   prove the set (7.2.4) is nonempty.

2. Consider the proof of Theorem 7.2.6.

   (a) Prove $K_S(\bar{x}) \subset K$.

   (b) If $\widetilde{K}$ is nonempty, prove $K = \operatorname{cl} \widetilde{K}$.

3. **(Linear constraints)** If the functions $g_i$ (for $i$ in $I(\bar{x})$) and $h$ are affine, prove the contingent cone formula (7.2.7) holds.

4. **(Bounded multipliers)** In Theorem 7.2.9 (Karush–Kuhn–Tucker conditions), prove the set of multiplier vectors $(\lambda, \mu)$ satisfying equation (7.2.10) is compact.

5. **(Slater condition)** Suppose the set $U$ is convex, the functions

$$g_1, g_2, \ldots, g_m : U \to \mathbf{R}$$

   are convex and Fréchet differentiable, and the function $h : \mathbf{E} \to \mathbf{Y}$ is affine and surjective. Suppose further there is a point $\hat{x}$ in $h^{-1}(0)$ satisfying $g_i(\hat{x}) < 0$ for $i = 1, 2, \ldots, m$. For any feasible point $\bar{x}$ for problem (7.2.1), prove the Mangasarian–Fromovitz constraint qualification holds.

6. **(Largest eigenvalue)** For a matrix $A$ in $\mathbf{S}^n$, use the Karush–Kuhn–Tucker theorem to calculate

$$\sup\{x^T A x \mid \|x\| = 1,\ x \in \mathbf{R}^n\}.$$

7. * **(Largest singular value [100, p. 135])** Given any $m \times n$ matrix $A$, consider the optimization problem

$$\alpha = \sup\{x^T A y \mid \|x\|^2 = 1,\ \|y\|^2 = 1\} \tag{7.2.11}$$

and the matrix

$$\widetilde{A} = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}.$$

(a) If $\mu$ is an eigenvalue of $\widetilde{A}$, prove $-\mu$ is also.

(b) If $\mu$ is a nonzero eigenvalue of $\widetilde{A}$, use a corresponding eigenvector to construct a feasible solution to problem (7.2.11) with objective value $\mu$.

(c) Deduce $\alpha \geq \lambda_1(\widetilde{A})$.

(d) Prove problem (7.2.11) has an optimal solution.

(e) Use the Karush–Kuhn–Tucker theorem to prove any optimal solution of problem (7.2.11) corresponds to an eigenvector of $\widetilde{A}$.

(f) **(Jordan [108])** Deduce $\alpha = \lambda_1(\widetilde{A})$. (This number is called the *largest singular value of $A$*.)

8. ** **(Hadamard's inequality [88])** The matrix with columns $x^1, x^2, \ldots, x^n$ in $\mathbf{R}^n$ we denote by $(x^1, x^2, \ldots, x^n)$. Prove $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ solves the problem

$$\begin{array}{lrcl} \inf & -\det(x^1, x^2, \ldots, x^n) & & \\ \text{subject to} & \|x^i\|^2 & = & 1 \ \text{ for } i = 1, 2, \ldots, n \\ & x^1, x^2, \ldots, x^n & \in & \mathbf{R}^n \end{array}$$

if and only if the matrix $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$ has determinant equal to one and has columns forming an orthonormal basis, and deduce the inequality

$$\det(x^1, x^2, \ldots, x^n) \leq \prod_{i=1}^{n} \|x^i\|.$$

9. **(Nonexistence of multipliers [77])** Define a function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ by

$$\mathrm{sgn}(v) = \begin{cases} 1 & \text{if } v > 0 \\ 0 & \text{if } v = 0 \\ -1 & \text{if } v < 0 \end{cases}$$

and a function $h : \mathbf{R}^2 \to \mathbf{R}$ by

$$h(u, v) = v - \mathrm{sgn}(v)(u^+)^2.$$

(a) Prove $h$ is Fréchet differentiable at $(0, 0)$ with derivative $(0, 1)$.

(b) Prove $h$ is not continuous on any neighbourhood of $(0,0)$, and deduce it is not strictly differentiable at $(0,0)$.

(c) Prove $(0,0)$ is optimal for the problem

$$\inf\{f(u,v) \mid h(u,v) = 0\},$$

where $f(u,v) = u$, and yet there is no real $\lambda$ satisfying

$$\nabla f(0,0) + \lambda \nabla h(0,0) = (0,0).$$

(Exercise 14 in Section 8.1 gives an approach to weakening the conditions required in this section.)

10. * **(Guignard optimality conditions [87])** Suppose the point $\bar{x}$ is a local minimizer for the optimization problem

$$\inf\{f(x) \mid h(x) \in R,\ x \in S\}$$

where $R \subset \mathbf{Y}$. If the functions $f$ and $h$ are strictly differentiable at $\bar{x}$ and the transversality condition

$$\nabla h(\bar{x}) T_S(\bar{x}) - T_R(h(\bar{x})) = \mathbf{Y}$$

holds, use Section 7.1, Exercise 6 (Transversality) to prove the optimality condition

$$0 \in \nabla f(\bar{x}) + \nabla h(\bar{x})^* N_R(h(\bar{x})) + N_S(\bar{x}).$$

11. ** **(Quadratic penalties [136])** Take the nonlinear program (7.2.1) in the case $\mathbf{Y} = \mathbf{R}^q$ and now let us assume all the functions

$$f, g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_q : U \to \mathbf{R}$$

are continuously differentiable on the set $U$. For positive integers $k$ we define a function $p_k : U \to \mathbf{R}$ by

$$p_k(x) = f(x) + k\Big(\sum_{i=1}^{m} (g_i^+(x))^2 + \sum_{j=1}^{q} (h_j(x))^2\Big).$$

Suppose the point $\bar{x}$ is a local minimizer for the problem (7.2.1). Then for some compact neighbourhood $W$ of $\bar{x}$ in $U$ we know $f(x) \geq f(\bar{x})$ for all feasible points $x$ in $W$. Now define a function $r_k : W \to \mathbf{R}$ by

$$r_k(x) = p_k(x) + \|x - \bar{x}\|^2,$$

and for each $k = 1, 2, \ldots$ choose a point $x^k$ minimizing $r_k$ on $W$.

(a) Prove $r_k(x^k) \le f(\bar{x})$ for each $k = 1, 2, \ldots$.

(b) Deduce
$$\lim_{k\to\infty} g_i^+(x^k) = 0 \quad \text{for } i = 1, 2, \ldots, m$$
and
$$\lim_{k\to\infty} h_j(x^k) = 0 \quad \text{for } j = 1, 2, \ldots, q.$$

(c) Hence show $x^k \to \bar{x}$ as $k \to \infty$.

(d) Calculate $\nabla r_k(x)$.

(e) Deduce
$$-2(x^k - \bar{x}) = \nabla f(x^k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(x^k) + \sum_{j=1}^{q} \mu_j^k \nabla h_j(x^k)$$
for some suitable choice of vectors $\lambda^k$ in $\mathbf{R}_+^m$ and $\mu^k$ in $\mathbf{R}^q$.

(f) By taking a convergent subsequence of the vectors
$$\|(1, \lambda^k, \mu^k)\|^{-1}(1, \lambda^k, \mu^k) \in \mathbf{R} \times \mathbf{R}_+^m \times \mathbf{R}^q,$$
show from parts (c) and (e) the existence of a nonzero vector $(\lambda_0, \lambda, \mu)$ in $\mathbf{R} \times \mathbf{R}_+^m \times \mathbf{R}^q$ satisfying the *Fritz John conditions*:

(i) $\lambda_i g_i(\bar{x}) = 0$ for $i = 1, 2, \ldots, m$.

(ii) $\lambda_0 \nabla f(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) + \sum_{j=1}^q \mu_j \nabla h_j(\bar{x}) = 0$.

(g) Under the assumption of the Mangasarian–Fromovitz constraint qualification (7.2.3), show that the Fritz John conditions in part (f) imply the Karush–Kuhn–Tucker conditions.

## 7.3 Metric Regularity and the Limiting Subdifferential

In Section 7.1 we presented a convenient test for the weak metric regularity of a function at a point in terms of the surjectivity of its strict derivative there (Theorem 7.1.5). This test, while adequate for most of our purposes, can be richly refined using the limiting subdifferential.

As before, we consider an open set $U \subset \mathbf{E}$, a Euclidean space $\mathbf{Y}$, a closed set $S \subset U$, and a function $h : U \to \mathbf{Y}$ which we assume throughout this section is locally Lipschitz. We begin with the full definition of metric regularity, strengthening the notion of Section 7.1. We say $h$ is *metrically regular on $S$ at* the point $x$ in $S$ if there is a real constant $k$ such that the estimate

$$d_{S \cap h^{-1}(y)}(z) \leq k\|h(z) - y\|$$

holds for all points $z$ in $S$ close to $x$ and all vectors $y$ in $\mathbf{Y}$ close to $h(x)$. (Before we only required this to be true when $y = h(x)$.)

**Lemma 7.3.1** *If $h$ is not metrically regular on $S$ at $x$ then there are sequences $(v_r)$ in $S$ converging to $x$, $(y_r)$ in $\mathbf{Y}$ converging to $h(x)$, and $(\epsilon_r)$ in $\mathbf{R}_{++}$ decreasing to zero such that, for each index $r$, we have $h(v_r) \neq y_r$ and the function*

$$\|h(\cdot) - y_r\| + \epsilon_r\| \cdot - v_r\|$$

*is minimized on $S$ at $v_r$.*

**Proof.** The proof is completely analogous to that of Lemma 7.1.3: we leave it as an exercise. □

We also need the following chain-rule-type result; we leave the proof as an exercise.

**Lemma 7.3.2** *At any point $x$ in $\mathbf{E}$ where $h(x) \neq 0$ we have*

$$\partial_a \|h(\cdot)\|(x) = \partial_a \langle \|h(x)\|^{-1} h(x), h(\cdot)\rangle(x).$$

Using this result and a very similar proof to Theorem 7.1.5, we can now extend the surjectivity and metric regularity result.

**Theorem 7.3.3 (Limiting subdifferential and regularity)** *If a point $x$ lies in $S$ and no nonzero element $w$ of $\mathbf{Y}$ satisfies the condition*

$$0 \in \partial_a \langle w, h(\cdot)\rangle(x) + N_S^a(x)$$

*then $h$ is metrically regular on $S$ at $x$.*

# Chapter 8

# Fixed Points

## 8.1 The Brouwer Fixed Point Theorem

Many questions in optimization and analysis reduce to solving a nonlinear equation $h(x) = 0$, for some function $h : \mathbf{E} \to \mathbf{E}$. Equivalently, if we define another map $f = I - h$ (where $I$ is the identity map), we seek a point $x$ in $\mathbf{E}$ satisfying $f(x) = x$; we call $x$ a *fixed point* of $f$.

The most potent fixed point existence theorems fall into three categories: "geometric" results, devolving from the Banach contraction principle (which we state below), "order-theoretic" results (to which we briefly return in Section 8.3), and "topological" results, for which the prototype is the theorem of Brouwer forming the main body of this section. We begin with Banach's result.

Given a set $C \subset \mathbf{E}$ and a continuous *self map* $f : C \to C$, we ask whether $f$ has a fixed point. We call $f$ a *contraction* if there is a real constant $\gamma_f < 1$ such that

$$\|f(x) - f(y)\| \le \gamma_f \|x - y\| \text{ for all } x, y \in C. \tag{8.1.1}$$

**Theorem 8.1.2 (Banach contraction)** *Any contraction on a closed subset of* $\mathbf{E}$ *has a unique fixed point.*

**Proof.** Suppose the set $C \subset \mathbf{E}$ is closed and the function $f : C \to C$ satisfies the contraction condition (8.1.1). We apply the Ekeland variational principle (7.1.2) to the function

$$z \in \mathbf{E} \mapsto \begin{cases} \|z - f(z)\| & \text{if } z \in C \\ +\infty & \text{otherwise} \end{cases}$$

at an arbitrary point $x$ in $C$, with the choice of constants

$$\epsilon = \|x - f(x)\| \text{ and } \lambda = \frac{\epsilon}{1 - \gamma_f}.$$

This shows there is a point $v$ in $C$ satisfying

$$\|v - f(v)\| < \|z - f(z)\| + (1 - \gamma_f)\|z - v\|$$

for all points $z \neq v$ in $C$. Hence $v$ is a fixed point, since otherwise choosing $z = f(v)$ gives a contradiction. The uniqueness is easy. □

What if the map $f$ is not a contraction? A very useful weakening of the notion is the idea of a *nonexpansive* map, which is to say a self map $f$ satisfying

$$\|f(x) - f(y)\| \leq \|x - y\| \text{ for all } x, y$$

(see Exercise 2). A nonexpansive map on a nonempty compact set or a nonempty closed convex set may not have a fixed point, as simple examples like translations on $\mathbf{R}$ or rotations of the unit circle show. On the other hand, a straightforward argument using the Banach contraction theorem shows this cannot happen if the set is nonempty, compact, *and* convex. However, in this case we have the following more fundamental result.

**Theorem 8.1.3 (Brouwer)** *Any continuous self map of a nonempty compact convex subset of* $\mathbf{E}$ *has a fixed point.*

In this section we present an "analyst's approach" to Brouwer's theorem. We use the two following important analytic tools concerning $C^{(1)}$ (continuously differentiable) functions on the closed unit ball $B \subset \mathbf{R}^n$.

**Theorem 8.1.4 (Stone–Weierstrass)** *For any continuous map* $f : B \to \mathbf{R}^n$*, there is a sequence of* $C^{(1)}$ *maps* $f_r : B \to \mathbf{R}^n$ *converging uniformly to* $f$.

An easy exercise shows that, in this result, if $f$ is a self map then we can assume each $f_r$ is also a self map.

**Theorem 8.1.5 (Change of variable)** *Suppose that the set* $W \subset \mathbf{R}^n$ *is open and that the* $C^{(1)}$ *map* $g : W \to \mathbf{R}^n$ *is one-to-one with* $\nabla g$ *invertible throughout* $W$*. Then the set* $g(W)$ *is open with measure*

$$\int_W |\det \nabla g|.$$

We also use the elementary topological fact that the open unit ball $\operatorname{int} B$ is *connected*; that is, it cannot be written as the disjoint union of two nonempty open sets.

The key step in our argument is the following topological result.

**Theorem 8.1.6 (Retraction)** *The unit sphere* $S$ *is not a* $C^{(1)}$ **retract** *of the unit ball* $B$*; that is, there is no* $C^{(1)}$ *map from* $B$ *to* $S$ *whose restriction to* $S$ *is the identity.*

**Proof.** Suppose there is such a retraction map $p : B \to S$. For real $t$ in $[0,1]$, define a self map of $B$ by $p_t = tp + (1-t)I$. As a function of the variables $x \in B$ and $t$, the function $\det \nabla p_t(x)$ is continuous and hence strictly positive for small $t$. Furthermore, $p_t$ is one-to-one for small $t$ (Exercise 7).

If we denote the open unit ball $B \setminus S$ by $U$, then the change of variables theorem above shows for small $t$ that $p_t(U)$ is open with measure

$$\nu(t) = \int_U \det \nabla p_t. \tag{8.1.7}$$

On the other hand, by compactness, $p_t(B)$ is a closed subset of $B$, and we also know $p_t(S) = S$. A little manipulation now shows we can write $U$ as a disjoint union of two open sets:

$$U = (p_t(U) \cap U) \cup (p_t(B)^c \cap U). \tag{8.1.8}$$

The first set is nonempty, since $p_t(0) = tp(0) \in U$. But as we observed, $U$ is connected, so the second set must be empty, which shows $p_t(B) = B$. Thus the function $\nu(t)$ defined by equation (8.1.7) equals the volume of the unit ball $B$ for all small $t$.

However, as a function of $t \in [0,1]$, $\nu(t)$ is a polynomial, so it must be constant. Since $p$ is a retraction we know that all points $x$ in $U$ satisfy $\|p(x)\|^2 = 1$. Differentiating implies $(\nabla p(x))p(x) = 0$, from which we deduce $\det \nabla p(x) = 0$, since $p(x)$ is nonzero. Thus $\nu(1)$ is zero, which is a contradiction. $\square$

**Proof of Brouwer's theorem.** Consider first a $C^{(1)}$ self map $f$ on the unit ball $B$. Suppose $f$ has no fixed point. A straightforward exercise shows there are unique functions $\alpha : B \to \mathbf{R}_+$ and $p : B \to S$ satisfying the relationship

$$p(x) = x + \alpha(x)(x - f(x)) \text{ for all } x \text{ in } B. \tag{8.1.9}$$

Geometrically, $p(x)$ is the point where the line extending from the point $f(x)$ through the point $x$ meets the unit sphere $S$. In fact $p$ must then be a $C^{(1)}$ retraction, contradicting the retraction theorem above. Thus we have proved that any $C^{(1)}$ self map of $B$ has a fixed point.

Now suppose the function $f$ is just continuous. The Stone–Weierstrass theorem (8.1.4) implies there is a sequence of $C^{(1)}$ maps $f_r : B \to \mathbf{R}^n$ converging uniformly to $f$, and by Exercise 4 we can assume each $f_r$ is a self map. Our argument above shows each $f_r$ has a fixed point $x^r$. Since $B$ is compact, the sequence $(x^r)$ has a subsequence converging to some point $x$ in $B$, which it is easy to see must be a fixed point of $f$. So any continuous self map of $B$ has a fixed point.

Finally, consider a nonempty compact convex set $C \subset \mathbf{E}$ and a continuous self map $g$ on $C$. Just as in our proof of Minkowski's theorem (4.1.8), we may as well assume $C$ has nonempty interior. Thus there is a *homeomorphism* (a continuous onto map with continuous inverse) $h : C \to B$ (see Exercise 11). Since the function $h \circ g \circ h^{-1}$ is a continuous self map of $B$, our argument above shows this function has a fixed point $x$ in $B$, and therefore $h^{-1}(x)$ is a fixed point of $g$. □

## Exercises and Commentary

Good general references on fixed point theory are [68, 174, 83]. The Banach contraction principle appeared in [7]. Brouwer proved the three-dimensional case of his theorem in 1909 [49] and the general case in 1912 [50], with another proof by Hadamard in 1910 [89]. A nice exposition of the Stone–Weierstrass theorem may be found in [16], for example. The Change of variable theorem (8.1.5) we use can be found in [177]; a beautiful proof of a simplified version, also sufficient to prove Brouwer's theorem, appeared in [118]. Ulam conjectured and Borsuk proved their result in 1933 [17].

1. **(Banach iterates)** Consider a closed subset $C \subset \mathbf{E}$ and a contraction $f : C \to C$ with fixed point $x^f$. Given any point $x_0$ in $C$, define a sequence of points inductively by

$$x_{r+1} = f(x_r) \quad \text{for } r = 0, 1, \ldots.$$

   (a) Prove $\lim_{r,s\to\infty} \|x_r - x_s\| = 0$. Since $\mathbf{E}$ is *complete*, the sequence $(x_r)$ converges. (Another approach first shows $(x_r)$ is bounded.) Hence prove in fact $x_r$ approaches $x^f$. Deduce the Banach contraction theorem.

   (b) Consider another contraction $g : C \to C$ with fixed point $x^g$. Use part (a) to prove the inequality

$$\|x^f - x^g\| \le \frac{\sup_{z\in C} \|f(z) - g(z)\|}{1 - \gamma_f}.$$

2. **(Nonexpansive maps)**

   (a) If the $n \times n$ matrix $U$ is orthogonal, prove the map $x \in \mathbf{R}^n \to Ux$ is nonexpansive.

   (b) If the set $S \subset \mathbf{E}$ is closed and convex then for any real $\lambda$ in the interval $[0, 2]$ prove the *relaxed projection*

$$x \in \mathbf{E} \mapsto (1 - \lambda)x + \lambda P_S(x)$$

   is nonexpansive. (Hint: Use the nearest point characterization in Section 2.1, Exercise 8(c).)

(c) **(Browder–Kirk [51, 112])** Suppose the set $C \subset \mathbf{E}$ is compact and convex and the map $f : C \to C$ is nonexpansive. Prove $f$ has a fixed point. (Hint: Choose an arbitrary point $x$ in $C$ and consider the contractions

$$z \in C \mapsto (1-\epsilon)f(z) + \epsilon x$$

for small real $\epsilon > 0$.)

(d)* In part (c), prove the fixed points form a nonempty compact convex set.

3. **(Non-uniform contractions)**

(a) Consider a nonempty compact set $C \subset \mathbf{E}$ and a self map $f$ on $C$ satisfying the condition

$$\|f(x) - f(y)\| < \|x - y\| \quad \text{for all distinct } x, y \in C.$$

By considering $\inf \|x - f(x)\|$, prove $f$ has a unique fixed point.

(b) Show the result in part (a) can fail if $C$ is unbounded.

(c) Prove the map $x \in [0,1] \mapsto xe^{-x}$ satisfies the condition in part (a).

4. In the Stone–Weierstrass theorem, prove that if $f$ is a self map then we can assume each $f_r$ is also a self map.

5. Prove the interval $(-1, 1)$ is connected. Deduce the open unit ball in $\mathbf{R}^n$ is connected.

6. In the Change of variable theorem (8.1.5), use metric regularity to prove the set $g(W)$ is open.

7. In the proof of the Retraction theorem (8.1.6), prove the map $p$ is Lipschitz, and deduce that the map $p_t$ is one-to-one for small $t$. Also prove that if $t$ is small then $\det \nabla p_t$ is strictly positive throughout $B$.

8. In the proof of the Retraction theorem (8.1.6), prove the partition (8.1.8), and deduce $p_t(B) = B$.

9. In the proof of the Retraction theorem (8.1.6), prove $\nu(t)$ is a polynomial in $t$.

10. In the proof of Brouwer's theorem, prove the relationship (8.1.9) defines a $C^{(1)}$ retraction $p : B \to S$.

11. **(Convex sets homeomorphic to the ball)** Suppose the compact convex set $C \subset \mathbf{E}$ satisfies $0 \in \operatorname{int} C$. Prove that the map $h : C \to B$ defined by
$$h(x) = \begin{cases} \gamma_C(x)\|x\|^{-1}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
(where $\gamma_C$ is the gauge function we defined in Section 4.1) is a homeomorphism.

12. * **(A nonclosed nonconvex set with the fixed point property)** Let $Z$ be the subset of the unit disk in $\mathbf{R}^2$ consisting of all lines through the origin with rational slope. Prove every continuous self map of $Z$ has a fixed point.

13. * **(Change of variable and Brouwer)** A very simple proof may be found in [118] of the formula
$$\int (f \circ g)|\nabla g| = \int f$$
when the function $f$ is continuous with bounded support and the function $g$ is differentiable, equaling the identity outside a large ball. Prove any such $g$ is surjective by considering an $f$ supported outside the range of $g$ (which is closed). Deduce Brouwer's theorem.

14. ** **(Brouwer and inversion)** The central tool of the last chapter, the Surjectivity and metric regularity theorem (7.1.5), considers a function $h$ whose *strict* derivative at a point satisfies a certain surjectivity condition. In this exercise, which comes out of a long tradition, we use Brouwer's theorem to consider functions $h$ which are merely *Fréchet* differentiable. This exercise proves the following result.

**Theorem 8.1.10** *Consider an open set $U \subset \mathbf{E}$, a closed convex set $S \subset U$, and a Euclidean space $\mathbf{Y}$, and suppose the continuous function $h : U \to \mathbf{Y}$ has Fréchet derivative at the point $x \in S$ satisfying the surjectivity condition*
$$\nabla h(x) T_S(x) = \mathbf{Y}.$$
*Then there is a neighbourhood $V$ of $h(x)$, a continuous, piecewise linear function $F : \mathbf{Y} \to \mathbf{E}$, and a function $g : V \to \mathbf{Y}$ that is Fréchet differentiable at $h(x)$ and satisfies $(F \circ g)(V) \subset S$ and*
$$h((F \circ g)(y)) = y \quad \textit{for all } y \in V.$$

**Proof.** We can assume $x = 0$ and $h(0) = 0$.

(a) Use Section 4.1, Exercise 20 (Properties of the relative interior) to prove $\nabla h(0)(\mathbf{R}_+ S) = \mathbf{Y}$.

(b) Deduce that there exists a basis $y_1, y_2, \ldots, y_n$ of $\mathbf{Y}$ and points $u_1, u_2, \ldots, u_n$ and $w_1, w_2, \ldots, w_n$ in $S$ satisfying

$$\nabla h(0)u_i = y_i = -\nabla h(0)w_i \quad \text{for } i = 1, 2, \ldots, n.$$

(c) Prove the set

$$B_1 = \Big\{ \sum_1^n t_i y_i \;\Big|\; t \in \mathbf{R}^n, \; \sum_1^n |t_i| \le 1 \Big\}$$

and the function $F$ defined by

$$F\Big(\sum_1^n t_i y_i\Big) = \sum_1^n \big(t_i^+ u_i + (-t_i)^+ w_i\big)$$

satisfy $F(B_1) \subset S$ and $\nabla(h \circ F)(0) = I$.

(d) Deduce there exists a real $\epsilon > 0$ such that $\epsilon B_{\mathbf{Y}} \subset B_1$ and

$$\|h(F(y)) - y\| \le \frac{\|y\|}{2} \quad \text{whenever } \|y\| \le 2\epsilon.$$

(e) For any point $v$ in the neighbourhood $V = (\epsilon/2)B_{\mathbf{Y}}$, prove the map

$$y \in V \mapsto v + y - h(F(y))$$

is a continuous self map of $V$.

(f) Apply Brouwer's theorem to deduce the existence of a fixed point $g(v)$ for the map in part (e). Prove $\nabla g(0) = I$, and hence complete the proof of the result.

(g) If $x$ lies in the interior of $S$, prove $F$ can be assumed linear.

(Exercise 9 (Nonexistence of multipliers) in Section 7.2 suggests the importance here of assuming $h$ continuous.)

15. * **(Knaster–Kuratowski–Mazurkiewicz principle [114])** In this exercise we show the equivalence of Brouwer's theorem with the following result.

**Theorem 8.1.11 (KKM)** *Suppose for every point $x$ in a nonempty set $X \subset \mathbf{E}$ there is an associated closed subset $M(x) \subset X$. Assume the property*

$$\operatorname{conv} F \subset \bigcup_{x \in F} M(x)$$

*holds for all finite subsets $F \subset X$. Then for any finite subset $F \subset X$ we have*

$$\bigcap_{x \in F} M(x) \neq \emptyset.$$

*Hence if some subset $M(x)$ is compact we have*

$$\bigcap_{x \in X} M(x) \neq \emptyset.$$

(a) Prove that the final assertion follows from the main part of the theorem using Theorem 8.2.3 (General definition of compactness).

(b) **(KKM implies Brouwer)** Given a continuous self map $f$ on a nonempty compact convex set $C \subset \mathbf{E}$, apply the KKM theorem to the family of sets

$$M(x) = \{y \in C \mid \langle y - f(y), y - x \rangle \leq 0\} \quad \text{for } x \in C$$

to deduce $f$ has a fixed point.

(c) **(Brouwer implies KKM)** With the hypotheses of the KKM theorem, assume $\cap_{x \in F} M(x)$ is empty for some finite set $F$. Consider a fixed point $z$ of the self map

$$y \in \operatorname{conv} F \mapsto \frac{\sum_{x \in F} d_{M(x)}(y) x}{\sum_{x \in F} d_{M(x)}(y)}$$

and define $F' = \{x \in F \mid z \notin M(x)\}$. Show $z \in \operatorname{conv} F'$ and derive a contradiction.

16. ** **(Hairy ball theorem [140])** Let $S_n$ denote the Euclidean sphere

$$\{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}.$$

A *tangent vector field* on $S_n$ is a function $w : S_n \to \mathbf{R}^{n+1}$ satisfying $\langle x, w(x) \rangle = 0$ for all points $x$ in $S_n$. This exercise proves the following result.

**Theorem 8.1.12** *For every even $n$, any continuous tangent vector field on $S_n$ must vanish somewhere.*

**Proof.** Consider a nonvanishing continuous tangent vector field $u$ on $S_n$.

(a) Prove there is a nonvanishing $C^{(1)}$ tangent vector field on $S_n$, by using the Stone–Weierstrass theorem (8.1.4) to approximate $u$ by a $C^{(1)}$ function $p$ and then considering the vector field

$$x \in S_n \mapsto p(x) - \langle x, p(x) \rangle x.$$

(b) Deduce the existence of a positively homogeneous $C^{(1)}$ function $w : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ whose restriction to $S_n$ is a *unit norm* $C^{(1)}$ tangent vector field: $\|w(x)\| = 1$ for all $x$ in $S_n$.

Define a set

$$A = \{x \in \mathbf{R}^{n+1} \mid 1 < 2\|x\| < 3\}$$

and use the field $w$ in part (b) to define functions $w_t : \mathbf{R}^{n+1} \to \mathbf{R}^{n+1}$ for real $t$ by

$$w_t(x) = x + tw(x).$$

(c) Imitate the proof of Brouwer's theorem to prove the measure of the image set $w_t(A)$ is a polynomial in $t$ when $t$ is small.

(d) Prove directly the inclusion $w_t(A) \subset \sqrt{1+t^2}A$.

(e) For any point $y$ in $\sqrt{1+t^2}A$, apply the Banach contraction theorem to the function $x \in kB \mapsto y - tw(x)$ (for large real $k$) to deduce in fact

$$w_t(A) = \sqrt{1+t^2}A \quad \text{for small } t.$$

(f) Complete the proof by combining parts (c) and (e). □

(g) If $f$ is a continuous self map of $S_n$ where $n$ is even, prove either $f$ or $-f$ has a fixed point.

(h) **(Hedgehog theorem)** Prove for even $n$ that any nonvanishing continuous vector field must be somewhere *normal*: $|\langle x, f(x)\rangle| = \|f(x)\|$ for some $x$ in $S_n$.

(i) Find examples to show the Hairy ball theorem fails for all odd $n$.

17. * **(Borsuk–Ulam theorem)** Let $S_n$ denote the Euclidean sphere

$$\{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}.$$

We state the following result without proof.

**Theorem 8.1.13 (Borsuk–Ulam)** *For any positive integers $m \le n$, if the function $f : S_n \to \mathbf{R}^m$ is continuous then there is a point $x$ in $S_n$ satisfying $f(x) = f(-x)$.*

(a) If $m \le n$ and the map $f : S_n \to \mathbf{R}^m$ is continuous and odd, prove $f$ vanishes somewhere.

(b) Prove any odd continuous self map $f$ on $S_n$ is surjective. (Hint: For any point $u$ in $S_n$, consider the function

$$x \in S_n \mapsto f(x) - \langle f(x), u \rangle u$$

and apply part (a).)

(c) Prove the result in part (a) is equivalent to the following result:

**Theorem 8.1.14** *For positive integers $m < n$ there is no continuous odd map from $S_n$ to $S_m$.*

(d) **(Borsuk–Ulam implies Brouwer [178])** Let $B$ denote the unit ball in $\mathbf{R}^n$, and let $S$ denote the boundary of $B \times [-1, 1]$:

$$S = \{(x, t) \in B \times [-1, 1] \mid \|x\| = 1 \text{ or } |t| = 1\}.$$

(i) If the map $g : S \to \mathbf{R}^n$ is continuous and odd, use part (a) to prove $g$ vanishes somewhere on $S$.

(ii) Consider a continuous self map $f$ on $B$. By applying part (i) to the function

$$(x, t) \in S \mapsto (2 - |t|)x - tf(tx),$$

prove $f$ has a fixed point.

18. ** **(Generalized Riesz lemma)** Consider a smooth norm $|||\cdot|||$ on $\mathbf{E}$ (that is, a norm which is continuously differentiable except at the origin) and linear subspaces $U, V \subset \mathbf{E}$ satisfying $\dim U > \dim V = n$. Denote the unit sphere in $U$ (in this norm) by $S(U)$.

(a) By choosing a basis $v_1, v_2, \ldots, v_n$ of $V$ and applying the Borsuk–Ulam theorem (see Exercise 17) to the map

$$x \in S(U) \mapsto (\langle \nabla |||\cdot|||(x), v_i \rangle)_{i=1}^n \in \mathbf{R}^n,$$

prove there is a point $x$ in $S(U)$ satisfying $\nabla |||\cdot|||(x) \perp V$.

(b) Deduce the origin is the nearest point to $x$ in $V$ (in this norm).

(c) With this norm, deduce there is a unit vector in $U$ whose distance from $V$ is equal to one.

(d) Use the fact that any norm can be uniformly approximated arbitrarily well by a smooth norm to extend the result of part (c) to arbitrary norms.

(e) Find a simpler proof when $V \subset U$.

19. ** **(Riesz implies Borsuk)** In this question we use the generalized Riesz lemma, Exercise 18, to prove the Borsuk–Ulam result, Exercise 17(a). To this end, suppose the map $f : S_n \to \mathbf{R}^n$ is continuous and odd. Define functions

$$\begin{aligned} u_i : S_n &\to \mathbf{R} \ \text{ for } i = 1, 2, \ldots, n+1 \\ v_i : \mathbf{R}^n &\to \mathbf{R} \ \text{ for } i = 1, 2, \ldots, n \end{aligned}$$

by $u_i(x) = x_i$ and $v_i(x) = x_i$ for each index $i$. Define spaces of continuous odd functions on $S_n$ by

$$\begin{aligned} U &= \text{span}\,\{u_1, u_2, \ldots .u_{n+1}\} \\ V &= \text{span}\,\{v_1 \circ f, v_2 \circ f, \ldots, v_n \circ f\} \\ \mathbf{E} &= U + V, \end{aligned}$$

with norm $\|u\| = \max u(S_n)$ (for $u$ in $\mathbf{E}$).

(a) Prove there is a function $u$ in $U$ satisfying $\|u\| = 1$ and whose distance from $V$ is equal to one.

(b) Prove $u$ attains its maximum on $S_n$ at a unique point $y$.

(c) Use the fact that for any function $w$ in $\mathbf{E}$, we have

$$(\nabla\| \cdot \|(u))w = w(y)$$

to deduce $f(y) = 0$.

## 8.2 Selection and the Kakutani–Fan Fixed Point Theorem

The Brouwer fixed point theorem in the previous section concerns functions from a nonempty compact convex set to itself. In optimization, as we have already seen in Section 5.4, it may be convenient to broaden our language to consider *multifunctions* $\Omega$ from the set to itself and seek a *fixed point*—a point $x$ satisfying $x \in \Omega(x)$. To begin this section we summarize some definitions for future reference.

We consider a subset $K \subset \mathbf{E}$, a Euclidean space $\mathbf{Y}$, and a multifunction $\Omega : K \to \mathbf{Y}$. We say $\Omega$ is *USC* at a point $x$ in $K$ if every open set $U$ containing $\Omega(x)$ also contains $\Omega(z)$ for all points $z$ in $K$ close to $x$.

Thus a multifunction $\Omega$ is USC if for any sequence of points $(x_n)$ approaching $x$, any sequence of elements $y_n \in \Omega(x_n)$ is eventually close to $\Omega(x)$. If $\Omega$ is USC at every point in $K$ we simply call it *USC*. On the other hand, as in Section 5.4, we say $\Omega$ is *LSC* if, for every $x$ in $K$, every neighbourhood $V$ of any point in $\Omega(x)$ intersects $\Omega(z)$ for all points $z$ in $K$ close to $x$.

We refer to the sets $\Omega(x)$ $(x \in K)$ as the *images* of $\Omega$. The multifunction $\Omega$ is a *cusco* if it is USC with nonempty compact convex images. Clearly such multifunctions are *locally bounded*: any point in $K$ has a neighbourhood whose image is bounded. Cuscos appear in several important optimization contexts. For example, the Clarke subdifferential of a locally Lipschitz function is a cusco (Exercise 5).

To see another important class of examples we need a further definition. We say a multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ is *monotone* if it satisfies the condition

$$\langle u - v, x - y \rangle \geq 0 \quad \text{whenever } u \in \Phi(x) \text{ and } v \in \Phi(y).$$

In particular, any (not necessarily self-adjoint) positive semidefinite linear operator is monotone, as is the subdifferential of any convex function. One multifunction *contains* another if the graph of the first contains the graph of the second. We say a monotone multifunction is *maximal* if the only monotone multifunction containing it is itself. The subdifferentials of closed proper convex functions are examples (see Exercise 16). Zorn's lemma (which lies outside our immediate scope) shows any monotone multifunction is contained in a maximal monotone multifunction.

**Theorem 8.2.1 (Maximal monotonicity)** *Maximal monotone multifunctions are cuscos on the interiors of their domains.*

**Proof.** See Exercise 16. □

Maximal monotone multifunctions in fact have to be single-valued *generically*, that is on sets which are "large" in a topological sense, specifically on a dense set which is a “$G_\delta$” (a countable intersection of open sets)—see Exercise 17.

Returning to our main theme, the central result of this section extends Brouwer's theorem to the multifunction case.

**Theorem 8.2.2 (Kakutani–Fan)** *If the set $C \subset \mathbf{E}$ is nonempty, compact and convex, then any cusco $\Omega : C \to C$ has a fixed point.*

Before we prove this result, we outline a little more topology. A *cover* of a set $K \subset \mathbf{E}$ is a collection of sets in $\mathbf{E}$ whose union contains $K$. The cover is *open* if each set in the collection is open. A *subcover* is just a subcollection of the sets which is also a cover. The following result, which we state as a theorem, is in truth the definition of compactness in spaces more general than $\mathbf{E}$.

**Theorem 8.2.3 (General definition of compactness)** *Any open cover of a compact set in $\mathbf{E}$ has a finite subcover.*

Given a finite open cover $\{O_1, O_2, \ldots, O_m\}$ of a set $K \subset \mathbf{E}$, a *partition of unity subordinate to* this cover is a set of continuous functions $p_1, p_2, \ldots, p_m : K \to \mathbf{R}_+$ whose sum is identically equal to one and satisfying $p_i(x) = 0$ for all points $x$ outside $O_i$ (for each index $i$). We outline the proof of the next result, a central topological tool, in the exercises.

**Theorem 8.2.4 (Partition of unity)** *There is a partition of unity subordinate to any finite open cover of a compact subset of $\mathbf{E}$.*

Besides fixed points, the other main theme of this section is the idea of a *continuous selection* of a multifunction $\Omega$ on a set $K \subset \mathbf{E}$, by which we mean a continuous map $f$ on $K$ satisfying $f(x) \in \Omega(x)$ for all points $x$ in $K$. The central step in our proof of the Kakutani–Fan theorem is the following “approximate selection” theorem.

**Theorem 8.2.5 (Cellina)** *Given any compact set $K \subset \mathbf{E}$, suppose the multifunction $\Omega : K \to \mathbf{Y}$ is USC with nonempty convex images. Then for any real $\epsilon > 0$ there is a continuous map $f : K \to \mathbf{Y}$ which is an “approximate selection” of $\Omega$ :*

$$d_{G(\Omega)}(x, f(x)) < \epsilon \text{ for all points } x \text{ in } K. \tag{8.2.6}$$

*Furthermore the range of $f$ is contained in the convex hull of the range of $\Omega$.*

**Proof.** We can assume the norm on $\mathbf{E} \times \mathbf{Y}$ is given by

$$\|(x, y)\|_{\mathbf{E}\times\mathbf{Y}} = \|x\|_{\mathbf{E}} + \|y\|_{\mathbf{Y}} \quad \text{for all } x \in \mathbf{E} \text{ and } y \in \mathbf{Y}$$

(since all norms are equivalent—see Section 4.1, Exercise 2). Now, since $\Omega$ is USC, for each point $x$ in $K$ there is a real $\delta_x$ in the interval $(0, \epsilon/2)$ satisfying

$$\Omega(x + \delta_x B_{\mathbf{E}}) \subset \Omega(x) + \frac{\epsilon}{2} B_{\mathbf{Y}}.$$

Since the sets $x + (\delta_x/2)\mathrm{int}\, B_{\mathbf{E}}$ (as the point $x$ ranges over $K$) comprise an open cover of the compact set $K$, there is a finite subset $\{x_1, x_2, \ldots, x_m\}$ of $K$ with the sets $x_i + (\delta_i/2)\mathrm{int}\, B_{\mathbf{E}}$ comprising a finite subcover (where $\delta_i$ is shorthand for $\delta_{x_i}$ for each index $i$).

Theorem 8.2.4 shows there is a partition of unity $p_1, p_2, \ldots, p_m : K \to \mathbf{R}_+$ subordinate to this subcover. We now construct our desired approximate selection $f$ by choosing a point $y_i$ from $\Omega(x_i)$ for each $i$ and defining

$$f(x) = \sum_{i=1}^{m} p_i(x) y_i \quad \text{for all points } x \text{ in } K. \tag{8.2.7}$$

Fix any point $x$ in $K$ and define the set $I = \{i | p_i(x) \neq 0\}$. By definition, $x$ satisfies $\|x - x_i\| < \delta_i/2$ for each $i$ in $I$. If we choose an index $j$ in $I$ maximizing $\delta_j$, the triangle inequality shows $\|x_j - x_i\| < \delta_j$, whence we deduce the inclusions

$$y_i \in \Omega(x_i) \subset \Omega(x_j + \delta_j B_{\mathbf{E}}) \subset \Omega(x_j) + \frac{\epsilon}{2} B_{\mathbf{Y}}$$

for all $i$ in $I$. In other words, for each $i$ in $I$ we know $d_{\Omega(x_j)}(y_i) \leq \epsilon/2$. Since the distance function is convex, equation (8.2.7) shows $d_{\Omega(x_j)}(f(x)) \leq \epsilon/2$. Since we also know $\|x - x_j\| < \epsilon/2$, this proves inequality (8.2.6). The final claim follows immediately from equation (8.2.7). □

**Proof of the Kakutani–Fan theorem.** With the assumption of the theorem, Cellina's result above shows for each positive integer $r$ there is a continuous self map $f_r$ of $C$ satisfying

$$d_{G(\Omega)}(x, f_r(x)) < \frac{1}{r} \quad \text{for all points } x \text{ in } C.$$

By Brouwer's theorem (8.1.3), each $f_r$ has a fixed point $x^r$ in $C$, which therefore satisfies

$$d_{G(\Omega)}(x^r, x^r) < \frac{1}{r} \quad \text{for each } r.$$

Since $C$ is compact, the sequence $(x^r)$ has a convergent subsequence, and its limit must be a fixed point of $\Omega$ because $\Omega$ is closed by Exercise 3(c) (Closed versus USC). □

In the next section we describe some variational applications of the Kakutani–Fan theorem. But we end this section with an *exact* selection theorem parallel to Cellina's result but assuming an LSC rather than a USC multifunction.

**Theorem 8.2.8 (Michael)** *Given any closed set $K \subset \mathbf{E}$, suppose the multifunction $\Omega : K \to \mathbf{Y}$ is LSC with nonempty closed convex images. Then given any point $(\bar{x}, \bar{y})$ in $G(\Omega)$, there is a continuous selection $f$ of $\Omega$ satisfying $f(\bar{x}) = \bar{y}$.*

We outline the proof in the exercises.

## Exercises and Commentary

Many useful properties of cuscos are summarized in [27]. An excellent general reference on monotone operators is [153]. The topology we use in this section can be found in any standard text (see [67, 106], for example). The Kakutani–Fan theorem first appeared in [109] and was extended in [74]. Cellina's approximate selection theorem appears, for example, in [4, p. 84]. One example of the many uses of the Kakutani–Fan theorem is establishing equilibria in mathematical economics. The Michael selection theorem appeared in [137].

1. **(USC and continuity)** Consider a closed subset $K \subset \mathbf{E}$ and a multifunction $\Omega : K \to \mathbf{Y}$.

   (a) Prove the multifunction

   $$x \in \mathbf{E} \mapsto \begin{cases} \Omega(x) & \text{for } x \in K \\ \emptyset & \text{for } x \notin K \end{cases}$$

   is USC if and only if $\Omega$ is USC.

   (b) Prove a function $f : K \to \mathbf{Y}$ is continuous if and only if the multifunction $x \in K \mapsto \{f(x)\}$ is USC.

   (c) Prove a function $f : \mathbf{E} \to [-\infty, +\infty]$ is lower semicontinuous at a point $x$ in $\mathbf{E}$ if and only if the multifunction whose graph is the epigraph of $f$ is USC at $x$.

2. * **(Minimum norm)** If the set $U \subset \mathbf{E}$ is open and the multifunction $\Omega : U \to \mathbf{Y}$ is USC, prove the function $g : U \to \mathbf{Y}$ defined by

   $$g(x) = \inf\{\|y\| \mid y \in \Omega(x)\}$$

   is lower semicontinuous.

3. **(Closed versus USC)**

   (a) If the multifunction $\Phi : \mathbf{E} \to \mathbf{Y}$ is closed and the multifunction $\Omega : \mathbf{E} \to \mathbf{Y}$ is USC at the point $x$ in $\mathbf{E}$ with $\Omega(x)$ compact, prove the multifunction

   $$z \in \mathbf{E} \mapsto \Omega(z) \cap \Phi(z)$$

   is USC at $x$.

(b) Hence prove that any closed multifunction with compact range is USC.

(c) Prove any USC multifunction with closed images is closed.

(d) If a USC multifunction has compact images, prove it is locally bounded.

4. **(Composition)** If the multifunctions $\Phi$ and $\Omega$ are USC prove their composition $x \mapsto \Phi(\Omega(x))$ is also.

5. $^*$ **(Clarke subdifferential)** If the set $U \subset \mathbf{E}$ is open and the function $f : U \to \mathbf{R}$ is locally Lipschitz, use Section 6.2, Exercise 12 (Closed subdifferentials) and Exercise 3 (Closed versus USC) to prove the Clarke subdifferential $x \in U \mapsto \partial_\circ f(x)$ is a cusco.

6. $^{**}$ **(USC images of compact sets)** Consider a given multifunction $\Omega : K \to \mathbf{Y}$.

(a) Prove $\Omega$ is USC if and only if for every open subset $U$ of $\mathbf{Y}$ the set $\{x \in K \mid \Omega(x) \subset U\}$ is open in $K$.

Now suppose $K$ is compact and $\Omega$ is USC with compact images. Using the general definition of compactness (8.2.3), prove the range $\Omega(K)$ is compact by following the steps below.

(b) Fix an open cover $\{U_\gamma \mid \gamma \in \Gamma\}$ of $\Omega(K)$. For each point $x$ in $K$, prove there is a finite subset $\Gamma_x$ of $\Gamma$ with

$$\Omega(x) \subset \bigcup_{\gamma \in \Gamma_x} U_\gamma.$$

(c) Construct an open cover of $K$ by considering the sets

$$\left\{ z \in K \,\middle|\, \Omega(z) \subset \bigcup_{\gamma \in \Gamma_x} U_\gamma \right\}$$

as the point $x$ ranges over $K$.

(d) Hence construct a finite subcover of the original cover of $\Omega(K)$.

7. $^*$ **(Partitions of unity)** Suppose the set $K \subset \mathbf{E}$ is compact with a finite open cover $\{O_1, O_2, \ldots, O_m\}$.

(i) Show how to construct another open cover $\{V_1, V_2, \ldots, V_m\}$ of $K$ satisfying $\mathrm{cl}\, V_i \subset O_i$ for each index $i$. (Hint: Each point $x$ in $K$ lies in some set $O_i$, so there is a real $\delta_x > 0$ with $x + \delta_x B \subset O_i$; now take a finite subcover of $\{x + \delta_x \mathrm{int}\, B \mid x \in K\}$ and build the sets $V_i$ from it.)

(ii) For each index $i$, prove the function $q_i : K \to [0,1]$ given by

$$q_i = \frac{d_{K \setminus O_i}}{d_{K \setminus O_i} + d_{V_i}}$$

is well-defined and continuous, with $q_i$ identically zero outside the set $O_i$.

(iii) Deduce that the set of functions $p_i : K \to \mathbf{R}_+$ defined by

$$p_i = \frac{q_i}{\sum_j q_j}$$

is a partition of unity subordinate to the cover $\{O_1, O_2, \ldots, O_m\}$.

8. Prove the Kakutani–Fan theorem is also valid under the weaker assumption that the images of the cusco $\Omega : C \to \mathbf{E}$ always intersect the set $C$ using Exercise 3(a) (Closed versus USC).

9. ** **(Michael's theorem)** Suppose all the assumptions of Michael's theorem (8.2.8) hold. We consider first the case with $K$ compact.

(a) Fix a real $\epsilon > 0$. By constructing a partition of unity subordinate to a finite subcover of the open cover of $K$ consisting of the sets

$$O_y = \{x \in \mathbf{E} \mid d_{\Omega(x)}(y) < \epsilon\} \ \text{ for } y \text{ in } Y,$$

construct a continuous function $f : K \to Y$ satisfying

$$d_{\Omega(x)}(f(x)) < \epsilon \ \text{ for all points } x \text{ in } K.$$

(b) Construct a sequence of continuous functions $f_1, f_2, \ldots : K \to Y$ satisfying

$$\begin{aligned} d_{\Omega(x)}(f_i(x)) &< 2^{-i} \ \text{ for } i = 1, 2, \ldots \\ \|f_{i+1}(x) - f_i(x)\| &< 2^{1-i} \ \text{ for } i = 1, 2, \ldots \end{aligned}$$

for all points $x$ in $K$. (Hint: Construct $f_1$ by applying part (a) with $\epsilon = 1/2$; then construct $f_{i+1}$ inductively by applying part (a) to the multifunction

$$x \in K \mapsto \Omega(x) \cap (f_i(x) + 2^{-i} B_{\mathbf{Y}})$$

with $\epsilon = 2^{-i-1}$.

(c) The functions $f_i$ of part (b) must converge uniformly to a continuous function $f$. Prove $f$ is a continuous selection of $\Omega$.

(d) Prove Michael's theorem by applying part (c) to the multifunction

$$\hat{\Omega}(x) = \begin{cases} \Omega(x) & \text{if } x \neq \bar{x} \\ \{\bar{y}\} & \text{if } x = \bar{x}. \end{cases}$$

(e) Now extend to the general case where $K$ is possibly unbounded in the following steps. Define sets $K_n = K \cap nB_{\mathbf{E}}$ for each $n = 1, 2, \ldots$ and apply the compact case to the multifunction $\Omega_1 = \Omega|_{K_1}$ to obtain a continuous selection $g_1 : K_1 \to \mathbf{Y}$. Then inductively find a continuous selection $g_{n+1} : K_{n+1} \to \mathbf{Y}$ from the multifunction

$$\Omega_{n+1}(x) = \begin{cases} \{g_n(x)\} & \text{for } x \in K_n \\ \Omega(x) & \text{for } x \in K_{n+1} \setminus K_n \end{cases}$$

and prove the function defined by

$$f(x) = g_n(x) \ \text{ for } x \in K_n, \ n = 1, 2, \ldots$$

is the required selection.

10. **(Hahn–Katetov–Dowker sandwich theorem)** Suppose the set $K \subset \mathbf{E}$ is closed.

(a) For any two lower semicontinuous functions $f, g : K \to \mathbf{R}$ satisfying $f \geq -g$, prove there is a continuous function $h : K \to \mathbf{R}$ satisfying $f \geq h \geq -g$ by considering the multifunction $x \mapsto [-g(x), f(x)]$. Observe the result also holds for extended-real-valued $f$ and $g$.

(b) **(Urysohn lemma)** Suppose the closed set $V$ and the open set $U$ satisfy $V \subset U \subset K$. By applying part (i) to suitable functions, prove there is a continuous function $f : K \to [0, 1]$ that is identically equal to one on $V$ and to zero on $U^c$.

11. **(Continuous extension)** Consider a closed subset $K$ of $\mathbf{E}$ and a continuous function $f : K \to \mathbf{Y}$. By considering the multifunction

$$\Omega(x) = \begin{cases} \{f(x)\} & \text{for } x \in K \\ \mathrm{cl}\,(\mathrm{conv}\, f(K)) & \text{for } x \notin K, \end{cases}$$

prove there is a continuous function $g : \mathbf{E} \to \mathbf{Y}$ satisfying $g|_K = f$ and $g(\mathbf{E}) \subset \mathrm{cl}\,(\mathrm{conv}\, f(K))$.

12. * **(Generated cuscos)** Suppose the multifunction $\Omega : K \to \mathbf{Y}$ is locally bounded with nonempty images.

(a) Among those cuscos containing $\Omega$, prove there is a unique one with minimal graph, given by

$$\Phi(x) = \bigcap_{\epsilon>0} \mathrm{cl\,conv}\,(\Omega(x+\epsilon B)) \quad \text{for } x \in K.$$

(b) If $K$ is nonempty, compact, and convex, $\mathbf{Y} = \mathbf{E}$, and $\Omega$ satisfies the conditions $\Omega(K) \subset K$ and

$$x \in \Phi(x) \Rightarrow x \in \Omega(x) \quad \text{for } x \in K,$$

prove $\Omega$ has a fixed point.

13. * **(Multifunctions containing cuscos)** Suppose the multifunction $\Omega : K \to \mathbf{Y}$ is closed with nonempty convex images, and the function $f : K \to \mathbf{Y}$ has the property that $f(x)$ is a point of minimum norm in $\Omega(x)$ for all points $x$ in $K$. Prove $\Omega$ contains a cusco if and only if $f$ is locally bounded. (Hint: Use Exercise 12 (Generated cuscos) to consider the cusco generated by $f$.)

14. * **(Singleton points)** For any subset $D$ of $\mathbf{Y}$, define

$$s(D) = \inf\{r \in \mathbf{R} \mid D \subset y + rB_{\mathbf{Y}} \text{ for some } y \in \mathbf{Y}\}.$$

Consider an open subset $U$ of $\mathbf{E}$.

(a) If the multifunction $\Omega : U \to \mathbf{Y}$ is USC with nonempty images, prove for any real $\epsilon > 0$ the set

$$S_\epsilon = \{x \in U \mid s(\Omega(x)) < \epsilon\}$$

is open. By considering the set $\cap_{n>1} S_{1/n}$, prove the set of points in $U$ whose image is a singleton is a $G_\delta$.

(b) Use Exercise 5 (Clarke subdifferential) to prove that the set of points where a locally Lipschitz function $f : U \to \mathbf{R}$ is strictly differentiable is a $G_\delta$. If $U$ and $f$ are convex (or if $f$ is regular throughout $U$), use Rademacher's theorem (in Section 6.2) to deduce $f$ is generically differentiable.

15. **(Skew symmetry)** If the matrix $A \in \mathbf{M}^n$ satisfies $0 \neq A = -A^T$, prove the multifunction $x \in \mathbf{R}^n \mapsto x^T A x$ is maximal monotone, yet is not the subdifferential of a convex function.

16. ** **(Monotonicity)** Consider a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$.

(a) **(Inverses)** Prove $\Phi^{-1}$ is monotone.

(b) Prove $\Phi^{-1}$ is maximal if and only if $\Phi$ is.

(c) **(Applying maximality)** Prove $\Phi$ is maximal if and only if it has the property

$$\langle u - v, x - y \rangle \geq 0 \ \text{ for all } (x, u) \in G(\Phi) \ \Rightarrow \ v \in \Phi(y).$$

(d) **(Maximality and closedness)** If $\Phi$ is maximal, prove it is closed with convex images.

(e) **(Continuity and maximality)** Supposing $\Phi$ is everywhere single-valued and *hemicontinuous* (that is, continuous on every line in $\mathbf{E}$), prove it is maximal. (Hint: Apply part (c) with $x = y + tw$ for $w$ in $\mathbf{E}$ and $t \downarrow 0$ in $\mathbf{R}$.)

(f) We say $\Phi$ is *hypermaximal* if $\Phi + \lambda I$ is surjective for some real $\lambda > 0$. In this case, prove $\Phi$ is maximal. (Hint: Apply part (c) and use a solution $x \in \mathbf{E}$ to the inclusion $v + \lambda y \in (\Phi + \lambda I)(x)$.) What if just $\Phi$ is surjective?

(g) **(Subdifferentials)** If the function $f : \mathbf{E} \to (\infty, +\infty]$ is closed, convex, and proper, prove $\partial f$ is maximal monotone. (Hint: For any element $\phi$ of $\mathbf{E}$, prove the function

$$x \in \mathbf{E} \mapsto f(x) + \|x\|^2 + \langle \phi, x \rangle$$

has a minimizer, and deduce $\partial f$ is hypermaximal.)

(h) **(Local boundedness)** By completing the following steps, prove $\Phi$ is locally bounded at any point in the core of its domain.

(i) Assume $0 \in \Phi(0)$ and $0 \in \operatorname{core} D(\Phi)$, define a convex function $g : \mathbf{E} \to (\infty, +\infty]$ by

$$g(y) = \sup\{\langle u, y - x \rangle \mid x \in B, \ u \in \Phi(x)\}.$$

(ii) Prove $D(\Phi) \subset \operatorname{dom} g$.

(iii) Deduce $g$ is continuous at zero.

(iv) Hence show $|g(y)| \leq 1$ for all small $y$, and deduce the result.

(j) **(Maximality and cuscos)** Use parts (d) and (h), and Exercise 3 (Closed versus USC) to conclude that any maximal monotone multifunction is a cusco on the interior of its domain.

(k) **(Surjectivity and growth)** If $\Phi$ is surjective, prove

$$\lim_{\|x\| \to \infty} \|\Phi(x)\| = +\infty.$$

(Hint: Assume the maximality of $\Phi$, and hence of $\Phi^{-1}$; deduce $\Phi^{-1}$ is a cusco on $\mathbf{E}$, and now apply Exercise 6 (USC images of compact sets).)

17. ** **(Single-valuedness and maximal monotonicity)** Consider a maximal monotone multifunction $\Omega : \mathbf{E} \to \mathbf{E}$ and an open subset $U$ of its domain, and define the minimum norm function $g : U \to \mathbf{R}$ as in Exercise 2.

    (a) Prove $g$ is lower semicontinuous. An application of the Baire category theorem now shows that any such function is generically continuous.

    (b) For any point $x$ in $U$ at which $g$ is continuous, prove $\Omega(x)$ is a singleton. (Hint: Prove $\|\cdot\|$ is constant on $\Omega(x)$ by first assuming $y, z \in \Omega(x)$ and $\|y\| > \|z\|$, and then using the condition

    $$\langle w - y, x + ty - x \rangle \geq 0 \;\text{ for all small } t > 0 \text{ and } w \in \Omega(x + ty)$$

    to derive a contradiction.)

    (c) Conclude that any maximal monotone multifunction is generically single-valued.

    (d) Deduce that any convex function is generically differentiable on the interior of its domain.

## 8.3 Variational Inequalities

At the very beginning of this book we considered the problem of minimizing a differentiable function $f : \mathbf{E} \to \mathbf{R}$ over a convex set $C \subset \mathbf{E}$. A necessary optimality condition for a point $x_0$ in $C$ to be a local minimizer is

$$\langle \nabla f(x_0), x - x_0 \rangle \geq 0 \text{ for all points } x \text{ in } C, \tag{8.3.1}$$

or equivalently

$$0 \in \nabla f(x_0) + N_C(x_0).$$

If the function $f$ is convex instead of differentiable, the necessary and sufficient condition for optimality (assuming a constraint qualification) is

$$0 \in \partial f(x_0) + N_C(x_0),$$

and there are analogous nonsmooth necessary conditions.

We call problems like (8.3.1) "variational inequalities". Let us fix a multifunction $\Omega : C \to \mathbf{E}$. In this section we use the fixed point theory we have developed to study the *multivalued variational inequality*

$VI(\Omega, C)$: Find points $x_0$ in $C$ and $y_0$ in $\Omega(x_0)$ satisfying $\langle y_0, x - x_0 \rangle \geq 0$ for all points $x$ in $C$.

A more concise way to write the problem is this:

$$\text{Find a point } x_0 \text{ in } C \text{ satisfying } 0 \in \Omega(x_0) + N_C(x_0). \tag{8.3.2}$$

Suppose the set $C$ is closed, convex, and nonempty. Recall that the projection $P_C : \mathbf{E} \to C$ is the (continuous) map that sends points in $\mathbf{E}$ to their unique nearest points in $C$ (see Section 2.1, Exercise 8). Using this notation we can also write the variational inequality as a fixed point problem:

$$\text{Find a fixed point of } P_C \circ (I - \Omega) : C \to C. \tag{8.3.3}$$

This reformulation is useful if the multifunction $\Omega$ is single-valued, but less so in general because the composition will often not have convex images.

A more versatile approach is to define the (multivalued) *normal mapping* $\Omega_C = (\Omega \circ P_C) + I - P_C$, and repose the problem as follows:

$$\text{Find a point } \bar{x} \text{ in } \mathbf{E} \text{ satisfying } 0 \in \Omega_C(\bar{x}). \tag{8.3.4}$$

Then setting $x_0 = P_C(\bar{x})$ gives a solution to the original problem. Equivalently, we could phrase this as follows:

$$\text{Find a fixed point of } (I - \Omega) \circ P_C : \mathbf{E} \to \mathbf{E}. \tag{8.3.5}$$

As we shall see, this last formulation lets us immediately use the fixed point theory of the previous section.

The basic result guaranteeing the existence of solutions to variational inequalities is the following.

**Theorem 8.3.6 (Solvability of variational inequalities)** *If the subset $C$ of $\mathbf{E}$ is compact, convex, and nonempty, then for any cusco $\Omega : C \to \mathbf{E}$ the variational inequality $VI(\Omega, C)$ has a solution.*

**Proof.** We in fact prove *Theorem 8.3.6 is equivalent to the Kakutani–Fan fixed point theorem (8.2.2).*

When $\Omega$ is a cusco its range $\Omega(C)$ is compact—we outline the proof in Section 8.2, Exercise 6. We can easily check that the multifunction $(I - \Omega) \circ P_C$ is also a cusco because the projection $P_C$ is continuous. Since this multifunction maps the compact convex set $\mathrm{conv}\,(C - \Omega(C))$ into itself, the Kakutani–Fan theorem shows it has a fixed point, which, as we have already observed, implies the solvability of $VI(\Omega, C)$.

Conversely, suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex. For any cusco $\Omega : C \to C$, the Solvability theorem (8.3.6) implies we can solve the variational inequality $VI(I - \Omega, C)$, so there are points $x_0$ in $C$ and $z_0$ in $\Omega(x_0)$ satisfying

$$\langle x_0 - z_0, x - x_0 \rangle \geq 0 \ \text{ for all points } x \text{ in } C.$$

Setting $x = z_0$ shows $x_0 = z_0$, so $x_0$ is a fixed point. □

An elegant application is von Neumann's minimax theorem, which we proved by a Fenchel duality argument in Section 4.2, Exercise 16. Consider Euclidean spaces $\mathbf{Y}$ and $\mathbf{Z}$, nonempty compact convex subsets $F \subset \mathbf{Y}$ and $G \subset \mathbf{Z}$, and a linear map $A : \mathbf{Y} \to \mathbf{Z}$. If we define a function $\Omega : F \times G \to \mathbf{Y} \times \mathbf{Z}$ by $\Omega(y, z) = (-A^* z, Ay)$, then it is easy to see that a point $(y_0, z_0)$ in $F \times G$ solves the variational inequality $VI(\Omega, F \times G)$ if and only if it is a *saddlepoint*:

$$\langle z_0, Ay \rangle \leq \langle z_0, Ay_0 \rangle \leq \langle z, Ay_0 \rangle \ \text{ for all } y \in F,\ z \in G.$$

In particular, by the Solvability of variational inequalities theorem, there exists a saddlepoint, so

$$\min_{z \in G} \max_{y \in F} \langle z, Ay \rangle = \max_{y \in F} \min_{z \in G} \langle z, Ay \rangle.$$

Many interesting variational inequalities involve a noncompact set $C$. In such cases we need to impose a growth condition on the multifunction to guarantee solvability. The following result is an example.

**Theorem 8.3.7 (Noncompact variational inequalities)** *If the subset $C$ of $\mathbf{E}$ is nonempty, closed, and convex, and the cusco $\Omega : C \to \mathbf{E}$ is* **coercive***, that is, it satisfies the condition*

$$\liminf_{\|x\|\to\infty,\ x\in C} \inf \langle x, \Omega(x) + N_C(x)\rangle > 0, \tag{8.3.8}$$

*then the variational inequality $VI(\Omega, C)$ has a solution.*

**Proof.** For any large integer $r$, we can apply the solvability theorem (8.3.6) to the variational inequality $VI(\Omega, C \cap rB)$ to find a point $x_r$ in $C \cap rB$ satisfying

$$\begin{aligned} 0 &\in \Omega(x_r) + N_{C\cap rB}(x_r) \\ &= \Omega(x_r) + N_C(x_r) + N_{rB}(x_r) \\ &\subset \Omega(x_r) + N_C(x_r) + \mathbf{R}_+ x_r \end{aligned}$$

(using Section 3.3, Exercise 10). Hence for all large $r$, the point $x_r$ satisfies

$$\inf \langle x_r, \Omega(x_r) + N_C(x_r)\rangle \le 0.$$

This sequence of points $(x_r)$ must therefore remain bounded, by the coercivity condition (8.3.8), and so $x_r$ lies in $\operatorname{int} rB$ for large $r$ and hence satisfies $0 \in \Omega(x_r) + N_C(x_r)$, as required. □

A straightforward exercise shows in particular that the growth condition (8.3.8) holds whenever the cusco $\Omega$ is defined by $x \in \mathbf{R}^n \mapsto x^T Ax$ for a matrix $A$ in $\mathbf{S}^n_{++}$.

The most important example of a noncompact variational inequality is the case when the set $C$ is a closed convex cone $S \subset \mathbf{E}$. In this case $VI(\Omega, S)$ becomes the *multivalued complementarity problem*:

$$\begin{array}{l} \text{Find points } x_0 \text{ in } S \text{ and } y_0 \text{ in } \Omega(x_0) \cap (-S^-) \\ \text{satisfying } \langle x_0, y_0\rangle = 0. \end{array} \tag{8.3.9}$$

As a particular example, we consider the dual pair of abstract linear programs (5.3.4) and (5.3.5):

$$\inf\{\langle c, z\rangle \mid Az - b \in H,\ z \in K\} \tag{8.3.10}$$

(where $\mathbf{Y}$ is a Euclidean space, the map $A : \mathbf{E} \to \mathbf{Y}$ is linear, the cones $H \subset \mathbf{Y}$ and $K \subset \mathbf{E}$ are closed and convex, and $b$ and $c$ are given elements of $\mathbf{Y}$ and $\mathbf{E}$ respectively), and

$$\sup\{\langle b, \phi\rangle \mid A^*\phi - c \in K^-,\ \phi \in -H^-\}. \tag{8.3.11}$$

As usual, we denote the corresponding primal and dual optimal values by $p$ and $d$. We consider a corresponding variational inequality on the space $\mathbf{E} \times \mathbf{Y}$:

$$VI(\Omega, K \times (-H^-)) \ \text{ with } \ \Omega(z, \phi) = (c - A^*\phi, Ax - b). \tag{8.3.12}$$

**Theorem 8.3.13 (Linear programming and variational inequalities)** *Any solution of the above variational inequality (8.3.12) consists of a pair of optimal solutions for the linear programming dual pair (8.3.10) and (8.3.11). The converse is also true, providing there is no duality gap ($p = d$).*

We leave the proof as an exercise.

Notice that the linear map appearing in the above example, namely $M : \mathbf{E} \times \mathbf{Y} \to \mathbf{E} \times \mathbf{Y}$ defined by $M(z, \phi) = (-A^*\phi, Az)$, is monotone. We study monotone complementarity problems further in Exercise 7.

To end this section we return to the complementarity problem (8.3.9) in the special case where $\mathbf{E}$ is $\mathbf{R}^n$, the cone $S$ is $\mathbf{R}^n_+$, and the multifunction $\Omega$ is single-valued: $\Omega(x) = \{F(x)\}$ for all points $x$ in $\mathbf{R}^n_+$. In other words, we consider the following problem:

Find a point $x_0$ in $\mathbf{R}^n_+$ satisfying $F(x_0) \in \mathbf{R}^n_+$ and $\langle x_0, F(x_0)\rangle = 0$.

The lattice operation $\wedge$ is defined on $\mathbf{R}^n$ by $(x \wedge y)_i = \min\{x_i, y_i\}$ for points $x$ and $y$ in $\mathbf{R}^n$ and each index $i$. With this notation we can rewrite the above problem as the following *order complementarity problem.*

$OCP(F)$: Find a point $x_0$ in $\mathbf{R}^n_+$ satisfying $x_0 \wedge F(x_0) = 0$.

The map $x \in \mathbf{R}^n \mapsto x \wedge F(x) \in \mathbf{R}^n$ is sometimes amenable to fixed point methods.

As an example, let us fix a real $\alpha > 0$, a vector $q \in \mathbf{R}^n$, and an $n \times n$ matrix $P$ with nonnegative entries, and define the map $F : \mathbf{R}^n \to \mathbf{R}^n$ by $F(x) = \alpha x - Px + q$. Then the complementarity problem $OCP(F)$ is equivalent to finding a fixed point of the map $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$\Phi(x) = \frac{1}{\alpha}(0 \vee (Px - q)), \tag{8.3.14}$$

a problem that can be solved iteratively (see Exercise 9).

## Exercises and commentary

A survey of variational inequalities and complementarity problems may be found in [93]. The normal mapping $\Omega_C$ is especially well studied when the multifunction $\Omega$ is single-valued with affine components and the set $C$ is polyhedral. In this case the normal mapping is piecewise affine (see [164]). More generally, if we restrict the class of multifunctions $\Omega$ we wish to consider in the variational inequality, clearly we can correspondingly restrict the versions of the Kakutani–Fan theorem or normal mappings we study. Order complementarity problems are studied further in [26]. The Nash equilibrium theorem (Exercise 10(d)), which appeared in [147], asserts the existence of a Pareto efficient choice for $n$ individuals consuming from $n$ associated convex sets with $n$ associated joint cost functions.

1. Prove the equivalence of the various formulations (8.3.2), (8.3.3), (8.3.4) and (8.3.5) with the original variational inequality $VI(\Omega, C)$.

2. Use Section 8.2, Exercise 4 (Composition) to prove the multifunction

$$(I - \Omega) \circ P_C$$

in the proof of Theorem 8.3.6 (Solvability of variational inequalities) is a cusco.

3. Consider Theorem 8.3.6 (Solvability of variational inequalities). Use the function

$$x \in [0,1] \mapsto \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ -1 & \text{if } x = 0 \end{cases}$$

to prove the assumption in the theorem—that the multifunction $\Omega$ is USC—cannot be weakened to $\Omega$ closed.

4. * **(Variational inequalities containing cuscos)** Suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex, and consider a multifunction $\Omega : C \to \mathbf{E}$.

   (a) If $\Omega$ contains a cusco, prove the variational inequality $VI(\Omega, C)$ has a solution.

   (b) Deduce from Michael's theorem (8.2.8) that if $\Omega$ is LSC with nonempty closed convex images then $VI(\Omega, C)$ has a solution.

5. Check the details of the proof of von Neumann's minimax theorem.

6. Prove Theorem 8.3.13 (Linear programming and variational inequalities).

7. **(Monotone complementarity problems)** Suppose the linear map $M : \mathbf{E} \to \mathbf{E}$ is monotone.

   (a) Prove the function $x \in \mathbf{E} \mapsto \langle Mx, x \rangle$ is convex.

   For a closed convex cone $S \subset \mathbf{E}$ and a point $q$ in $\mathbf{E}$, consider the optimization problem

$$\inf\{\langle Mx + q, x\rangle \mid Mx + q \in -S^-,\ x \in S\}. \tag{8.3.15}$$

   (b) If the condition $-q \in \operatorname{core}(S^- + MS)$ holds, use the Fenchel duality theorem (3.3.5) to prove problem (8.3.15) has optimal value zero.

(c) If the cone $S$ is polyhedral, problem (8.3.15) is a convex "quadratic program": when the optimal value is finite, it is known that there is no duality gap for such a problem and its (Fenchel) dual, and that both problems attain their optimal value. Deduce that when $S$ is polyhedral and contains a point $x$ with $Mx+q$ in $-S^-$, there is such a point satisfying the additional complementarity condition $\langle Mx + q, x\rangle = 0$.

8. * Consider a compact convex set $C \subset \mathbf{E}$ satisfying $C = -C$ and a continuous function $f : C \to \mathbf{E}$. If $f$ has no zeroes, prove there is a point $x$ on the boundary of $C$ satisfying $\langle f(x), x\rangle < 0$. (Hint: For positive integers $n$, consider $VI(f + I/n, C)$.)

9. **(Iterative solution of OCP [26])** Consider the order complementarity problem $OCP(F)$ for the function $F$ that we defined before equation (8.3.14). A point $x^0$ in $\mathbf{R}^n_+$ is *feasible* if it satisfies $F(x^0) \geq 0$.

    (a) Prove the map $\Phi$ in equation (8.3.14) is *isotone*: $x \geq y$ implies $\Phi(x) \geq \Phi(y)$ for points $x$ and $y$ in $\mathbf{R}^n$.

    (b) Suppose the point $x^0$ in $\mathbf{R}^n_+$ is feasible. Define a sequence $(x^r)$ in $\mathbf{R}^n_+$ inductively by $x^{r+1} = \Phi(x^r)$. Prove this sequence decreases monotonically: $x_i^{r+1} \leq x_i^r$ for all $r$ and $i$.

    (c) Prove the limit of the sequence in part (b) solves $OCP(F)$.

    (d) Define a sequence $(y^r)$ in $\mathbf{R}^n_+$ inductively by $y^0 = 0$ and $y^{r+1} = \Phi(y^r)$. Prove this sequence increases monotonically.

    (e) If $OCP(F)$ has a feasible solution, prove the sequence in part (d) converges to a limit $\bar{y}$ which solves $OCP(F)$. What happens if $OCP(F)$ has no feasible solution?

    (f) Prove the limit $\bar{y}$ of part (e) is the *minimal* solution of $OCP(F)$: any other solution $x$ satisfies $x \geq \bar{y}$.

10. * **(Fan minimax inequality [74])** We call a real function $g$ on a convex set $C \subset \mathbf{E}$ *quasiconcave* if the set $\{x \in C \,|\, g(x) \geq \alpha\}$ is convex for all real $\alpha$.

    Suppose the set $C \subset \mathbf{E}$ is nonempty, compact, and convex.

    (a) If the function $f : C \times C \to \mathbf{R}$ has the properties that the function $f(\cdot, y)$ is quasiconcave for all points $y$ in $C$ and the function $f(x, \cdot)$ is lower semicontinuous for all points $x$ in $C$, prove *Fan's inequality*:

$$\min_y \sup_x f(x, y) \leq \sup_x f(x, x).$$

(Hint: Apply the KKM theorem (Section 8.1, Exercise 15) to the family of sets

$$\{y \in C \mid f(x, y) \le \beta\} \text{ for } x \in C,$$

where $\beta$ denotes the right hand side of Fan's inequality.)

(b) If the function $F : C \to \mathbf{E}$ is continuous, apply Fan's inequality to the function $f(x, y) = \langle F(y), y - x\rangle$ to prove the variational inequality $VI(F, C)$ has a solution.

(c) Deduce Fan's inequality is equivalent to the Brouwer fixed point theorem.

(d) **(Nash equilibrium)** Define a set $C = C_1 \times C_2 \times \ldots \times C_n$, where each set $C_i \subset \mathbf{E}$ is nonempty, compact, and convex. For any continuous functions $f_1, f_2, \ldots, f_n : C \to \mathbf{R}$, if each function

$$x_i \in C_i \mapsto f_i(y_1, \ldots, x_i, \ldots, y_n)$$

is convex for all elements $y$ of $C$, prove there is an element $y$ of $C$ satisfying the inequalities

$$f_i(y) \le f_i(y_1, \ldots, x_i, \ldots, y_n) \text{ for all } x_i \in C_i,\ i = 1, 2, \ldots, n.$$

(Hint: Consider the function

$$f(x, y) = \sum_i (f_i(y) - f_i(y_1, \ldots, x_i, \ldots, y_n))$$

and apply Fan's inequality.)

(e) **(Minimax)** Apply the Nash equilibrium result from part (d) in the case $n = 2$ and $f_1 = -f_2$ to deduce the Kakutani minimax theorem (Section 4.3, Exercise 14).

11. **(Bolzano–Poincaré–Miranda intermediate value theorem)** Consider the box

$$J = \{x \in \mathbf{R}^n \mid 0 \le x_i \le 1 \text{ for all } i\}.$$

We call a continuous map $f : J \to \mathbf{R}^n$ *reversing* if it satisfies the condition

$$f_i(x)f_i(y) \le 0 \text{ whenever } x_i = 0,\ y_i = 1, \text{ and } i = 1, 2, \ldots, n.$$

Prove any such map vanishes somewhere on $J$ by completing the following steps:

(a) Observe the case $n = 1$ is just the classical intermediate value theorem.

(b) For all small real $\epsilon > 0$, prove the function $f^\epsilon = f + \epsilon I$ satisfies for all $i$

$$x_i = 0 \text{ and } y_i = 1 \;\Rightarrow\; \begin{cases} \text{either} & f_i^\epsilon(y) > 0 \text{ and } f_i^\epsilon(x) \leq 0 \\ \text{or} & f_i^\epsilon(y) < 0 \text{ and } f_i^\epsilon(x) \geq 0. \end{cases}$$

(c) ¿From part (b), deduce there is a function $\widetilde{f}^\epsilon$, defined coordinatewise by $\widetilde{f}_i^\epsilon = \pm f_i^\epsilon$, for some suitable choice of signs, satisfying the conditions (for each $i$)

$$\widetilde{f}_i^\epsilon(x) \leq 0 \text{ whenever } x_i = 0 \text{ and}$$
$$\widetilde{f}_i^\epsilon(x) > 0 \text{ whenever } x_i = 1.$$

(d) By considering the variational inequality $VI(\widetilde{f}^\epsilon, J)$, prove there is a point $x^\epsilon$ in $J$ satisfying $\widetilde{f}^\epsilon(x^\epsilon) = 0$.

(e) Complete the proof by letting $\epsilon$ approach zero.

12. **(Coercive cuscos)** Consider a multifunction $\Omega : \mathbf{E} \to \mathbf{E}$ with nonempty images.

(a) If $\Omega$ is a coercive cusco, prove it is surjective.

(b) On the other hand, if $\Omega$ is monotone, use Section 8.2, Exercise 16 (Monotonicity) to deduce $\Omega$ is hypermaximal if and only if it is maximal. (We generalize this result in Exercise 13 (Monotone variational inequalities).)

13. ** **(Monotone variational inequalities)** Consider a continuous function $G : \mathbf{E} \to \mathbf{E}$ and a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$.

(a) Given a nonempty compact convex set $K \subset \mathbf{E}$, prove there is point $x_0$ in $K$ satisfying

$$\langle x - x_0, y + G(x_0)\rangle \geq 0 \text{ for all } x \in K,\ y \in \Phi(x)$$

by completing the following steps:

(i) Assuming the result fails, show the collection of sets

$$\{x \in K \mid \langle z - x, w + G(x)\rangle < 0\} \text{ for } z \in K,\ w \in \Phi(z)$$

is an open cover of $K$.

(ii) For a partition of unity $p_1, p_2, \ldots, p_n$ subordinate to a finite subcover $K_1, K_2, \ldots K_n$ corresponding to points $z_i \in K$ and $w_i \in \Phi(z_i)$ (for $i = 1, 2, \ldots, n$), prove the function

$$f(x) = \sum_i p_i(x) z_i$$

is a continuous self map of $K$.

(iii) Prove the inequality

$$\begin{aligned}\langle f(x) - x, \textstyle\sum_i p_i(x) w_i + G(x)\rangle \\ &= \sum_{i,j} p_i(x) p_j(x) \langle z_j - x, w_i + G(x)\rangle \\ &< 0\end{aligned}$$

by considering the terms in the double sum where $i = j$ and sums of pairs where $i \neq j$ separately.

(iv) Deduce a contradiction with part (ii).

(b) Now assume $G$ satisfies the growth condition

$$\lim_{\|x\| \to \infty} \|G(x)\| = +\infty \quad \text{and} \quad \liminf_{\|x\| \to \infty} \frac{\langle x, G(x)\rangle}{\|x\| \|G(x)\|} > 0.$$

(i) Prove there is a point $x_0$ in $\mathbf{E}$ satisfying

$$\langle x - x_0, y + G(x_0)\rangle \geq 0 \text{ whenever } y \in \Phi(x).$$

(Hint: Apply part (a) with $K = nB$ for $n = 1, 2, \ldots$.)

(ii) If $\Phi$ is maximal, deduce $-G(x_0) \in \Phi(x_0)$.

(c) Apply part (b) to prove that if $\Phi$ is maximal then for any real $\lambda > 0$, the multifunction $\Phi + \lambda I$ is surjective.

(d) **(Hypermaximal $\Leftrightarrow$ maximal)** Using Section 8.2, Exercise 16 (Monotonicity), deduce a monotone multifunction is maximal if and only if it is hypermaximal.

(e) **(Resolvent)** If $\Phi$ is maximal then for any real $\lambda > 0$ and any point $y$ in $\mathbf{E}$ prove there is a unique point $x$ satisfying the inclusion

$$y \in \Phi(x) + \lambda x.$$

(f) **(Maximality and surjectivity)** Prove a maximal $\Phi$ is surjective if and only if it satisfies the growth condition

$$\lim_{\|x\| \to \infty} \inf \|\Phi(x)\| = +\infty.$$

(Hint: The "only if" direction is Section 8.2, Exercise 16(k) (Monotonicity); for the "if" direction, apply part (e) with $\lambda = 1/n$ for $n = 1, 2, \ldots$, obtaining a sequence $(x_n)$; if this sequence is unbounded, apply maximal monotonicity.)

14. * **(Semidefinite complementarity)** Define $F : \mathbf{S}^n \times \mathbf{S}^n \to \mathbf{S}^n$ by

$$F(U, V) = U + V - (U^2 + V^2)^{1/2}.$$

For any function $G : \mathbf{S}^n \to \mathbf{S}^n$, prove $U \in \mathbf{S}^n$ solves the variational inequality $VI(G, \mathbf{S}^n_+)$ if and only if $F(U, G(U)) = 0$. (Hint: See Section 5.2, Exercise 11.)

## Monotonicity via convex analysis

Many important properties of monotone multifunctions can be derived using convex analysis, without using the Brouwer fixed point theorem (8.1.3). The following sequence of exercises illustrates the ideas. Throughout, we consider a monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$. The point $(u, v) \in \mathbf{E} \times \mathbf{E}$ is *monotonically related* to $\Phi$ if $\langle x - u, y - v \rangle \geq 0$ whenever $y \in \Phi(x)$: in other words, appending this point to the graph of $\Phi$ does not destroy monotonicity. Our main aim is to prove a central case of the *Debrunner-Flor extension theorem* [59]. The full theorem states that if $\Phi$ has range contained in a nonempty compact convex set $C \subset \mathbf{E}$, and the function $f : C \to \mathbf{E}$ is continuous, then there is a point $c \in C$ such that the point $(f(c), c)$ is monotonically related to $\Phi$. For an accessible derivation of this result from Brouwer's theorem, see [154]: the two results are in fact equivalent (see Exercise 19).

We call a convex function $\mathcal{H} : \mathbf{E} \times \mathbf{E} \to (\infty, +\infty]$ *representative* for $\Phi$ if all points $x, y \in \mathbf{E}$ satisfy $\mathcal{H}(x, y) \geq \langle x, y \rangle$, with equality if $y \in \Phi(x)$. Following [79], the *Fitzpatrick function* $\mathcal{F}_\Phi : \mathbf{E} \times \mathbf{E} \to [-\infty, +\infty]$ is defined by

$$\mathcal{F}_\Phi(x, y) = \sup\{\langle x, v \rangle + \langle u, y \rangle - \langle u, v \rangle \mid v \in \Phi(u)\},$$

while [171, 150] the *convexified representative* $\mathcal{P}_\Phi : \mathbf{E} \times \mathbf{E} \to [-\infty, +\infty]$ is defined by

$$\mathcal{P}_\Phi(x, y) = \inf\Big\{ \sum_{i=1}^m \lambda_i (x_i, y_i) \,\Big|\, m \in \mathbf{N},\ \lambda \in \mathbf{R}^m_+, \\ \sum_{i=1}^m \lambda_i (x_i, y_i, 1) = (x, y, 1),\ y_i \in \Phi(x_i)\ \forall i \Big\}.$$

These constructions are explored extensively in [30, 43, 172].

15. **(Fitzpatrick representatives)**

    (a) Prove the Fitzpatrick function $\mathcal{F}_\Phi$ is closed and convex.

    (b) Prove $\mathcal{F}_\Phi(x, y) = \langle x, y \rangle$ whenever $y \in \Phi(x)$.

    (c) Prove $\mathcal{F}_\Phi$ is representative providing $\Phi$ is maximal.

    (d) Find an example where $\mathcal{F}_\Phi$ is not representative.

16. **(Convexified representatives)** Consider points $x \in \mathbf{E}$ and $y \in \Phi(x)$.

    (a) Prove $\mathcal{P}_\Phi(x, y) \leq \langle x, y \rangle$.

    Now consider any points $u, v \in \mathbf{E}$.

(b) Prove $\mathcal{P}_\Phi(u,v) \geq \langle u,y\rangle + \langle x,v\rangle - \langle x,y\rangle$.

(c) Deduce $\mathcal{P}_\Phi(x,y) = \langle x,y\rangle$.

(d) Deduce $\mathcal{P}_\Phi(x,y) + \mathcal{P}_\Phi(u,v) \geq \langle u,y\rangle + \langle x,v\rangle$.

(e) Prove $\mathcal{P}_\Phi(u,v) \geq \langle u,v\rangle$ if $(u,v) \in \operatorname{conv} G(\Phi)$ and is $+\infty$ otherwise.

(f) Deduce that convexified representatives are indeed both convex and representative.

(g) Prove $\mathcal{P}_\Phi^* = \mathcal{F}_\Phi \leq \mathcal{F}_\Phi^*$.

17. * **(Monotone multifunctions with bounded range)** Suppose that the monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ has bounded range $R(\Phi)$, and let $C = \operatorname{cl}\operatorname{conv} R(\Phi)$. Apply Exercise 16 to prove the following properties.

(a) Prove the convexity of the function $f : \mathbf{E} \to [-\infty, +\infty]$ defined by
$$f(x) = \inf\{\mathcal{P}_\Phi(x,y) \mid y \in C\}.$$

(b) Prove that the function $g = \inf_{y\in C} \langle \cdot, y\rangle$ is a continuous concave minorant of $f$.

(c) Apply the Sandwich theorem (Exercise 13 in Section 3.3) to deduce the existence of an affine function $\alpha$ satisfying $f \geq \alpha \geq g$.

(d) Prove that the point $(0, \nabla\alpha)$ is monotonically related to $\Phi$.

(e) Prove $\nabla\alpha \in C$.

(f) Given any point $x \in \mathbf{E}$, show that $\Phi$ is contained in a monotone multifunction $\Phi'$ with $x$ in its domain and $R(\Phi') \subset C$.

(g) Give an alternative proof of part (f) using the Debrunner-Flor extension theorem.

(h) Extend part (f) to monotone multifunctions with unbounded ranges, by assuming that the point $x$ lies in the set $\operatorname{int}\operatorname{dom} f - \operatorname{dom} \delta_C^*$. Express this condition explicitly in terms of $C$ and the domain of $\Phi$.

18. ** **(Maximal monotone extension)** Suppose the monotone multifunction $\Phi : \mathbf{E} \to \mathbf{E}$ has bounded range $R(\Phi)$.

(a) Use Exercise 17 and Zorn's lemma to prove that $\Phi$ is contained in a monotone multifunction $\Phi'$ with domain $\mathbf{E}$ and range contained in $\operatorname{cl}\operatorname{conv} R(\Phi)$.

(b) Deduce that if $\Phi$ is in fact maximal monotone, then its domain is $\mathbf{E}$.

(c) Using Exercise 16 (Local boundedness) in Section 8.2, prove that the multifunction $\Phi'' : \mathbf{E} \to \mathbf{E}$ defined by

$$\Phi''(x) = \bigcap_{\epsilon > 0} \operatorname{cl} \operatorname{conv} \Phi'(x + \epsilon B)$$

is both monotone and a cusco.

(d) Prove that a monotone multifunction is a cusco on the interior of its domain if and only if it is maximal monotone.

(e) Deduce that $\Phi$ is contained in a maximal monotone multifunction with domain $\mathbf{E}$ and range contained in $\operatorname{cl}\operatorname{conv} R(\Phi)$.

(f) Apply part (e) to $\Phi^{-1}$ to deduce a parallel result.

19. ** **(Brouwer via Debrunner-Flor)** Consider a nonempty compact convex set $D \subset \operatorname{int} B$ and a continuous self map $g : D \to D$. By applying the Debrunner-Flor extension theorem in the case where $C = B$, the multifunction $\Phi$ is the identity map, and $f = g \circ P_D$ (where $P_D$ is the nearest point projection), prove that $g$ has a fixed point.

In similar fashion one may establish that the sum of two maximal monotone multifunctions $S$ and $T$ is maximal assuming the condition $0 \in \operatorname{core}(\operatorname{dom} T - \operatorname{dom} S)$. One commences with the *Fitzpatrick inequality* that

$$\mathcal{F}_T(x, x^*) + \mathcal{F}_S(x, -x^*) \geq 0,$$

for all $x, x^*$ in $\mathbf{E}$. This and many other applications of representative functions are described in [30].

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