# Discrete Power Series Methods

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**Introduction.**

The starting point is two papers by D. H. Armitage and I. J. Maddox.

(1) *A new type of Cesàro mean,* Analysis **9**(1989), 195-204.

(2) *Discrete Abel means,* Analysis **10**(1990), 177-186.

They studied discrete versions of the classical Cesàro and Abel summability methods.

To see what they did set

$$s_n = a_0 + a_1 + \cdots + a_n, \quad n \geq 0.$$

$$f(x) = (1-x)\sum_0^\infty s_k x^k, \quad \rho = 1$$

$$1 \leq \lambda_0 < \lambda_1 < \cdots \to \infty$$

$$x_n = 1 - \frac{1}{\lambda_n}, \quad n \geq 0$$

Recall that $\sum_0^\infty = s(A)$ if $\rho = 1$ and $f(x) \to s$ as $x \to 1^-$.

**Definition 0.1. Discrete Abel** *(Armitage and Maddox* $\sum_0^\infty = s(A_\lambda)$ *if* $f(x_n)$ *exists for all* $n$ *and* $f(x_n) \to s$ *as* $n \to \infty$.

## Known Results for $(A_\lambda)$, (DHA and IJM)

Define $E_\lambda = \{\lambda_n \colon n \geq 0\}$.

(1) Since $(A) \subseteq (A_\lambda)$, regularity is inherited from $(A)$.

(2) $(A_\lambda) \subseteq (A_\mu)$ if and only of $E_\mu - E_\lambda$ is finite.

(3) $(A_\lambda)$ is equivalent to $(A_\mu)$ if and only if $E_\mu \Delta E_\lambda$ is finite.

(4) If $\frac{\lambda_{n+1}}{\lambda_n} \to 1$ (and $\{s_n\}$ is real) then the slow decrease of $\{s_n\}$ is a tauberian condition for $(A_\lambda)$ (to ordinary convergence).

**Extension to Power Series Methods.**

(i) $\{p_k\}_0^\infty$ is a nonnegative sequence with $p_0 > 0$.

(ii) $p(x) := \sum_{k=0}^{\infty} p_k x^k$ has radius of convergence $\rho > 0$

(iii) $p_s(x) := \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k$

**Definition 0.2.** *Suppose that $p_s(x)$ exists for each $x \in (0, \rho)$. $\sum_0^\infty = s(P)$ if $p_s(x) \to s as\ x \to \rho^-$.*

This is also called $J - p$ or $(J, p_n)$ summability in the literature.

**Examples.**

(1) **Abel.** Take $p_k = 1$ for all $k$. Then $\rho = 1$, $p(x) = \frac{1}{1-x}$ and $p_s(x) = (1-x)\sum_0^\infty s_k x^k$.

(2) **Borel.** Take $p_k = \frac{1}{k!}$ for all $k$. Then $\rho = \infty$, $p(x) = e^x$ and $p_s(x) = e^{-x}\sum_0^\infty \frac{s_k}{k!} x^k$.

**Discrete Power Series Methods.** Assume as before $1 \le \lambda_0 < \lambda_1 < \cdots \to \infty$ and set

$$x_n = \begin{cases} \rho - \frac{1}{\lambda_n} & \text{if } 0 < \rho < \infty, \\ \lambda_n & \text{if } \rho = \infty. \end{cases}$$

**Definition 0.3.** *Suppose* $p_s(x_n)$ *exists for all* $n \ge 0$. $\sum\limits_0^\infty = s(P_\lambda)$ *if* $p_s(x_n) \to s$ *as* $n \to \infty$.

**Examples.**

(1) **Discrete Abel.** $p_s(x_n) = (1 - x_n) \sum\limits_0^\infty s_k x_n^k = \dfrac{1}{\lambda_n} \sum\limits_0^\infty s_k (1 - \dfrac{1}{\lambda_n})^k$.

(2) **Borel.** $p_s(x_n) = e^{-x_n} \sum\limits_0^\infty \dfrac{s_k}{k!} x_n^k = e^{-\lambda_n} \sum\limits_0^\infty \dfrac{s_k}{k!} \lambda_n^k$.

## Results for Discrete Power Series Methods.

Since $(P) \subseteq (P_\lambda)$, the regularity of $(P_\lambda)$ is inherited from $(P)$. The latter is well–known but was summarized concisely by Borwein in

*On Methods of Summability Based on Power Series*, Proc. Royal Soc. Edinburgh, **64**(1957).

**Theorem 0.1.**

(1) *If* $0 < \rho < \infty$ *then* $(P_\lambda)$ *is regular if and only if* $\sum_{k=0}^{\infty} p_k \rho^k = \infty$.

(2) *If* $\rho = \infty$ *then* $(P_\lambda)$ *is regular.*

## Abelian Results.

As in Armitage and Maddox, $E_\lambda = \{\lambda_n : n \geq 0\}$.

**Theorem 0.2.** *Suppose that $(P_\lambda)$ is regular and that $p_k > 0$ for all $k \geq 0$.*

(1) *$P_\lambda \subseteq P_\mu$ if and only if $E_\mu - E_\lambda$ is finite.*
(2) *$P_\mu = P_\lambda$ if and only if $E_\mu \Delta E_\lambda$ is finite.*

**Corollary 0.3.** *If $(P_\lambda)$ is regular and $p_k > 0$ for all $k \geq 0$, then $(P) \subset (P_\lambda)$ strictly.*

**Proof.**
Set $\mu_n = \dfrac{\lambda_n + \lambda_{n+1}}{2}$ for $n \geq 0$. Then $E_\lambda \cap E_\mu = \emptyset$. Hence we cannot have $(P_\lambda) \subseteq (P_\mu)$. Therefore there exists a sequence $\{s_k\}$ such that $s_k \to s(P_\lambda)$ but $\{s_k\}$ is not limitable $(P_\mu)$. But we always have $(P) \subseteq (P_\mu)$. Therefore $s_k \to s(P_\lambda)$ but $s_k \not\to s(P)$.

## Tauberian Results

Assume that $\rho = 1$ so that $x_n = 1 - \frac{1}{\lambda_n}$.

Motivation for the result here is Theorem 3 in the paper of K. Ishiguro,

*A tauberian theorem for* $(J, p_n)$ *summability*, Proc. Japan Acad., **40**(1964).

**Theorem 0.4.** *Suppose that*

(i) $\dfrac{P_n}{p(x_n)} = \mathrm{O}(1)$ *as* $n \to \infty$,

(ii) $0 < p_k \leq M$ *for all* $k \geq 0$,

(iii) $\dfrac{\lambda_n}{P_n} = \mathrm{O}(1)$,

(iv) $\sum_{k=0}^{\infty} a_k = s(P_\lambda)$ *and*

(v) $a_k = \mathrm{o}(\dfrac{p_k}{P_k})$ *as* $k \to \infty$.

*Then* $\sum_{k=0}^{\infty} a_k = s$.

**Corollary 0.5. For** $(A_\lambda)$ *If* $na_n \to 0$ *and there exist positive constants,* $\gamma_1$ *and* $\gamma_2$*, such that* $\gamma_1 \leq \dfrac{\lambda_n}{n} \leq \gamma_2$ *then* $\sum_{k=0}^{\infty} a_k = s(A_\lambda)$ *implies* $\sum_{k=0}^{\infty} a_k = s$.

## Open Questions.

(1) O(), one–sided or $\sum_1^n ka_k = \mathrm{o}(n)$ type tauberian theorems.

(2) Tauberian results for $\rho = \infty$.

(3) Tauberian results between discrete weighted mean and discrete power series methods.

(4) A "slow–decrease"–type result.

## References


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