A Plethora of Remarkable Concurrences

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In an article in this journal last year [1], I showed that the lines through the mid-points of the sides of a triangle with slopes, a constant multiple of the slopes of the corresponding sides of the triangle, were concurrent. Here, I show that there are many more points with a similar property.

Take the triangle with coordinates $(0,0)$, (a,b) and (c,d). The lines through the points with coordinates (at,bt), (cu,du) and $(aw+c(1-w), bw+d(1-w))$ with slopes $\frac{\lambda a}{b}, \frac{\lambda c}{d}$ and $\frac{\lambda(a-c)}{b-d}$, respectively, are:

$$y = \frac{\lambda a x}{b} + \left(bt - \frac{a^2\lambda t}{b}\right) \tag{1}$$

$$y = \frac{\lambda c x}{d} + \left(du - \frac{c^2\lambda u}{d}\right) \tag{2}$$

$$y = \frac{\lambda x(a-c)}{(b-d)} + \frac{a^2\lambda w + ac\lambda(1-2w) - b^2 w + bd(2w-1) + (w-1)(c^2\lambda - d^2)}{d-b} \tag{3}$$

Solving (1) and (2), we get

$$x = \frac{a^2 d\lambda t - b(bdt + u(c^2\lambda - d^2))}{\lambda(ad-bc)}$$

$$y = \frac{bc(bt-du)(c^2\lambda - d^2)}{(d^2(ad-bc))} + \frac{ac\lambda t}{d} + \frac{bc^2\lambda t - du(c^2\lambda - d^2)}{d^2}$$

Substituting into (3) gives:

$$\begin{aligned}\frac{bc(bt-du)(c^2\lambda - d^2)}{d^2(ad-bc)} + \frac{ac\lambda t}{d} + \frac{(bc^2\lambda t - du(c^2\lambda - d^2))}{d^2} \\ = \lambda\left(\frac{a^2 d\lambda t - b(bdt + u(c^2\lambda - d^2))}{\lambda(ad-bc)}\right)\frac{a-c}{b-d} \\ + \frac{a^2\lambda w + ac\lambda(1-2w) - b^2 w + bd(2w-1) + (w-1)(c^2\lambda - d^2)}{d-b}\end{aligned}$$

This simplifies to:

$$\lambda\frac{a^2(t-w) + ac(2w-1) - c^2(u+w-1)}{d-b} + \frac{b^2(t-w) + bd(2w-1) - d^2(u+w-1)}{b-d} = 0, \tag{4}$$

or

$$\lambda = \frac{b^2(t-w) + bd(2w-1) - d^2(u+w-1)}{a^2(t-w) + ac(2w-1) - c^2(u+w-1)}. \tag{5}$$

If $t = u = w = \frac{1}{2}$, then left side of (4) is zero. This means that the lines always concur.

If we make the same subsitution into (4), we get:

$$\lambda = \frac{b^2(\frac{1}{2}-\frac{1}{2}) + bd(2(\frac{1}{2})-1) - d^2(\frac{1}{2}+\frac{1}{2}-1)}{a^2(\frac{1}{2}-\frac{1}{2}) + ac(2(\frac{1}{2})-1) - c^2(\frac{1}{2}+\frac{1}{2}-1)} = \frac{0}{0}$$

There are other values of u, v and w that result in the same result. Note that we will have two linear equations in the three unknowns t, u and w. Thus, unless a, b, c and d are "peculiar", we have infinitely many such triples.

We call the set of all values of u, v and w that make the left side of (5) zero, the **Concurrence Set** of the triangle. This is given by

$$\begin{aligned} \frac{T}{W} &= 1 - \frac{2cd}{ad+bc}, \\ \frac{U}{W} &= \frac{2ab}{ad+bc} - 1, \end{aligned}$$

where $t = \frac{1}{2} - T$, $u = \frac{1}{2} - U$ and $w = \frac{1}{2} - W$. Thus, there are infinitely many points in this set.

For points not in the concurrence set, we have that there is one unique value of λ such that the three lines concur

The value is

$$\frac{b^2(t-w) + bd(2w-1) - d^2(u+w-1)}{a^2(t-w) + ac(2w-1) - c^2(u+w-1)}$$

Replace t by $\frac{1}{2} - T$, etc.; the value is

$$\frac{b^2(T-W) + 2bdW - d^2(U+W)}{a^2(T-W) + 2acW - c^2(U+W)}$$

The point of concurrence is

$x = \dfrac{N_x}{D_x}$, where

$$\begin{aligned} N_x &= (TU(2ad(b-d) + 2bc(b-d)) + TW(2ad(2b-d) - 2bcd) - bT(ad+bc) \\ &\quad -2bUW(ad + c(b-2d)) + U(ad^2 + bcd) + W(d-b)(ad-bc)), \end{aligned}$$

and

$$D_x = (2b^2T - 2d^2U - 2W(b^2 - 2bd + d^2));$$

and $y = \dfrac{N_y}{D_y}$, where

$$\begin{aligned} N_y &= (2TU(a^2d + ac(b-d) - b^2c^2) + 2cTW(a(2b-d) - b^2c) - aT(ad+bc) \\ &\quad -2aUW(ad + c(b-2d)) + cU(ad + b^2c) + W(a^2d - ac(b+d) + b^2c^2)) \end{aligned}$$

and

$$D_y = (2a^2T - 2c^2U - 2W(a^2 - 2ac + c^2)).$$

For fixed T and U, or any other pair, the locus of this may reduce to a single point or to a rectangular hyperbola, or even a straight line.

References

[1] Shawyer, Bruce, *Some Remarkable Concurrences*, Forum Geometricorum (2001), pp. 69-74.

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