David Borwein's 80th Birthday CMS Halifax June 2004

1 Bohr-Hardy Theorems

In his doctoral thesis, David investigated some extensions and refinements of the "Bohr-Hardy" theorems. To set this in context we first recall a familiar theorem of Abel and Dirichlet.

Theorem 1. *If (i) a series $\sum a_n$ is convergent or has bounded partial sums, and (ii) the sequence $\{f_n\}$ decreases monotonically to 0, or, more generally, $f_n \to 0$ and $\sum |\Delta f_n| < \infty$, then $\sum a_n f_n$ is convergent.*

For α real and n a nonnegative integer, let

$$\epsilon_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Let $\alpha > -1$. Given an infinite series $\sum a_n$, let $s_n = a_0 + a_1 + \cdots + a_n$ and define

$$s_n^\alpha = \frac{1}{\epsilon_n^\alpha} \sum_{\nu=0}^{n} \epsilon_{n-\nu}^{\alpha-1} s_\nu.$$

We say that $\sum a_n$ is **summable** (C,α) to s if $\lim_{n\to\infty} s_n^\alpha = s$, and that $\sum a_n$ is **bounded** (C,α) if $s_n^\alpha = O(1)$.

Hardy in 1908, and Bohr in 1909, independently established the following theorem, which became known subsequently as the "Bohr-Hardy Theorem." See, for example, G. H. Hardy *Divergent Series* Oxford 1949 (Theorem 71).

Theorem 2. *If*

$$\sum_{n=0}^{\infty} a_n \text{ is summable, or bounded, } (C,\alpha) \tag{1}$$

where α is a nonnegative integer,

$$f_n \to 0; \text{ and } \sum_{n=0}^{\infty} (n+1)^\alpha |\Delta^{\alpha+1} f_n| < \infty; \tag{2}$$

then $\sum a_n f_n$ is summable (C,α). Moreover

$$\sum_{n=0}^{\infty} a_n f_n = \sum_{n=0}^{\infty} s_n^\alpha \Delta^{\alpha+1} f_n,$$

where the left hand sum is interpreted in the (C, α) sense and the right hand series is absolutely convergent.

The Bohr-Hardy Theorem is an example of a "summability factor" theorem. There is a rich literature on summability factor theorems which, incidentally, is of interest in connection with the computation of β-duals of sequence spaces.

Fekete in 1917, showed that the Bohr-Hardy conditions above were also necessary, and A. F. Andersen, in 1921, established a certain generalization of the Bohr-Hardy Theorem. L. S. Bosanquet, in 1942, in an article in the Journal of the London Mathematical Society Volume 17, proved a definitive generalized Bohr-Hardy Theorem, which established both the necessity and sufficiency of the Bohr-Hardy conditions.

Other authors had considered Bohr-Hardy type theorems for Cesàro summability of integrals.

Let a be Lebesgue measurable and integrable on every finite interval $(0, X)$. Let $\alpha > -1$, and

$$A_0(x) = \int_0^x a(t)\, dt \quad \textit{and} \quad A_\alpha(x) = \int_0^x A_{\alpha-1}(t)\, dt.$$

If

$$\lim_{x \to \infty} \Gamma(\alpha+1) x^{-\alpha} A_\alpha(x) = A,$$

we write

$$\int_0^\infty a(x)\, dx = A\ (C, \alpha),$$

and say that the integral is summable (C, α) to A. If $\Gamma(\alpha+1)x^{-\alpha}A_\alpha(x) = O(1)$ as $x \to \infty$, then we say that

$$\int_0^\infty a(x)\, dx \text{ is bounded } (C, \alpha).$$

David established in his doctoral thesis a 'necessity' Bohr-Hardy integral type theorem and published it in *A summability factor theorem*, J. London Math. Soc. 25 (1950), 302-315. His main theorem is the following. In this theorem α need not be an integer, and $\phi^{(\alpha)}(t)$ is the α-th fractional derivative of $\phi(t)$.

Theorem 3. *For $\alpha \geq 0$, if $\phi^{(\alpha)}(t)$ is absolutely continuous and*

$$\int_1^\infty f(t)\phi(t)\, dt \textit{ is bounded } (C) \textit{ [or summable } (C)\textit{]}$$

whenever

$$\int_1^\infty f(t)\,dt \textit{ is summable } (C,\alpha) \textit{ [or bounded } (C,\alpha)\textit{]},$$

then there is an absolutely continuous function $\psi(t)$ *such that* $\psi(t) = \phi(t)$ *a.e. in* $(1,\infty)$ *and* $\psi(t) \to l$ *[or is* $o(1)$*] as* $t \to \infty$*, and*

$$\int_1^\infty t^\alpha |\phi^{(\alpha+1)}(t)|\,dt < \infty.$$

The proof of this theorem required imagination and considerable technical skill and demonstrates at an early stage in his career, David's mastery of classical analysis, a mastery indeed that remains undiminished to this day.

Summability factor theorems have held an interest for David throughout his career and he has made many significant contributions in this area.

2 Strong Summability

Let $\alpha > -1$ and $\lambda \geq 1$. A series $\sum a_n$ is said to be **strongly summable** $(C, \alpha+1)$ **with sum** s if

$$\frac{1}{n+1}\sum_{k=0}^{n} |s_k^\alpha - s|^\lambda = o(1).$$

We denote this by

$$\sum_{n=0}^{\infty} a_n = s[C,\alpha+1]_\lambda \text{ or } s_n \to s[C,\alpha+1]_\lambda.$$

This definition can be made for $\lambda > 0$, but certain pathologies, that need not detain us here, occur in the range $0 < \lambda < 1$.

It is easy to prove that, for $\alpha > -1$,

$$(C,\alpha) \Rightarrow [C,\alpha+1]_\lambda \Rightarrow (C,\alpha+1). \tag{3}$$

Strong summability, particularly $[C,1]_\lambda$, played a significant role in the study of Cesàro summability of Fourier series. T. M. Flett (Some remarks on strong summability, *Quarterly J. of Math.* **10** (1959), 115-139) made a detailed study of strong and absolute Cesàro summability, obtaining a number of interesting and significant results. Much of the analysis in Flett's paper was complicated by the fact that the $(C,1)$ mean of the sequence $\{|s_n^\alpha - s|^\lambda\}$ is used independently of α.

David, in a seminal paper (On strong and absolute summability, *Proceedings of the Glasgow Mathematical Association* Volume IV, Part III, (1960)), introduced the following definition of strong summability which applies quite generally, simplifies and clarifies many of Flett's results as well as considerably broadenung and enriching the study of strong summability.

Let $Q = (q_{n,r}) \quad (n, r = 0, 1, \ldots)$ be a (summability) matrix, and let

$$\sigma_n = Q(s_n) = \sum_{r=0}^{\infty} q_{n,r} s_r.$$

Let $P = (p_{n,r})$ be a matrix with $p_{n,r} \geq 0 \quad (n, r = 0, 1 \ldots)$. The series $\sum a_n$ is said to be summable $[P, Q]_\lambda \quad (\lambda > 0)$ to s, if

$$P(|\sigma_n - s|^\lambda) = \sum_{r=0}^{\infty} p_{n,r} |\sigma_n - s|^\lambda$$

is defined for all n and tends to 0 and n tends to ∞.

The series $\sum a_n$ is summable $|Q, \gamma|_\lambda$ (γ real and $\lambda > 0$) if

$$\sum_{n=1}^{\infty} n^{\gamma\lambda+\lambda-1} |\sigma_n - \sigma_{n-1}|^\lambda < \infty.$$

Let Q be any matrix and P satisfy

$$\sup_{n \geq 0} \sum_{r=0}^{\infty} p_{n,r} < \infty.$$

If $\lambda > \mu > 0$, then, a standard application of Hölder's inequality shows that $[P, Q]_\lambda \Rightarrow [P, Q]_\mu$.

The following result generalizes (3) and has a virtually trivial proof.

Theorem 4. *If P and Q are matrices and P is regular, then*

$$Q \Rightarrow [P, Q]_\lambda \text{ for } \lambda > 0 \text{ and } [P, Q]_\lambda \Rightarrow PQ \text{ for } \lambda \geq 1.$$

David's definition of strong summability has been applied successfully to Nörlund summability and in the particular case of Cesàro summability the $(C, 1)$ matrix in the rôle of P is replaced by the weighted mean matrix (M, ϵ_n^α).

More precisely, the definition of strong Cesáro summability arising from the specialization of the definition of strong Nörlund summability corresponding to $[C,\alpha+1]_\lambda$ is the method

$$[P,Q]_\lambda = [(M,\epsilon_n^\alpha),(C,\alpha)]_\lambda.$$

Considerable facility is gained from the fact that

$$PQ = (M,\epsilon_n^\alpha)(C,\alpha) = (C,\alpha+1).$$

Indeed,

$$\begin{aligned} \frac{1}{\epsilon_n^{\alpha+1}}\sum_{\nu=0}^{n}\epsilon_\nu^\alpha s_\nu^\alpha &= \frac{1}{\epsilon_n^{\alpha+1}}\sum_{\nu=0}^{n}\sum_{k=0}^{\nu}\epsilon_{\nu-k}^{\alpha-1}s_k \\ &= \frac{1}{\epsilon_n^{\alpha+1}}\sum_{k=0}^{n}\sum_{\nu=k}^{n}\epsilon_{\nu-k}^{\alpha-1}s_k \\ &= \frac{1}{\epsilon_n^{\alpha+1}}\sum_{k=0}^{n}s_k\sum_{\nu=k}^{n}\epsilon_{\nu-k}^{\alpha-1} \\ &= \frac{1}{\epsilon_n^{\alpha+1}}\sum_{k=0}^{n}\epsilon_{n-k}^{\alpha}s_k \\ &= s_n^{\alpha+1} \end{aligned} \tag{4}$$

It is easy to show that $(C,1) \iff (M,\epsilon_n^\alpha)$ for $\alpha > -1$, so the two definitions of strong Cesáro summability are eqivalent. So David's more complicated looking definition yields considerable simplification and clarification in the details of the proofs of a number of results.

The advantage of David's definition of strong summability is thus seen to lie in being able to choose the matrix P in a suitable relation to the matrix Q that certain properties follow easily and naturally. Unfortunately, this paper of David's does not seem to have been read widely. A number of authors have continued to define strong summability using the $(C,1)$ matrix in place of the matrix P but with any matrix Q. They have obtained thereby only limited and sometimes peculiar results.

3 Bounded Operators on l_p

Establishing good representation theorems for bounded linear operators on Banach spaces had always been important. Summability provides many examples of such theorems, the celebrated Toeplitz Theorem concening operators on the spaces l_∞, c_0 and c being perhaps the most famous and the important Schur Theorem characterizing the bounded linear operators in $B(l_\infty, c)$ has been noted for the fact that it does not appear to have a "functional analytic" proof. The apparent dichotomy and attendant competition between 'classical' and 'functional analytic' approaches to questions arising from summability seems to be often overblown and sometimes silly. This is not to try to gainsay the importance of the illumination often shed on results by diverse approaches to particular results, but there is a danger of losing contact with parts of mathematics through a too strict adherance to the dictates of fashions of the moment. R. Hermann, in his introduction to Klein's *Development of mathematics in the nineteenth century* aptly observed that 'we are so used to thinking in terms of "progress" of science that it is hard to remember that certain matters were better understood a hundred years ago.' David's work on bounded operators on l_p for $1 < p < \infty$ illustrates well the symbiosis that can exist between the 'classical' and 'functional analytic' approach to operator theory. Without his effort, the penetrating analysis that he brought to the problem could well have lain long unrealized.

The spaces l_p for $1 \leq p < \infty$ have the sequences $e_n = \{\delta_{mn}\}$ as a Schauder basis. Consequently all operators in $B(l_p)$ are represented by infinite matrices in the obvious way. In the case of $B(l_1)$, the representation of these operators by means of a condition involving only the entries of the matrix is easily achieved. Indeed,

Theorem 5 (Knopp-Lorentz). *Let $A = \{a_{nk}\} \quad n,k = 0,1,2,\ldots$ be a given matrix and let*

$$y_n = \sum_{k=0}^{\infty} a_{nk}x_k$$

be convergent for $n = 0,1,2,\ldots$ and for every sequence $\{x_k\} \in l_1$. Then $A \in B(l_1)$ if and only if

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{nk}| = M < \infty.$$

Moreover, $\|A\| = M$.

This simplicity in representing operators on l_1 does not extend to l_p for $1 < p < \infty$. In fact, to date there are no known necessary and sufficient conditions expressed in terms on the entries of the matrix for membership in $B(l_p)$ except for the case $p = 2$, and there the conditions are quite intractable. Laurence Crone, *A characterization of matrix operators on* l^2 Mathematische Zeitschrift 123, 315-317 (1971), established the following theorem. (A^* stands for the conjugate transpose of A).

Theorem 6 (Crone 1971). *The matrix* $A \in B(l_2)$ *if and only if*

$$\text{the rows of } A \text{ are in } l_2; \tag{5}$$

$$(A^*A)^n \text{ is defined for } n = 1, 2, \ldots; \tag{6}$$

$$\sup_n \sup_i |[(A^*A)^n]_{ii}|^{1/n} = K < \infty. \tag{7}$$

One might make good progress, however, by restricting attention to particular families of matrices to try to determine simple criteria for belonging to $B(l_p)$. Hardy in *An inequality for Hausdorff means* J. London Math. Soc. 18, 46-50 (1943), establishes a result that can readily be interpreted to show that, for $p > 1$ and $\alpha > 0$, the Cesàro matrix (C, α) is a bounded operator on l_p with norm $\Gamma(1+\alpha)\Gamma(1-1/p)/\Gamma(1+\alpha-1/p)$. Borwein and Jakimovski have established results about generalized Hausdorff matrices as bounded operators on l_p and Cass and Kratz, building on Borwein's and Jakimovski's work, looked at the case of Nörlund and weighted mean operators on l_p where the entries of the matrices were associated with logarithmico-exponential functions (a restriction that was integral to the analysis given by Cass and Kratz) and, along with norm estimates, obtained reasonable conditions for matrices to be in $B(l_p)$. But the final triumph belongs to David who was able to show, surprisingly, that the dependence on logarithmico-exponential functions was redundant and obtained a far less restrictive monotonicity condition.

A definition and some notation need to be introduced at this point. Let $1 < p < \infty$ and $1/p + 1/q = 1$.

Definition 1. *Let* $a_0 > 0$ *and* $a_n \geq 0$ *for* $n = 1, 2, \ldots$ *and* $A_n = \sum_{k=0}^n a_k$. *Then the* N_a *transform* $\{t_n\}$ *of a sequence* $\{s_n\}$ *is given by*

$$t_n = \frac{1}{A_n} \sum_{\nu=0}^{n} a_{n-\nu} s_\nu.$$

Also

$$\sigma_1(n) = \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} \left(\frac{n+1}{k+1}\right)^{1/p},$$

$$\sigma_2(k) = \sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n} \left(\frac{k+1}{n+1}\right)^{1/q},$$

$$M_1 = \sup_{n\geq 0} \sigma_1(n), \quad M_2 = \sup_{k\geq 0} \sigma_2(k).$$

David, in *Nörlund operators on* l_p Canadian Mathematical Bulletin Vol.36(1) (1993) 8-14, obtained the following theorem.

Theorem 7. *Suppose that* $na_n/A_n \to \alpha$. *Then the Nörlund matrix* $N_a \in B(l_p)$ *iff* $\alpha < \infty$. *Moreover, if* $\alpha < \infty$, *then*

$$\|N_a\|_p \leq M_1^{1/q} M_2^{1/p} < \infty,$$

and if, in addition, $\{n^c a_n\}$ *is eventually monotonic for every constant* $c \neq 1-\alpha$, *then,*

$$\lim_{n\to\infty} \sigma_1(n) = \frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)} \leq \|N_a\|_p.$$

Further, the monotonicity condition is redundant when $\alpha = 0$.

David also shows that it is possible to have $N_a \in B(l_p)$ for $1 < p < \infty$ when $\sup na_n/A_n = \infty$.

Notwithstanding a gracious acknowledgement to Dr. Xiaopeng Gao, a former graduate student at The University of Western Ontario, for a certain refinement that led to the final form of the above theorem, these results stand as signal testament to David's penetrating skills and the thoroughness that is evident in all his mathematical research.