THE HILBERT MATRIX

$$H = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \\ \frac{1}{2} & \frac{1}{3} & & \\ \frac{1}{3} & & & \\ & & & \end{pmatrix}$$

$$\|Hx\|_p \leq \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|x\|_p \qquad \begin{pmatrix} p > 1 \\ x \in \ell^p \end{pmatrix}$$

$$|||H|||_{p,p} = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$$

FROBENIUS

$$\sup_j \sum_k |a_{jk}| = R < \infty$$

$$\sup_k \sum_j |a_{jk}| = C < \infty$$

$$\Longrightarrow \quad \|A\|_{p,p} \leq C^{\frac{1}{p}} R^{\frac{1}{p^*}}$$

$$\left(\frac{1}{p} + \frac{1}{p^*} = 1\right)$$

$$H(\theta) = \begin{pmatrix} 1 & \theta & \theta^2 & \theta^3 \\ 1-\theta & 2(1-\theta)\theta & 3(1-\theta)\theta^2 & \\ (1-\theta)^2 & 3(1-\theta)^2\theta & & \\ (1-\theta)^3 & & & \end{pmatrix}$$

$(0<\theta<1)$

$$\|H(\theta)\|_{p,p} \leqslant \theta^{-\frac{1}{p}} (1-\theta)^{-\frac{1}{p^*}}$$

$$H_{n,k} = \int_0^1 H(\theta)_{n,k}\, d\theta$$

$$\begin{aligned} \|H\|_{p,p} &= \left\| \int_0^1 H(\theta)\, d\theta \right\|_{p,p} \\ &\leqslant \int_0^1 \|H(\theta)\|_{p,p}\, d\theta \\ &= \Gamma\left(1-\tfrac{1}{p}\right) \Gamma\left(\tfrac{1}{p}\right) \\ &= \pi \operatorname{cosec}\left(\tfrac{\pi}{p}\right) \end{aligned}$$

Hardy

$$C = \begin{pmatrix} 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\|C\|_{p,p} = p^* \quad (p>1)$$

$$E(\theta) = \begin{pmatrix} 1 & & & \\ 1-\theta & \theta & & \\ (1-\theta)^2 & 2(1-\theta)\theta & \theta^2 & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\|Hx\|_p \leq \frac{\pi}{p^* \sin\left(\frac{\pi}{p}\right)} \|Cx\|_p \quad (p>1)$$

Hausdorff matrices

$$\int_0^1 \begin{pmatrix} 1 & & & \\ 1-\theta & \theta & & \\ (1-\theta)^2 & 2(1-\theta)\theta & \theta^2 & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} d\mu(\theta)$$

If

$$\mu_n = \int_0^1 \theta^n \, d\mu(\theta)$$

$$\begin{pmatrix} \mu_0 & & & \\ \mu_0-\mu_1 & \mu_1 & & \\ \mu_0-2\mu_1+\mu_2 & 2(\mu_1-\mu_2) & \mu_2 & \\ \mu_0-3\mu_1+3\mu_2-\mu_3 & 3(\mu_1-2\mu_2+\mu_3) & 3(\mu_2-\mu_3) & \mu_3 \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Hausdorff matrices commute.

Is there a simple "counting" argument to show that

$$C(\alpha) \leftrightarrow C(\beta) \quad (\alpha, \beta = 1, 2, \ldots).$$

Cesaro matrices

$$C(2) = \begin{pmatrix} 1 & & & & \\ \frac{2}{3} & \frac{1}{3} & & & \\ \frac{3}{6} & \frac{2}{6} & \frac{1}{6} & & \\ \frac{4}{10} & \frac{3}{10} & \frac{2}{10} & \frac{1}{10} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$C(3) = \begin{pmatrix} 1 & & & & \\ \frac{3}{4} & \frac{1}{4} & & & \\ \frac{6}{10} & \frac{3}{10} & \frac{1}{10} & & \\ \frac{10}{20} & \frac{6}{20} & \frac{3}{20} & \frac{1}{20} & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Halmos New Orleans 1986

"Matrices I have met"

Does

$$C = \begin{pmatrix} 1 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

have a square root?

Rhoades uses Hölder matrices

$$H(\alpha)\,H(\beta) = H(\alpha+\beta) \qquad H(1) = C$$

so $\sqrt{C} = H(\tfrac{1}{2})$

$$H(\alpha) \sim \mu_n = \frac{1}{n^\alpha}$$

$$= \int_0^1 \theta^n \,|\log\theta|^{\alpha-1}\, d\theta$$

Suppose $y = Cx$, i.e.

$$(y_1, y_2, y_3, \ldots) = \left(x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \ldots\right)$$

Then

$x \geq 0 \Rightarrow y \geq 0$

$x \downarrow \Rightarrow y \downarrow$

(OZEKI) x convex $\Rightarrow$ y convex

(TOADER) x n-convex $\Rightarrow$ y n-convex

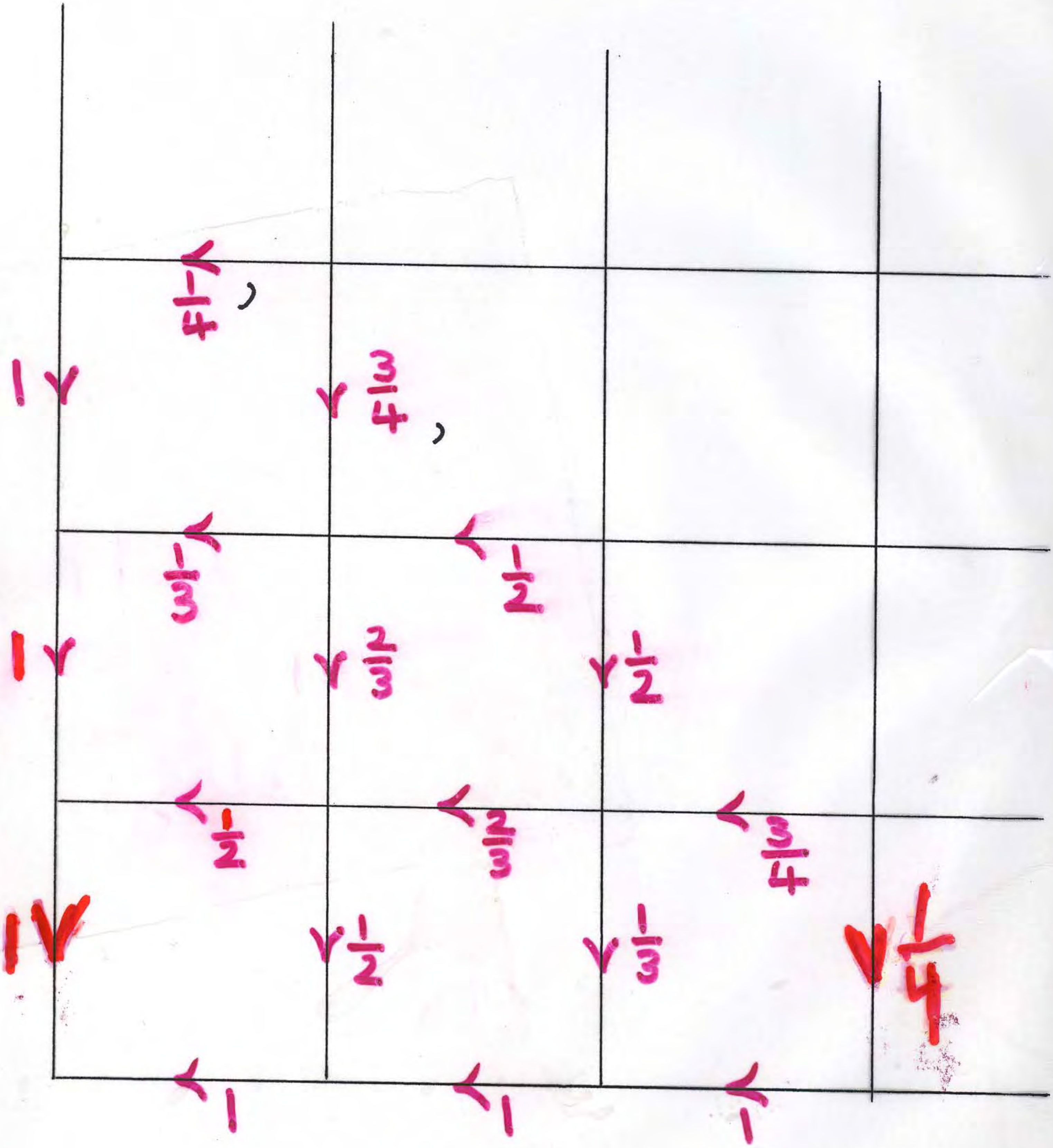
Which matrices preserve all monotonicities ... positive, negative, increasing, decreasing, convex, concave, ... ?

$$y = Hx \iff \Delta^n y_0 = \mu_n \Delta^n x_0$$

$$\Delta^n y_k = \sum_{j=0}^{k} \binom{k}{j} \left(\Delta^{k-j} \mu_{n+j}\right)\left(\Delta^n x_j\right)$$

The Hausdorff means do

... and only they.


1/4,
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MINKOWSKI'S INEQUALITY $(p \geq 1)$

$$\left(\sum_j \left(\sum_k |a_{jk}|\right)^p\right)^{\frac{1}{p}} \leq \sum_k \left(\sum_j |a_{jk}|^p\right)^{\frac{1}{p}}$$

$$\left(\int\left(\int f(x\,y)\,d\nu(y)\right)^p d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq \int\left(\int f^p(x\,y)\,d\mu(x)\right)^{\frac{1}{p}} d\nu(y)$$

Let

$$Af(y) = \int_0^1 f(xy)\,d\mu(x)$$

and

$$Bf(y) = \int_0^1 f(xy)\,d\nu(x)$$

Then

$$\left(A(Bf)^p\right)^{\frac{1}{p}} \leq B(Af^p)^{\frac{1}{p}}$$

Recall $\quad \Delta^n y_0 = \mu_n \Delta^n x_0 \iff y = Hx$

$$g^{(n)}(0) = \mu_n f^{(n)}(0)$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$g(z) = \sum_{n=0}^{\infty} \frac{\mu_n f^{(n)}(0)}{n!} z^n$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{f^{(n)}(0)}{n!} (\theta z)^n \, d\mu(\theta)$$

$$= \int_0^1 f(\theta z) \, d\mu(\theta)$$

This suggests

$$\left(A (Bx)^p\right)^{\frac{1}{p}} \leq B (Ax^p)^{\frac{1}{p}}$$

$(x \geq 0,\ p \geq 1)$ whenever A, B are Hausdorff matrices

IS IT TRUE THAT

$$\left(\frac{\left(\frac{x}{1}\right)^p + \left(\frac{x+y}{2}\right)^p + \left(\frac{x+y+z}{3}\right)^p}{3}\right)^{\frac{1}{p}}$$

$$\leq \frac{\left(\frac{x^p}{1}\right)^{\frac{1}{p}} + \left(\frac{x^p+y^p}{2}\right)^{\frac{1}{p}} + \left(\frac{x^p+y^p+z^p}{3}\right)^{\frac{1}{p}}}{3}$$

$\forall\, p \geq 1;\ x, y, z \geq 0$?

$$\begin{matrix} (x & x & x & x & x & x) \\ (x & x & x & x & x & x) \\ (x & x & x & y & y & y) \\ (x & x & y & x & y & y) \\ (x & x & y & y & z & z) \\ (x & x & y & y & z & z) \end{matrix}$$

Six vectors in $\mathbb{R}^6$

$$\|(6x, 6x, 3(x+y), 3(x+y), 2(x+y+z), 2(x+y+z))\|$$

$$\leq 2\,\|(x,x,x,x,x,x)\| + \|(x,x,x,y,y,y)\|$$

$$+ \|(x,x,y,x,y,y)\| + 2\,\|(x,x,y,y,z,z)\|$$

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	2	2	2
1	1	2	1	2	2
1	1	2	2	3	3
1	1	2	2	3	3

City Planning

Meanie-apolis

1

1	1
1	2

Teeny-weeny Meanie-apolises

When do two summability matrices commute? (They are then <u>consistent</u>.)

$$A \leftrightarrow B \qquad \begin{array}{l} Ax \to l \\ Bx \to l' \end{array}$$

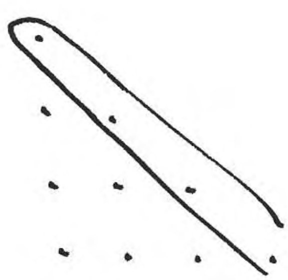
$$\begin{array}{l} BAx \to l \\ ABx \to l' \end{array} \implies l = l'$$

<u>Norlund means</u>

$$\begin{pmatrix} \frac{a_0}{A_0} & & \\ \frac{a_1}{A_1} & \frac{a_0}{A_1} & \\ \frac{a_2}{A_2} & \frac{a_1}{A_2} & \frac{a_0}{A_2} \end{pmatrix} \overset{?}{\longleftrightarrow} \begin{pmatrix} \frac{b_0}{B_0} & & \\ \frac{b_1}{B_1} & \frac{b_0}{B_1} & \\ \frac{b_2}{B_2} & \frac{b_1}{B_2} & \frac{b_0}{B_2} \end{pmatrix}$$

$$A_n = a_0 + \cdots + a_n \qquad B_n = b_0 + \cdots + b_n$$

$$a_n \geq 0, \; b_n \geq 0 \qquad a_0 = b_0 = 1.$$



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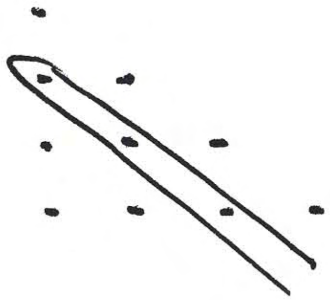
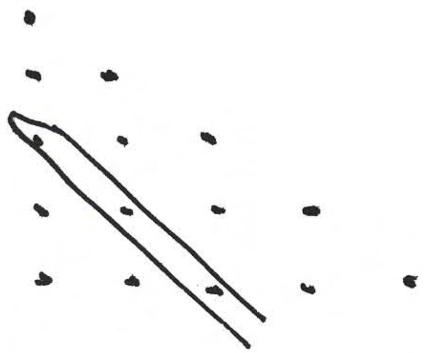


Express
b's in terms of a's
and b_1.



Quadratic in b_1

$$(\quad)b_1^2 + (\quad)b_1 + (\quad) = 0$$

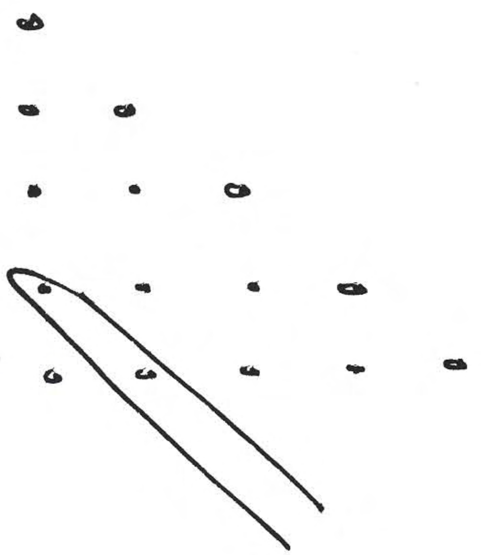
Two roots

$b_1 = 0 \Downarrow B = I$ $\qquad$ $b_1 = a_1 \Downarrow B = A$

All quadratics could be trivial

Express a's in terms of a_1 & a_2.

$$a_3 = \frac{a_1 a_2^2 (1 + a_1 + a_2)}{a_1^4 + 2a_1^3 + a_1^2 - a_1 a_2 - a_1^2 a_2 + a_2^2}$$



IMPASSE!

q-MATH

$$n = \lim_{q \to 1} \frac{q^n - 1}{q - 1}$$

$$(n!)_q = \frac{q-1}{q-1} \frac{q^2-1}{q-1} \cdots \frac{q^n-1}{q-1}$$

$$\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q \, (n-k)!_q}$$

$q = (\text{prime})^n$ for Combinatorics

$|q| < 1$ for asymptotics

Change variables

$$q_2 = \frac{q_1(1+q_1)}{1+q}$$

$$-1 < q_1 < \infty$$

$$a_3 = \frac{a_1(a_1+1)(a_1+1+q)}{1\,(1+q)\,(1+q+q^2)}$$

$a_4 =$ etc.

Leads to q-Hausdorff matrices

A, B Norlund

$A \Leftrightarrow B$ iff $A = C_q(\alpha)$

$B = C_q(\beta)$

for some $\alpha, \beta > 0$ and $-1 < q < \infty$.