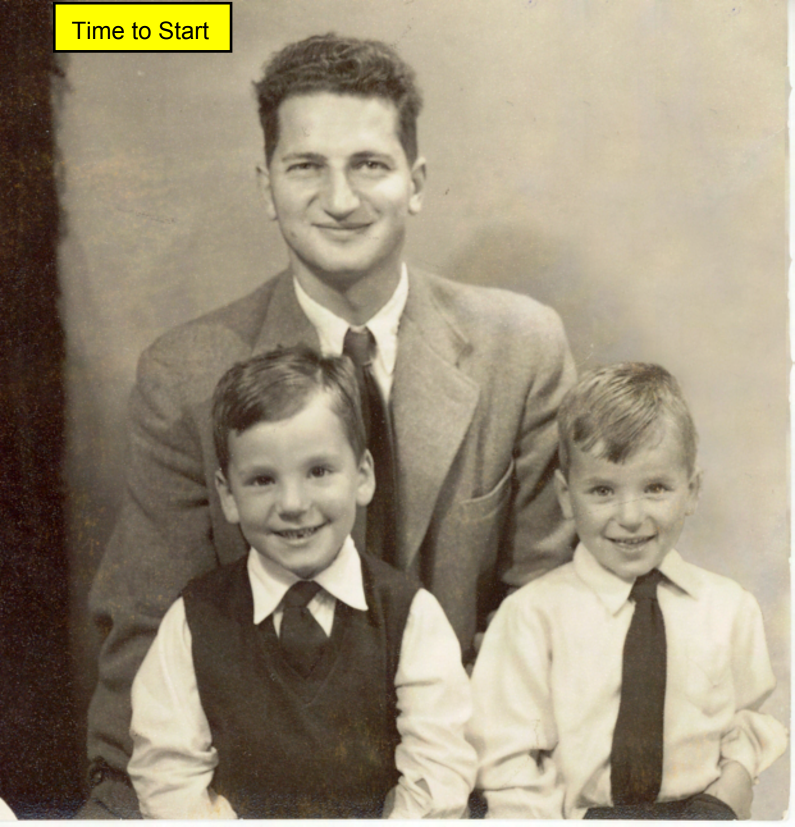
Time to Start



David and Me: a Chronology

With apologies to
Michael Moore (*Roger*) & Paul Erdos (*Ramanujan*)

Jonathan M. Borwein, FRSC

Canada Research Chair in IT
Dalhousie University

Halifax, Nova Scotia, Canada

Special Session in Analysis

In Honour of

David Borwein's 80th Birthday

CMS, Halifax, June 14, 2004

Topics: Analysis, Continuation, Renormalization, Summation, Summability, ℓ^p, $\mathcal{L}^p$

Qualities: Elegance, Power, Tenacity (*EPT*)

Birthday: 24-03-1924 **Talk Revised**: 09-06-2004

FREEMAN DYSON on STYLES (1995)

I see some parallels between the shifts of fashion in mathematics and in music. In music, the popular new styles of jazz and rock became fashionable a little earlier than the new mathematical styles of chaos and complexity theory. Jazz and rock were long despised by classical musicians, but have emerged as art-forms more accessible than classical music to a wide section of the public. Jazz and rock are no longer to be despised as passing fads. Neither are chaos and complexity theory.

But still, classical music and classical mathematics are not dead. Mozart lives, and so does Euler. When the wheel of fashion turns once more, quantum mechanics and hard analysis will once again be in style.

The Best Teacher I ever Had Was . . .

1924 DB born **'29** SA **'48** UK **1950**

1951 JB arrives **1953** PB arrives

1955 Bruce Shawyer arrives

1957 **DB + JB +** $2 \times 2 \rightarrowtail$ **Cheese +** £5.00

1958 DB secretary ICM GA **1963**

MATHAWOCKY

(with apologies to C.L.D.)

'Twas curvig and the graphley trace
Did max and minim in the plane;
All normless was the function space,
And the neighbourhoods insane.

"Beware the Mathawock, my son!
The rules that fright, the proofs that wrack!
Beware the ab ab surd, and shun
The booleous Algebrack!"

He took his abstract sword in hand:
Long time the symbful foe he sought -
So rested he by the Geoma tree,
And stood awhile on nought.

And, as on voidful nought he stood,
The Mathawock, with pi's aflame,
Transcended o'er the ringley wood,
And factored as it came!

P,Q! P,Q! and lambda mu
The abstract blade did lemmas hack!
He left it dead, and with its zed
He went permuting back.

"And hast thou slain the Mathawock?
Come to my arms, vectorious boy!
O baseless day! Arroo! - Array!"
He tupled in his joy.

'Twas curvig, and the graphley trace
Did max and minim in the plane;
All normless was the function space,
And the neighbourhoods insane.

D. Borwein


E. Smet
Rizwi
1968
Wm Penn
MARCHANT


1968
1200
1300
1400
1500
1600
1700
1800
Eddy Smet
Peter Cass
Rizvi
Bruce Shawyer
L.S. Bosanquet
David Borwein

1968 First D&JB *MAA Monthly* solutions

- JB has ‘only‘ class from DB (for BS)

- LB ‘makes‘ JB a functional analyst

1971 DB and JB's first *joint* AMS meeting. DB ‘discovers’ $\log 2$—on a Wang

1975 Victor Klee gets confused at UNB

1977 DB and JB's first joint AMS meeting as father and son professors

1983 December 8

E.J.Barbeau, Problem Editor
Canadian Math.Bulletin
Dept.Math.,Univ.of Toronto
Toronto, Ont. M5S 1A1.

Problem P.338 : Can.Math.Bull. 26 (2) 1983.

For $p > 1$, define $\pi_p = \frac{2}{p} \int_0^1 [t^{1-p} + (1-t)^{1-p}]^{1/p}\, dt.$

Compute π_2. If $p^{-1} + q^{-1} = 1$, show that $\pi_p = \pi_q$.

Solution by D.Borwein (Univ.of Western Ontario) and D.Russell (York Univ.).

Let $I_p := \frac{1}{p} \int_0^{\frac{1}{2}} [t^{1-p} + (1-t)^{1-p}]^{1/p}\, dt \; (= \tfrac{1}{4}\pi_p)$, $p > 1$, $p^{-1} + q^{-1} = 1$.

The value $\pi_2 = \pi$ is trivial (substitute $t = \sin^2\theta$).

Integration by parts, rearrangement, gives

$$I_p = 1 - \frac{1}{q} \int_0^{\frac{1}{2}} (1-t)^{-p} [t^{1-p} + (1-t)^{1-p}]^{-1/q}\, dt .$$

Substitute $t = (1 + u^q)^{-1}$ in this last integral to get

$$I_p = 1 - \int_1^\infty u^{-2} (1 + u^p)^{-1/q} (1 + u^q)^{-1/p}\, du$$

from which the symmetry is evident.

1983 Mv ~/DB

An elegant arclength duality

OUR JOINT CORPUS

∮∮ Then we started writing papers together ··· just before DB *retired*

- He has not changed much—before or since!

1985 DB, JB and K.F. Taylor, "Convergence of lattice sums and **Madelung's constant**," *J. Math. Phys.*, **26** (1985), 2999–3009.

$$\mathcal{M}_2(s) := \sum_{\substack{m,n \in \mathbf{Z} \\ (m,n) \neq 0}} \frac{(-1)^{m+n}}{(m^2+n^2)^{s/2}},$$

$$\mathcal{M}_3(s) := \sum_{\substack{(m,n,p) \in \mathbf{Z} \\ (m,n,p) \neq 0}} \frac{(-1)^{m+n+p}}{(m^2+n^2+p^2)^{s/2}}$$

Theorem 1. *Converges in 2D, 3D over increasing cubes (*Re $s > 0$*)* *(*ℓ^∞*)*
For $s = 1$, one may sum over circles in 2D but not in 3D *(*ℓ^2*)*
For $s = 1$, one may not *sum over diamonds in 2D* *(*ℓ^1*)*

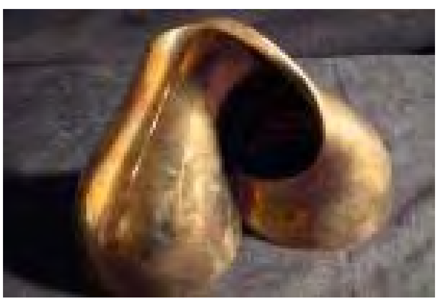
- $\otimes$ **Electrochemical stability of NaCl.** It upset many chemists that $\mathcal{M}_3(1) \overset{\text{Benson}}{=} 3\pi \times \sum_n o_3(n)\,\mathrm{sech}^2(\pi\sqrt{n}/2) \neq \sum_n (-1)^n r_3(n)/\sqrt{n}$ which diverges. Indeed, $r_3(n)$ is not $O\left(n^{1/2}\right)$

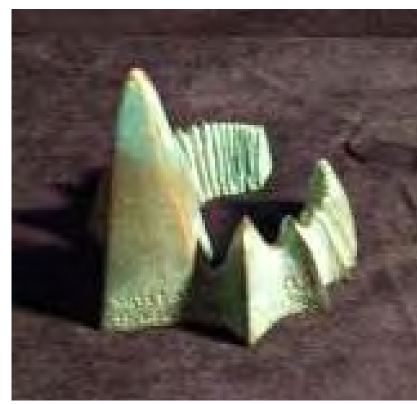
We are pleased to announce

The David Borwein Distinguished Career Award(s)

The awards intend to recognize *exceptional, broad and continued contributions to Canadian Mathematics*. One or two to be awarded every even year at the summer Meeting of the Society with recipients chosen by the *Advancement of Math Committee* (thus, President, President-elect and Treasurer are central to the decision). A presentation about the prize and winners(s) (perhaps a short montage, audio and video clips) will occur at the banquet.

The award will be a sculpture by Helaman Ferguson entitled something like Mathematics of Salt.





Based on Benson's formula for Madelung's constant for NaCl, the sculpture aims to reflect David's love and appreciation of mathematics in general and classical analysis in particular. It will be polished bronze and resemble the Clay Institute Prize (1999), the SIGGRAPH Prize (2003) and ICIAM03 Memorial each designed by Ferguson.

GH HARDY, 1887-1947

- Hardy— agreeing with Lorenz 1879—proved

$$\mathcal{M}_2(s) = -4 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

1986 JB & DB, "Alternating sums in several dimensions," *Monthly*, **93** (1986), 531–539.

- We needed general alternating series results (à la Riemann and Hardy) to explain why the physical universe is so perverse

1988 JB, DB, R. Shail and J. Zucker, "Energy of static electron lattices," *J. Phys. A: Math Gen.* **21** (1988), 1519–1531.

- Looked at **Wigner sums** such as

$$\mathcal{W}_3(s) \;:=\; \sum_{\substack{(m,n,p)\in\mathbf{Z}\\(m,n,p)\neq 0}} \frac{1}{(m^2+n^2+p^2)^{s/2}} \;-\; \mathbf{c}\iiint_{R^3} \frac{dV}{(x^2+y^2+z^2)^{s/2}}$$

- Klaus Fuchs—the mathematical physicist and East German atom spy—worked with the wrong sums (even though he attended an Edinburgh Colloquium with DB)

- Analytic continuation "rocks and rules". But on the boundary of convergence of $\mathcal{W}_3$

$$\mathcal{W}_3(1) \neq \lim_{s\downarrow 1} \mathcal{W}_3(s)$$

1989 JB, DB, and R. Shail, "Analysis of certain lattice sums," *JMAA* **143**(1989),126–137.

- More of the pure math driven by the previous physical and chemical lattice sums

1991 DB and JB, "Fixed points of real functions revisited," *JMAA*, **151** (1991), 112–126.

Theorem 2. (Mann 1953, 1971) *Let* $I := [0,1]$. *Let* f *be a be continuous self-map of* I. *For every* $z_0 \in I$,

$$z_{n+1} \hookleftarrow f\left(\frac{1}{n}\sum_{k=1}^{n} z_k\right)$$

converges to a fixed point of f.

- Compare Banach-Picard and Sarkovsky's theorem "**period 3 implies chaos**"

- We made a delicate analysis of what happens for regular summability methods. Also for Lipschitz functions

- The result holds for Hölder means H_2

⊞ It fails for some methods. What happens for C_2?

1994 DB and JB, "Some exponential and trigonometric lattice sums," *JMAA*, **188** (1994), 209–218.

- JB's unhealthy addiction—to lattice sums, analytic continuation and conditional convergence —continues and reinfects DB

1995 DB and JB, "On some intriguing sums involving $\zeta(4)$," *PAMS*, **123**(1995), 111-118.

- Ignorance was bliss as we met *Euler sums*:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \zeta(s,t) \quad \sum_{n>0} \frac{H_n^t}{n^s}$$

where

$$H_n^t := 1 + 1/2^t + \cdots + 1/(n-1)^t,$$

and we write $H_n = H_n^1$

- Euler studied these because Goldbach made a transcription error:
Si non errasset, fecerat ille minnus

Theorem 3.

$$\sum_{n>0} \left\{ \frac{H_n}{n} \right\}^2 = \frac{\zeta(4)}{4}$$

- Since, via *Parseval*

$$\frac{1}{2\pi}\int_0^\pi (\pi - t)^2 \log^2(2\sin\frac{t}{2})\,dt$$

$$= \sum_{n=1}^\infty \frac{(H_n)^2}{n^2}.$$

1995 DB, JB and R. Girgensohn, "Explicit evaluation of Euler sums," *Proc. Edin Math. Soc*, **38** (1995), 273–294.

David is a Fellow of the EMS and the Editor was the son of an old colleague

– Oh, and the analysis is pretty subtle

1996 DB, JB, PB, and R. Girgensohn, "Giuga's conjecture on primality," *MAA Monthly*, **103** (1996), 40–50.

✓ The only DJP-Borwein paper

- **Giuga's 1951 conjecture.** $N > 1$ is prime if (and only if)

$$\sigma_N = \sum_{k=1}^{N-1} k^{N-1} \equiv N - 1 \pmod{N}$$

⊞ We *broke* the Tokyo supercomputer

1998 DB, JB and C. Pinner, "Convergence of Madelung-like lattice sums," *Trans. AMS*, **350** (1998), 3131–3167.

- Culmination of work on lattice sums started in 1985, it relies on subtle estimates of *average orders* and *discrepancy*

1999 UWO Family Alumni Award

2000 DB, JB and P. Maréchal, "Surprise maximization," *American Math. Monthly*, **107** 2000, 527–537. [CECM Preprint 98:116]

- We analyzed the distribution of events such that surprise—appropriately defined—is maximized. This is the Paradox of the Unintended Hanging.

2001 DB, JB, G. Fee and R. Girgensohn, "Refined convexity and special cases of the Blaschke-Santalo inequality," *Math Ineq. Appl.* **4** (2001), 631–638. [CECM Preprint 00:146]

- $p \mapsto \mathrm{Vol}_N(B_p) = (2\Gamma)^N(1+1/p)/\Gamma(1+N/p)$ used, echoing '83 CMB solution, to show

$$\sqrt{\mathrm{Vol}_\mathrm{N}(B_p)\,\mathrm{Vol}_\mathrm{N}(B_q)} \geq \mathrm{Vol}_\mathrm{N}(B_2)$$

2002 DB and J, "Some remarkable properties of sinc and related integrals," The Ramanujan Journal, **5** (2001), 73–90. [CECM Preprint 99:142]

Example. For the *sinc* function

$$\mathrm{sinc}(x) := \frac{\sin(x)}{x},$$

consider, the seven —hard to compute numerically accurately—highly oscillatory integrals

$$I_1 := \int_0^\infty \mathrm{sinc}(x)\, dx = \frac{\pi}{2},$$

$$I_2 := \int_0^\infty \mathrm{sinc}(x)\mathrm{sinc}\left(\frac{x}{3}\right)\, dx = \frac{\pi}{2},$$

$$I_3 := \int_0^\infty \mathrm{sinc}(x)\mathrm{sinc}\left(\frac{x}{3}\right)\mathrm{sinc}\left(\frac{x}{5}\right)\, dx = \frac{\pi}{2},$$

$$\ddots$$

$$I_7 := \int_0^\infty \mathrm{sinc}(x)\mathrm{sinc}\left(\frac{x}{3}\right)\cdots\mathrm{sinc}\left(\frac{x}{13}\right)\, dx = \frac{\pi}{2}.$$

However,

$$I_8 := \int_0^\infty \text{sinc}(x)\text{sinc}\left(\frac{x}{3}\right) \cdots \text{sinc}\left(\frac{x}{15}\right)\, dx$$

$$= \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi$$

$$\approx 0.499999999992646\pi.$$

- ▶ When a researcher, using a well-known computer algebra package, checked this he—and the makers—diagnosed a "bug" in the software. Not so!

- Our analysis, via Parseval, links the integral

$$I_N := \int_0^\infty \text{sinc}(a_1 x)\text{sinc}\,(a_2 x) \cdots \text{sinc}\,(a_N x)\, dx$$

with the volume of the polyhedron

$$P_N := \{x : \left|\sum_{k=2}^N a_k x_k\right| \le a_1, |x_k| \le 1, 2 \le k \le N\}.$$

where $x := (x_2, x_3, \cdots, x_N)$

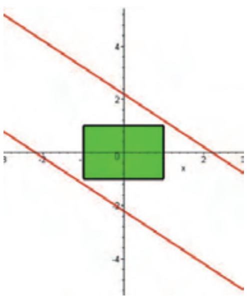
If we let

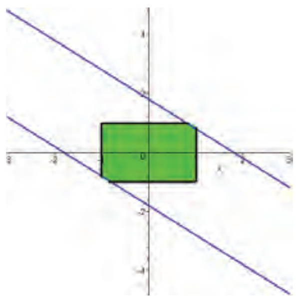
$$C_N := \{(x_2, x_3, \cdots, x_N) : -1 \le x_k \le 1, 2 \le k \le N\},$$

then

$$\boxed{I_N = \frac{\pi}{2a_1} \frac{Vol(P_N)}{Vol(C_N)}.}$$

- The value drops precisely when the constraint $\sum_{k=2}^{N} a_k x_k \le a_1$ becomes *active* and bites the hypercube C_N, as occurs when $\sum_{k=2}^{N} a_k > a_1$ Above, $\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$, but on adding $\frac{1}{15}$, the sum exceeds 1, the volume drops, and $I_N = \frac{\pi}{2}$ no longer holds





- A warning about inferring patterns from seemingly compelling computation

- DB's analytic power proved well suited to proving all this:

Theorem 4. Suppose $a_n > 0$. Let $s_n := \sum_{k=1}^{n} a_k$ and

$$\tau_n := \int_0^\infty \prod_{k=0}^{n} \operatorname{sinc}(a_k x)\, dx.$$

1. Then $0 < \tau_n \leq \pi/(2a_0)$, with equality if $n = 0$, or if $a_0 \geq s_n$ when $n \geq 1$

2. If $a_{n+1} \leq a_0 < s_n$ with $n \geq 1$, then $0 < \tau_{n+1} \leq \tau_n < \pi/(2a_0)$ (the hard part)

3. If $a_0 < s_{n_0}$ with $n_0 \geq 1$, and $\sum_{k=0}^{\infty} a_k^2 < \infty$, then there is $n_1 \geq n_0$ such that

$$\tau_n \geq \int_0^\infty \prod_{k=0}^{\infty} \operatorname{sinc}(a_k x)\, dx \geq \int_0^\infty \prod_{k=0}^{\infty} \operatorname{sinc}^2(a_k x)\, dx$$

for $n \geq n_1$

- Harder versions led to

2002 DB, JB and Bernard (BJ) Mares, "Multivariable sinc integrals and volumes of polyhedra," *The Ramanujan Journal*, **6** (2002), 189–208. [CECM Preprint: 01:159].

- BJ was 18 and DB was 77 when research took place. They had a fine few weeks together at CECM—mastering some scary determinants, and volumes of polytopes.

2004 DB, JB, and W. F. Galway, "Finding and Excluding b-ary Machin-Type BBP Formulae," *CJM*, Galleys June 2004. [CECM Preprint 2003:195]

BBP FORMULAE

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left\{ \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right\}$$

Ferguson's subtractive image of the **BBP Pi formula**

⊙ DB snookers PB by proving very cleanly:

Theorem 5. *Given $b > 2$ and not a proper power, then there is no* Q-**linear Machin-type BBP arctangent** *formula for* π

2004 DB, JB, R. Crandall and R. Mayer, "On the dynamics of certain recurrence relations," *Ramanujan Journal* (Issue for Dick Askey's 70th birthday), accepted June 10, 2004. [CoLab Preprint 253]

- This first paper written in DB's 80th year uses almost every trick in the continued fraction and special function book. One of the deepest any of us has produced, it is based on **non-symmetric word analysis**:

Theorem 6. *In any matrix algebra if*

$$A_n A_{n-1} \cdots A_1 \to L$$

with L non-singular, *but perhaps only conditionally convergent, then*

$$(A_n + B_n)(A_{n-1} + B_{n-1}) \cdots (A_1 + B_1) \to M$$

whenever

$$\sum_n \|B_n\| < \infty$$

From VISUAL DYNAMICS

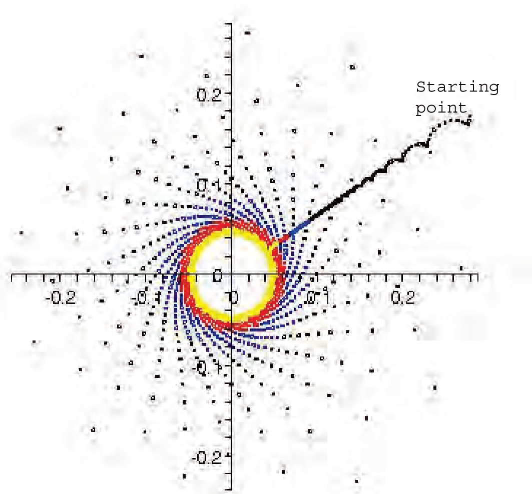
♠ In prior work on continued fractions of Ramanujan, Crandall and JB needed to study the *dynamical system* $t_0 := t_1 := 1$:

$$t_n \hookleftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

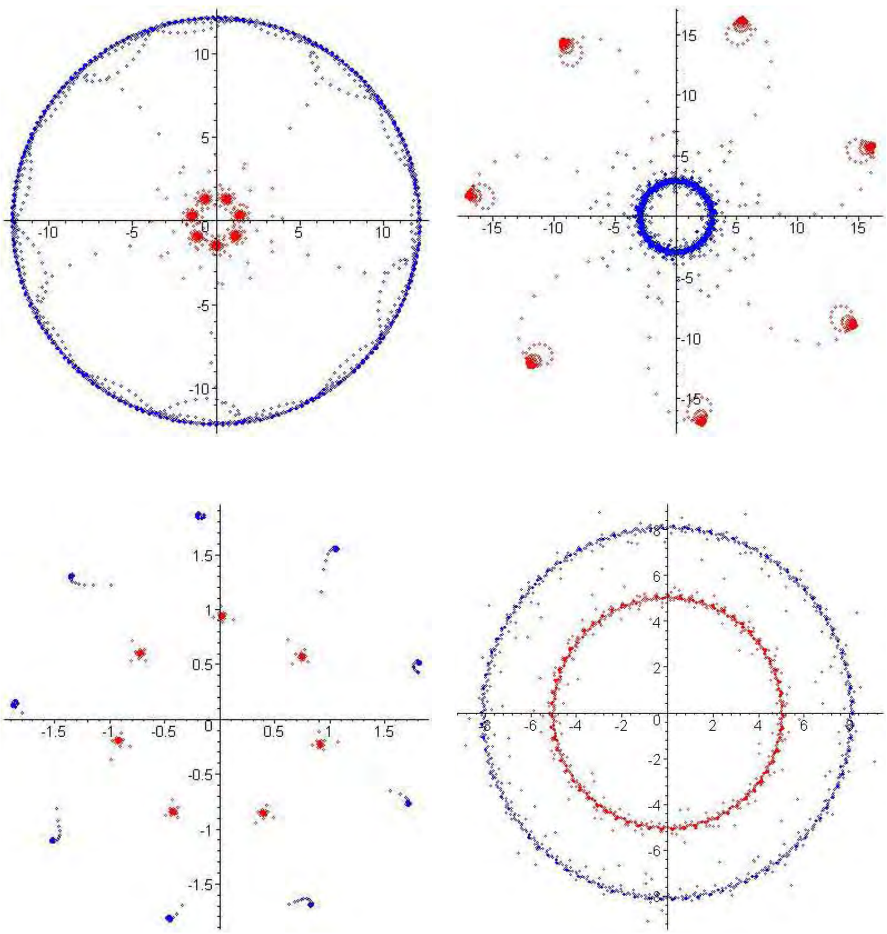
where $\omega_n = a^2, b^2$ for n even, odd respectively. Which we may view as a **black box**.

× Numerically all one *sees* is $t_n \to 0$ slowly

✓ Pictorially we see significantly more:

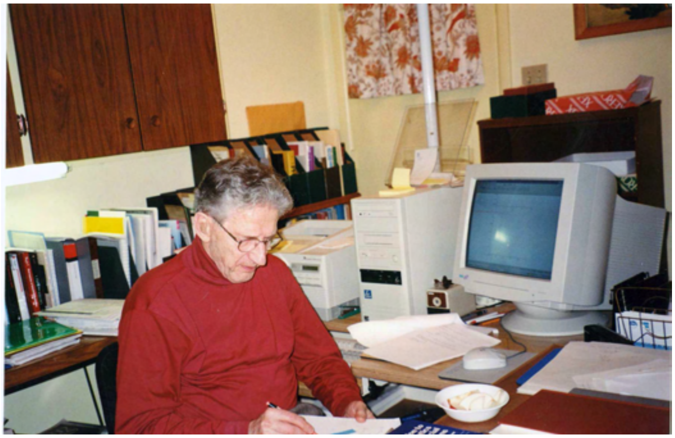


- Scaling by $\sqrt{n}$, and coloring odd and even iterates, **fine structure** appears



The attractors for various $|a| = |b| = 1$

★ This is now fully explained with a *lot* of work—the rate of convergence in some cases by a fine *singular-value* argument



2004 DB and JB are at work on 16th joint paper

- **Euler's reduction formula** is

$$\zeta(s,1) = \frac{s}{2}\zeta(s+1) - \frac{1}{2}\sum_{k=1}^{s-2}\zeta(k+1)\zeta(s+1-k)$$

This is equivalent to is

$$(1) \quad \sum_{n=1}^{\infty} \frac{1}{n^2\,(n-x)} = \sum_{n=1}^{\infty} \frac{\sum_{m=1}^{n-1} \frac{1}{(m-x)}}{n\,(n-x)}$$

- DB led to slick direct proof of (1) by experimentation in Maple—many vistas opened!

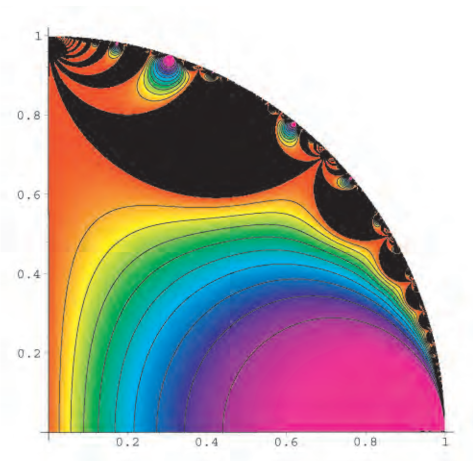
GAUSS, HERZBERG and DB

- Boris Stoicheff's often enthralling biography of Gerhard Herzberg* records:

  It is not knowledge, but the act of learning, not possession but the act of getting there which generates the greatest satisfaction. (Gauss)

Fractal similarity in Gauss' discovery of

Modular Functions



*Herzberg (1903-99) fled Germany for Saskatchewan in 1935 and won the 71 Chemistry Nobel: 12 (6 math/stats) NSERC Grantees have 1st degree pre-45.

A DETAILED MAP IS AT
WWW.CS.DAL.CA/~JBORWEIN/CMS/BRIDGES-MAP.PDF

Judith and Jonathan Borwein
invite you to a ***Reception***

TO CELEBRATE DAVID BORWEIN'S EIGHTIETH BIRTHDAY

❖ ❖ ❖

TURN ON ATLANTIC OFF TOWER ROAD ACROSS
FROM ST MARY'S UNIVERSITY AND THEN ONTO BRIDGES.

MONDAY JUNE 14, 2004

8.30 - 11.00 PM

857 BRIDGES STREET, HALIFAX

(902) 422-4131 OR 412-1228

❖ ❖ ❖