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Fixed Point Iterations for Real Functions

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We give proofs of general results on the computation of fixed points of a continuous function or a Lipschitz function on the real line. We also show how completely these results can fail to hold in spaces of more than one dimension.

1. SEGMENTING MANN ITERATIONS

Let $[a, b]$ be a closed bounded interval on the real line and consider a continuous mapping $f: [a, b] \to [a, b]$. Let $\{t_n\}$ be an arbitrary sequence of real numbers in $[0, 1]$ and consider the sequence of iterates $\{x_n\}$ in $[a, b]$ generated by

$$x_1 \in [a, b]$$

$$x_{n+1} := (1 - t_n)\, x_n + t_n f(x_n). \tag{1}$$

This iteration is often said to be a *segmenting Mann iteration* [12, 2, 5] or to be of *Krasnoselski-type* [11, 4, 7, 8]. More general Mann iterations are discussed in Section 3.

PROPOSITION 1. *Suppose* (i) *that* $\{x_n\}$ *converges to* z *and* (ii) *that*

$$\sum_{n=1}^{\infty} t_n = \infty. \tag{2}$$

Then $f(z) = z$ *so that* z *is a fixed point of* f.

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Proof. Suppose $f(z) \neq z$. Let $\varepsilon_n := f(x_n) - x_n$. Then $\{\varepsilon_n\}$ tends to a non-zero limit. Since $\Sigma\, t_n$ diverges so also does $\Sigma\, t_n \varepsilon_n$. As

$$x_n - x_1 = \sum_{k=1}^{n-1} t_k\, \varepsilon_k$$

this contradicts the convergence of $\{x_n\}$.

It is obvious that Proposition 1 can fail for a convergent series Σt_n with sum s, since $0 < |x_1 - z| < s \max |f(x_n) - x_n| < \operatorname{dist}(x_1, F)$ may well occur. (Here F denotes the fixed points of f.) Less trivially, the following converse of Proposition 1 holds.

PROPOSITION 2. *Suppose that for each continuous function $f: [a, b] \to [a, b]$ convergence of the iteration $\{x_n\}$ given by* (1), *say to z, implies that z is a fixed point of f. Suppose also that* $\sup t_n < 1$. *Then* (2) *must hold.*

Proof. Suppose without loss of generality that $a = 0$ and $b = 1$. Consider $f(x) := 1 - cx$ with c chosen so that $0 < c < \inf(1 - t_n)$. Then iteration (1) becomes

$$1 - (c+1)\, x_{n+1} = (1 - (c+1)\, x_n)(1 - (c+1)\, t_n)$$

and so

$$1 - (c+1)\, x_{n+1} = (1 - (c+1)\, x_1) \prod_{k=1}^{n} (1 - (c+1)\, t_k),$$

which tends to $(1 - (c+1)\, x_1)\, p$ where

$$p := \prod_{k=1}^{\infty} (1 - (c+1)\, t_k).$$

Note that p always exists as the limit of a decreasing positive sequence. Hence, $\{x_n\}$ converges to $w := z + (x_1 - z)\, p$, where $z := 1/(c+1)$ is the unique fixed point of f. Suppose that $x_1 \neq z$ and that the series Σt_n converges. Since no term in the infinite product is zero, p is non-zero and $\{x_n\}$ converges to $w \neq z$.

PROPOSITION 3. *Suppose that $\{t_n\}$ tends to zero. Then the sequence $\{x_n\}$ given by* (1) *converges.*

Proof. The proof is essentially that given in [3] for the case $t_n := 1/(n+1)$. Let $s := \limsup x_n$ and $i := \liminf x_n$. Suppose that $s > i$ and that c is any point with $s > c > i$. Then c is a fixed point of f. Suppose not. We may assume that $f(c) > c$ and so find δ such that $s - i > \delta > 0$ and

$$f(x) > x \qquad \text{whenever} \quad |x - c| < \delta. \tag{3}$$

Select m large enough so that

$$|x_{n+1}-x_n|<\delta \qquad \text{for} \quad n>m. \tag{4}$$

Now select $N>m$ with $x_N>c$ as is possible since s is the limit superior of $\{x_n\}$. It follows that $x_n>c$ for $n>N$. Indeed if $x_n>c+\delta$ then, using (4), $x_{n+1}>c$; while if $c+\delta \geqslant x_n>c$ then (1) and (3) combine to show that

$$x_{n+1}=x_n+t_n[f(x_n)-x_n]>x_n>c.$$

Hence, by induction $x_n>c$ for $n>N$ and so $i \geqslant c$. This contradiction shows $f(c)=c$. Now $i<x_n<s$ forces $x_{n+1}=x_n+t_n[f(x_n)-x_n]=x_n$ which implies that $s=i$. So for $n>N$ we must have $x_n>s$ or $x_n<i$. Since $s-i>\delta$ we must have $x_n>s$ for all $n>N$ or $x_n<i$ for all $n>N$. Both possibilities imply that $i \geqslant s$. Thus $i>s$ is impossible and $\{x_n\}$ converges as claimed.

There is another natural condition ensuring that (i) of Proposition 1 holds. Recall that f is *L-Lipschitz* if $|f(x)-f(y)| \leqslant L|x-y|$ for all x and y in $[a, b]$. The key lies in the next lemma.

LEMMA 4. *Suppose that f is L-Lipschitz and that $f(x_n)-x_n$ and $f(x_{n+1})-x_{n+1}$ have opposite signs. Then there is at least one fixed point in the interval between x_n and x_{n+1} and for each such fixed point z we have*

$$|x_{n+1}-z| \leqslant [(t_n(1+L)-1]\,|x_n-z|. \tag{5}$$

Proof. We may assume $f(x_n)-x_n \geqslant 0 \geqslant f(x_{n+1})-x_{n+1}$. Then $x_n \leqslant x_{n+1}$ and the Intermediate Value theorem guarantees the existence of a fixed point z in $[x_n, x_{n+1}]$. Thus we have

$$\begin{aligned} x_{n+1}-z &= (1-t_n)[x_n-z]+t_n[f(x_n)-f(z)] \\ &= (t_n-1)[z-x_n]+t_n[f(x_n)-f(z)] \\ &\leqslant (t_{n-1})[z-x_n]+t_nL[z-x_n] \\ &= [(t_n(1+L)-1)]\,|z-x_n|. \end{aligned}$$

Let us say that $\{x_n\}$ *switches directions at* x_{n+1} if either

$$x_n<x_{n+1}>x_{n+2} \qquad \text{or} \qquad x_n>x_{n+1}<x_{n+2}.$$

Observe that a switch occurs exactly when $f(x_n)-x_n$ and $f(x_{n+1})-x_{n+1}$ have opposite signs.

LEMMA 5. *Suppose that f is L-Lipschitz and $\{x_n\}$ has successive switches* *of direction at* $x_{n(1)+1}$ *and* $x_{n(2)+1}$ *and that for* $k := 1$ *or* 2, $t_{n(k)} = (2-\varepsilon_k)/(L+1)$ *for some* ε_k *in* $(0, 1)$. *Then*

(a) *for* $n(2)+1 \geqslant n \geqslant n(1)$ x_n *lies between* $x_{n(1)}$ *and* $x_{n(1)+1}$, *and*

(b) $|x_{n(2)} - x_{n(2)+1}| \leqslant (1-\varepsilon_2/2|\, x_{n(1)} - x_{n(1)+1}|$.

Proof. We may suppose that $x_{n(1)} < x_{n(1)+1} > x_{n(1)+2}$ and $x_{n(2)} > x_{n(2)+1} < x_{n(2)+2}$. Since $f(x_{n(1)}) > x_{n(1)}$ and $f(x_{n(1)+1}) < x_{n(1)+1}$ there are fixed points in $[x_{n(1)}, x_{n(1)+1}]$. Let

$$m := \inf\{x : f(x) = x,\ x_{n(1)} \leqslant x \leqslant x_{n(1)+1}\}.$$

Then $m > x_{n(1)}$ and Lemma 4 implies that

$$|m - x_{n(1)+1}| < (1-\varepsilon_1)\, |m - x_{n(1)}|.$$

Hence

$$m \geqslant [(1-\varepsilon_1)\, x_{n(1)} + x_{n(1)+1}]/(2-\varepsilon_1)$$

and so

$$x_{n(1)+1} - m \leqslant [x_{n(1)+1} - x_{n(1)}]/2. \tag{6}$$

Since x_n decreases for $n(1)+1 \leqslant n \leqslant n(2)$, either

(i) $x_n \geqslant x_{n(2)+1} \geqslant m \geqslant x_{n(1)}$; or

(ii) $x_n \geqslant x_{n(2)} \geqslant m \geqslant x_{n(2)+1}$

and (6) and Lemma 4 together imply that

$$x_{n(2)+1} \geqslant (1-\varepsilon_2)\, x_{n(1)} + \varepsilon_2 m.$$

In either case for $n(1)+1 \leqslant n \leqslant n(2)+1$

$$x_{n(1)+1} \geqslant x_n \geqslant (1-\varepsilon_2)\, x_{n(1)} + \varepsilon_2 m \geqslant x_{n(1)} \tag{7}$$

which establishes (a). Since $x_{n(1)+1} > x_{n(2)}$, (6) and (7) show that

$$\begin{aligned} 0 &\leqslant x_{n(2)} - x_{n(2)+1} \\ &\leqslant (1-\varepsilon_2)[x_{n(1)+1} - x_{n(1)}] + \varepsilon_2(x_{n(1)+1} - m) \\ &\leqslant (1-\varepsilon_2/2)[x_{n(1)+1} - x_{n(1)}], \end{aligned} \tag{8}$$

which establishes (b).

PROPOSITION 6. *Suppose that* f *is L-Lipschitz and that, for some* $\varepsilon > 0$ *and all* n,

$$t_n \leqslant \frac{2-\varepsilon}{L+1}. \tag{9}$$

Then $\{x_n\}$ *converges to some point* z. *Moreover, if there are infinitely many switches,* z *lies between* x_m *and* x_{m+1} *whenever there is a switch at* x_{m+1}.

In addition, if for all n

$$t_n \leqslant \frac{1}{L+1},$$

then convergence is monotone.

Proof. If $\{x_n\}$ switches directions only finitely often then convergence follows since the sequence is eventually monotone. Suppose therefore that the sequence switches directions infinitely often at $x_{n(1)+1}, x_{n(2)+1}, \ldots, x_{n(k)+1}, \ldots$. Lemma 5 shows that, for $n(k+1)+1 \geqslant n \geqslant n(k)$, x_n lies between $x_{n(k)}$ and $x_{n(k)+1}$ and that

$$|x_{n(k+1)} - x_{n(k+1)+1}| \leqslant (1-\varepsilon/2)\,|x_{n(k)} - x_{n(k)+1}|.$$

Inductively, we see that the intervals of switching are nested and that for $n, m > n(k+1)$

$$|x_n - x_m| < (1-\varepsilon/2)^k\,(b-a)$$

so that $\{x_n\}$ is a Cauchy sequence and hence has limit z.

Finally, if $\sup t_n \leqslant 1/(L+1)$ then Lemma 4 shows that no change of direction is possible.

Note that to establish convergence it is only necessary to assume that $\limsup t_n < 2/(L+1)$. Note also that we have only used the fact that f is *quasi L-Lipschitz*: $|f(x)-f(z)| \leqslant L|x-z|$ whenever z is a fixed point of f. We have now proved:

THEOREM 7. *Suppose that* t_n *lies in* $[0, 1]$, *that* Σt_n *is divergent, and that either*

(a) $\{t_n\}$ *converges to zero; or*

(b) f *is L-Lipschitz and* $\limsup t_n < 2/(L+1)$.

Then the iteration (1) *converges to a fixed point of* f.

In [7] Hillam states Theorem 6 (b), without proof, for constant t_n and proves the monotone result for constant $t_n \leqslant 1/(L+1)$. He also gives a simple example to show that the result may fail for $t_n = 2/(L+1)$. The whole of Theorem 7(a) can be found in Rhoades [13] from a different vantage point. In [14] Rhoades shows that (a) is not needed when f is increasing. It is reasonably easy to give an example to show that in (a) it does not suffice that $\liminf t_n = 0$.

2. Negative Results

There is a rich literature on the behavior of iteration (1) for non-expansive functions in normed space [1, 2, 4–6, 9–11]. We next show Theorem 7(a) and Theorem 7(b) have no obvious generalizations to functions of more than one variable.

Proposition 8. *Let D be the closed unit disk in the complex plane. Fix strictly positive constants a and α with $\alpha < \pi/2$. Consider the mapping $f: D \to D$ given by*

$$f(re^{i\theta}) := \frac{(a+1)\,r}{r+a}\, e^{i(\theta+\alpha)}$$

for $1 \geqslant r \geqslant 0$ and $0 \leqslant \theta < 2\pi$. Then f is Lipschitz with Euclidean constant $(1+1/a)^2$ and has a unique fixed point at the origin. Suppose that $\Sigma\, t_n = \infty$.

(a) *For $x_1 \neq 0$ the iteration fails to converge if $(a+1)\cos(\alpha) > a$.*

(b) *Suppose that $\{t_n\}$ tends to zero. For $x_1 \neq 0$ the cluster points of iteration (1) form a circle around the origin of radius*

$$r^* := \max\{(1+a)\cos(\alpha) - a, 0\}.$$

In particular, the iteration (1) converges if and only if $(a+1)\cos(\alpha) \leqslant a$.

(c) *Suppose that $\{t_n\}$ has constant value t in $(0, 1)$. For $x_1 \neq 0$ the cluster points of iteration (1) all lie on the circle around the origin of radius*

$$r^{**} := \max\{(1+a)\, c(t, \alpha) - a, 0\},$$

where $c(t, \alpha) := [(1-t)\cos(\alpha) + \{1 - [(1-t)\sin(\alpha)]^2\}^{1/2}]/(2-t) \geqslant \cos(\alpha)$.

Proof. It is clear that f has a unique fixed point at the origin and one easily checks the Lipschitz estimate. We consider the increasing function $g(r) := (r+a)/(a+1)$ and note that the angle between z and $f(z) - z$ is obtuse if and only if $g(r) \leqslant \cos(\alpha)$. This holds if and only if $r \leqslant (1+a)\cos(\alpha) - a$. Hence, if $|x_n| < r^*$ it follows that $|x_n| < |x_{n+1}|$ for any t_n in $[0, 1]$.

(a) Suppose that $\{x_n\}$ converges to x. Since $\Sigma\, t_n$ diverges, $x = 0$ is the unique fixed point. (This can be seen from [12], or from the argument in Proposition 1, or from the discussion in Section 3.) Hence for large n, x_n lies within radius r^* of the origin and so the sequence is ultimately increasing in norm. This is a contradiction except if eventually $x_n = 0$. Since $x_n = 0$ implies $x_{n-1} = 0$ this can only happen when $x_1 = 0$.

A careful but tedious argument is needed to make all the details of (b) and (c) explicit. We thus only indicate the method.

(b) A refinement of the argument in (a) shows that given $\varepsilon > 0$, for n sufficiently large if $|x_n| > r^* + \varepsilon$ then $|x_n| > |x_{n+1}|$ because $\{t_n\}$ tends to 0. Also divergence of Σt_n means that $\{|x_n|\}$ cannot converge monotonically to $r \neq r^*$. Thus either $\{|x_n|\}$ converges monotonically to r^*, or oscillates to r^*. In any event all cluster points of the iterates lie on the circle of radius r^*. A result in [12] is that the cluster point set, A, of a Mann iterative sequence is closed and connected (as a compact ε-chainable subset of a compact metric space). Also A is not singleton since 0 is the unique fixed point. Thus A is a non-trivial arc on $|z| = r^*$. Finally, since $f(z)$ is always anticlockwise of z, A cannot miss any segment of arc.

(c) The value of r^{**} $(> r^*)$ is computed by solving for r such that $|x_n| = r$ implies $|x_{n+1}| = r$. Again for $|x_n| < r^{**}$ we have $|x_n| < |x_{n+1}|$. Moreover if $|x_n| > r^{**}$ then $|x_n| > |x_{n+1}| > r^{**}$. Thus $\{|x_n|\}$ is eventually monotonic and, much as in (b), must converge to r^{**}.

An explicit example is afforded by taking $a := \frac{1}{3}$ and $\alpha := \pi/3$. In this case the Cesaro iterates $(t_n := 1/(n+1))$, cluster on $|z| = \frac{1}{3}$, while the Krasnoselski iterates $(t_n := \frac{1}{2})$, cluster on $|z| = (2\sqrt{13} - 1)/9$.

PROPOSITION 9. *Let C be a closed convex subset of a Hilbert space. Suppose that $\{t_n\}$ is such that $\Sigma t_n = \infty$. If iteration* (1) *converges for all continuous $f: C \to C$ then C is a compact line segment.*

Proof. Suppose not.

Case (i). C has affine dimension of one. In this case C is a set of points of the form $a + tb$ where $t \geqslant 0$, or $t \in R$, for points a and b, $b \neq 0$. Consider the mapping $f(a + tb) := a + (t+1)b$ which maps C to itself. Iteration (1) becomes $x_{n+1} = x_n + t_n b$ and fails to converge.

Case (ii). C has affine dimension greater than one. In this case C contains a simplex $[a, b, c]$ and hence a closed disk D. Let $f: C \to C$ be defined by

$$f(x) := g(P_D(x)),$$

where $g : D \to D$ is constructed by the recipe of Proposition 8 (or as in [6]) so that iteration (1) fails to converge, and $P_D(x)$ is the unique nearest point to x in D in the Hilbert norm. Since D is compact, P_D is continuous and hence so is f. Then iteration (1) fails to converge.

This construction works in any normed space with an equivalent rotund norm.

3. General Mann Iterations

Consider now a summability transformation given by

$$x_n := \sum_{k=1}^{n} a_{n,k} u_k,$$

where $a_{n,k} \geqslant 0$, for all k and n, and where

$$\lim_{n \to \infty} a_{n,k} = 0, \qquad \sum_{k=1}^{n} a_{n,k} = 1.$$

These conditions make the triangular matrix $[a_{n,k}]$ *regular* (i.e. $u_n \to s$ implies $x_n \to s$ [13, 15].) Following Dotson [2] we somewhat non-standardly call a summability matrix (*non-trivially*) *normal* if

$$a_{n+1,k} = (1 - a_{n+1,n+1})\, a_{n,k} \qquad \text{for} \quad 1 \leqslant k \leqslant n, \tag{10}$$

and $a_{n+1,n+1} < 1$ for $n = 1, 2, \ldots$.

Consider a nonnegative sequence $\{d_n\}$ with $d_1 \neq 0$ and set $D_n := d_1 + d_2 + \cdots + d_n$. Then the triangular matrix with entries

$$a_{n,k} := d_k / D_n \tag{11}$$

corresponds to a weighed mean and satisfies (10). In addition it is regular exactly when $\Sigma d_n = \infty$. Conversely, if we define

$$D_n := \prod_{k=2}^{n} \frac{1}{1 - a_{k,k}} \qquad \text{for} \quad n > 1 \text{ and } D_1 := 1,$$

and set $d_n := a_{n,n} D_n$ then (11) follows from (10).

We observe that (10) is equivalent, for matrices with $a_{n+1,n+1} < 1$, to the method being *stationary*:

$$u_{n+1} = x_n \Rightarrow x_{n+1} = x_n; \tag{12}$$

or to the method being *interpolatory*:

$$\min\{x_n, u_{n+1}\} \leqslant x_{n+1} \leqslant \max\{x_n, u_{n+1}\}. \tag{13}$$

Indeed (13) implies (12) implies (10) implies (11) implies (13). We have established:

Proposition 10. *A triangular row stochastic summability matrix* $[a_{n,k}]$ *with* $a_{n+1,n+1} < 1$ *is normal if and only if it is a weighted mean matrix given by* (11).

A *Mann iterative process* for a continuous function $f: C \to C$ mapping a closed convex set to itself is given by

$$u_{n+1} := f(x_n),$$

where

$$x_n := \sum_{k=1}^{n} a_{n,k} u_k$$

with $u_1 := x_1$ in C. Given that the matrix is regular, if either $\{x_n\}$ or $\{u_n\}$ converges, then both converge to a common limit which is a fixed point of f. For weighted mean iterations we have:

THEOREM 11. *Let $f: [a, b] \to [a, b]$ and let Σd_n be a divergent series of non-negative terms with partial sums $D_n := d_1 + d_2 + \cdots + d_n$ and $d_1 > 0$. Suppose that f is continuous and d_n/D_n tends to zero, or that f is L-Lipschitz and* $\limsup d_n/D_n < 2/(L+1)$.

Then both the weighted mean *iterations*

$$\text{(a)} \quad x_{n+1} := \frac{1}{D_n} \sum_{k=1}^{n} d_k f(x_k), \qquad x_1 \in [a, b]$$

and

$$\text{(b)} \quad x_{n+1} := f\left(\frac{1}{D_n} \sum_{k=1}^{n} d_k x_k\right), \qquad x_1 \in [a, b]$$

converge to fixed points of f.

Proof. We apply Theorem 7. An easy calculation shows that (a) is precisely iteration (1) for $t_n := d_n/D_n$. Abel's Theorem (2.41 in [15]) shows that Σt_n is divergent. To establish (b) we let

$$c_n := \frac{1}{D_n} \sum_{k=1}^{n} d_k x_k$$

and $t_n := d_{n+1}/D_{n+1}$ and we observe that the iteration becomes $c_{n+1} = (1 - t_n) c_n + t_n f(c_n)$. It follows that $\{c_n\}$ converges to a fixed point z. Since $x_{n+1} = f(c_n)$, x_n also converges to z.

The requirement that Σd_n be divergent is precisely the condition for the weighted average summability method to be regular [15], whether or not d_n/D_n tends to zero. The Franks and Marzec result in [3] is (b) for the C_1-method: the simplest Cesaro means with $d_n := 1$. The non-Lipschitz version of Theorem 11 (b) is also to be found in Rhoades [13].

In the proof of Theorem 11 a sequence $\{t_n\}$ corresponding to any weighted mean matrix is constructed. (See also [2].) Conversely, given $\{t_n\}$ with $0 \leqslant t_n < 1$ a corresponding weighted mean matrix is given by $D_1 := d_1 := 1$ and for $n \geqslant 2$

$$D_n := \prod_{k=1}^{n-1} \frac{1}{1-t_k} \qquad \text{and} \qquad d_n := t_{n-1} D_n.$$

Thus, as known, segmenting Mann iterations correspond to weighted mean matrices. However, the weighted mean iterations are not the most general of the classical summability transformations for which the Mann iteration process works. We illustrate this now. We consider *product means* given by

$$y_n := \frac{1}{P_n} \sum_{k=1}^{n} p_k x_k, \qquad z_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_k y_k,$$

where Σp_n and Σq_n are series of non-negative terms with partial sums $P_n := p_1 + p_2 + \cdots + p_n$ and $Q_n := q_1 + q_2 + \cdots + q_n$, and with $p_1, q_1 > 0$. The corresponding matrix, which transforms $\{x_n\}$ into $\{z_n\}$, has entries

$$a_{n,k} := \frac{p_k}{Q_n} \sum_{r=k}^{n} \frac{q_r}{P_r}$$

and is regular when both series diverge. We may rewrite the Mann iteration for this matrix as

$$\begin{aligned} & x_{n+1} := f(z_n), \\ \text{(i)} \quad & y_{n+1} := (1-\alpha_n)\, y_n + \alpha_n\, x_{n+1}, \\ \text{(ii)} \quad & z_{n+1} := (1-\beta_n)\, z_n + \beta_n y_{n+1}, \end{aligned} \tag{14}$$

with $y_1 = z_1 \in [a, b]$ and where, working as above, $\alpha_n := p_{n+1}/P_{n+1}$ and $\beta_n := q_{n+1}/Q_{n+1}$.

If α_n or β_n are constantly 1 then (14) reduces to (1), while if neither α_n nor β_n is ever 1 we may reconstruct the product mean from (14). The iteration (14) is often susceptible to the next result whose proof is entirely analogous to that of Theorem 7(a).

PROPOSITION 12. *Let $f: [a, b] \to [a, b]$ be continuous. Suppose that t_n lies in $[0, 1]$, that Σt_n is divergent, and that $\{t_n\}$ converges to zero. Suppose also that $x_1 \in [a, b]$ and*

$$x_{n+1} := (1-t_n)\, x_n + t_n w_n, \tag{15}$$

where $w_n - f(x_n) \to 0$. *Then either* (i) $\{x_n\}$ *converges to a fixed point of* f, *or* (ii) *every point of* (i, s) *is a fixed point of* f, *where* $i := \liminf x_n$ *and* $s := \limsup x_n$.

THEOREM 13. *Let* $f: [a, b] \to [a, b]$ *be continuous. Let* Σp_n *and* Σq_n *diverge.*

(a) *Iteration* (14) *converges to a fixed point of* f *is either*

$$\text{(A) (i) } \frac{p_n}{P_n} \to 0 \quad \textit{and} \quad \text{(ii) } \frac{1}{Q_n} \sum_{k=1}^{n} \frac{Q_k}{P_k} p_k \to 0,$$

or if

$$\text{(B) (i) } \frac{q_n}{Q_n} \to 0 \quad \textit{and} \quad \text{(ii) } \frac{1}{P_n} \sum_{k=2}^{n} P_k \, |f(z_k) - f(z_{k-1})| \to 0.$$

(b) *In particular,* (A) (ii) *holds if* $\{q_k/Q_k\}$ *is bounded away from zero and* (B) (ii) *holds if* $\{p_k/P_k\}$ *is bounded away from zero. Moreover, if* f *is Lipschitz then* (B) (ii) *may be replaced by*

$$\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{Q_k} q_k \to 0.$$

Proof. We suppose the interval $[a, b]$ is $[0, 1]$ without loss of generality.

Case (A). (a) Let $\delta_n := |z_n - y_n|$. We show $\delta_n \to 0$. It follows by uniform continuity that $f(z_n) - f(y_n) \to 0$ and that Proposition 12 applies to (14) (i). Now

$$\begin{aligned} \delta_{n+1} &\leqslant (1 - \beta_n)(\delta_n + |y_{n+1} - y_n|) \\ &\leqslant (1 - \beta_n)(\delta_n + \alpha_n) = (Q_n/Q_{n+1})(\delta_n + \alpha_n) \end{aligned}$$

and inductively,

$$\begin{aligned} \delta_{n+1} &\leqslant \frac{\delta_1 Q_1}{Q_{n+1}} + \frac{1}{Q_{n+1}} \sum_{k=1}^{n} Q_{n+1-k} \alpha_{n+1-k} \\ &= \frac{\delta_1 Q_1}{Q_{n+1}} + \frac{1}{Q_{n+1}} \sum_{k=1}^{n} Q_k \alpha_k \end{aligned}$$

which converges to zero whenever (A) holds.

Assume that $\{y_n\}$ does not converge, and let i and s be its inferior and superior limits respectively. Since $y_{n+1} - y_n \to 0$ and $z_n - y_n \to 0$, we must have both y_n and z_n arbitrarily close to $(i+s)/2$ for infinitely many n.

Hence there is an n for which both y_n and z_n lie in (i, s). But then, by Proposition 12 (ii), $f(z_n) = z_n$ and it follows from (14) (i) and (14) (ii) that y_{n+1} and z_{n+1} lie between y_n and z_n. Induction now yields that both $\{y_n\}$ and $\{z_n\}$ converge, contrary to the assumption.

(b) We observe that since $\alpha_n \to 0$ (ii) will hold if

$$\sum_{k=1}^{n} Q_k = O(Q_{n+1}).$$

This is true in particular if $q_k \geqslant \varepsilon Q_k$ for some $\varepsilon > 0$.

Case (B). (a) Let $\delta_n := |y_{n+1} - f(z_n)|$. We show $\delta_n \to 0$. It follows that Proposition 12 applies to (14) (ii). Let $\Delta_n := |f(z_n) - f(z_{n-1})|$. Then, arguing as in Case (A),

$$\delta_n \leqslant \frac{\delta_1 P_2}{P_{n+1}} + \frac{1}{P_{n+1}} \sum_{k=2}^{n} P_k \Delta_k$$

which converges to zero whenever (B) holds. The proof of Case (B) can now be completed in much the same way as in Case (A).

(b) We observe that $\Delta_n \to 0$ since $\beta_n \to 0$, and so (ii) holds if

$$\sum_{k=1}^{n} P_k = O(P_{n+1}).$$

This is true in particular if $p_k \geqslant \varepsilon P_k$ for some $\varepsilon > 0$.

Last, if f is L-Lipschitz $\Delta_{n+1} \leqslant L\beta_n$ and we obtain the final sufficient condition as in Case (A).

EXAMPLE 14. (a) Let $p_n := 1/n$ and $q_n := 1$. Then

$$a_{n,k} := \frac{1}{nk} \sum_{r=k}^{n} \frac{1}{P_r} \qquad \text{where} \quad P_r \sim \log(r)$$

and so $p_n/P_n \to 0$ and Theorem 13 (A) (ii) holds since $1/\log(n) \to 0$.

(b) Let $p_1 := 1$ and $p_n := 2^{n-2}$ for $n \geqslant 2$, and $q_n := 1$. Then $\beta_n = 1/(n+1)$ and $\alpha_n = 1/2$. Thus Theorem 13 (B) holds while

$$a_{n,k} := \frac{1}{n}\left(1 - \frac{2^k}{2^{n+1}}\right), \qquad n \geqslant k \geqslant 2$$

and

$$a_{n,1} := \frac{2}{n}\left(1 - \frac{1}{2^n}\right).$$

Exchanging the roles of p_n and q_n leads to an application of Theorem 13 (A) with

$$a_{n,k} := \frac{1}{2^{n+1}} \sum_{r=k}^{n} \frac{2^r}{r}, \qquad n \geqslant k \geqslant 2$$

and

$$a_{n,1} := \frac{1}{2^{n-1}} \left(1 + \sum_{r=2}^{n} \frac{2^{r-2}}{r}\right).$$

(c) Theorem 13 fails to apply to two very natural iterations. Let $p_n := 1$ and $q_n := 1$. The underlying mean is the Hölder mean of order 2, H_2 [13], for which $\alpha_n = \beta_n = 1/(n+1)$. This mean takes the Cesaro average of Cesaro averages.

Correspondingly let $p_n := 1$ and $q_n := P_n = n$. The underlying mean is the Cesaro mean of order 2, C_2 [13], for which $\alpha_n = 1/(n+1)$ and $\beta_n = 2/(n+2)$. On beginning indexing at $k = 0$ as is conventional the mean has $a_{n,k} := 2(n+1-k)/[(n+1)(n+2)]$. This mean is equivalent in the summability sense to H_2 and is also a simple Nörlund mean [13].

We leave open the question of whether (14) converges in these cases. Observe, however, that if $p_n := 1/n$ and $q_n := P_n$ then Theorem 13 (A) does apply and $a_{n,k} := (n+1-k)/(kQ_n)$ with $Q_n \sim n \log(n)$.

The following gives an example of a simple regular triangular row stochastic matrix and a continuous function $f\colon [0, 1] \to [0, 1]$ for which the Mann iteration, $u_{n+1} := f(x_n)$, fails to converge while the difference between successive terms goes to zero, so that the cluster set of $\{x_n\}$ is connected for all continuous $f\colon [0, 1] \to [0, 1]$.

EXAMPLE 15. For $1 \leqslant k \leqslant n$ let

$$a_{n,k} = \begin{cases} \dfrac{1}{m} & \text{when } 3^{m-1} \leqslant n < 3^m, \quad m = 1, 2, ..., \\ 0 & \text{otherwise.} \end{cases}$$

This is the C_1 matrix with its mth row repeated $2 \cdot 3^{m-1}$ times. The corresponding Mann iterative process is given by

$$u_{n+1} := f(x_n)$$

with $u_1 = x_1 \in [0, 1]$ and

$$x_n := \frac{1}{m} \sum_{k=1}^{m} u_k \qquad \text{for} \quad 3^{m-1} \leqslant n < 3^m.$$

Clearly $x_{n+1} - x_n \to 0$. Let

$$v_n := \frac{1}{n} \sum_{k=1}^{n} u_k.$$

Then $x_k = v_n$ for $3^{n-1} \leqslant k < 3^n$ and so

$$\begin{aligned} v_{3^n} &= \frac{1}{3^n} \sum_{k=1}^{3^n} u_k = \frac{1}{3}\left(\frac{1}{3^{n-1}} \sum_{k=1}^{3^{n-1}} u_k\right) + \frac{1}{3^n} \sum_{k=3^{n-1}+1}^{3^n} f(x_{k-1}) \\ &= \frac{1}{3} v_{3^{n-1}} + \frac{2}{3} f(v_n). \end{aligned}$$

Now take

$$f(x) := \begin{cases} 1, & 0 \leqslant x \leqslant \frac{1}{3} \\ 2 - 3x, & \frac{1}{3} < x < \frac{2}{3} \\ 0, & \frac{2}{3} \leqslant x \leqslant 1. \end{cases}$$

Since $0 \leqslant v_n \leqslant 1$, we see that $v_n \geqslant \frac{2}{3} \Rightarrow v_{3^n} \leqslant \frac{1}{3}$ and $v_n \leqslant \frac{1}{3} \Rightarrow v_{3^n} \geqslant \frac{2}{3}$. Thus if we take $v_1 = u_1 = x_1$ in either $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$ then the sequence $\{v_n\}$ has infinitely many terms in each interval and so cannot converge. Thus the sequence $\{u_n\}$ does not converge.

We note finally that the function $f(x) := 1 - x^p (p \geqslant 1)$ is Lipschitz and decreasing on $[0, 1]$. Thus, both parts of Theorems 7 and 11 apply. By contrast, the *mean ergodic estimate*

$$x_n := [f(x) + f^{(2)}(x) + \cdots + f^{(n)}(x)]/n$$

need not converge to a fixed point unless $p = 1$ in which case f is non-expansive [8]. Clearly, for $x := 0$ or $x := 1$, $\{x_n\}$ converges to $\frac{1}{2}$ not to the fixed point. In fact, for any x other than the fixed point and for any $p > 1$, $\{x_n\}$ converges to $\frac{1}{2}$ not to the fixed point.

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