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# Asymptotic Relationships between Dirichlet Series*

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It is shown, inter alia, that under certain conditions the asymptotic relationhip

$$\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \sim l \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \qquad \text{as} \quad x \to 0+$$

between two Dirichlet series implies the same relationship with $\lambda_n$ replaced by $\lambda_n^c$, $0 < c < 1$. 

## 1. INTRODUCTION

Suppose throughout that $\lambda := \{\lambda_n\}$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geqslant 0$, and that $a := \{a_n\}$ is a sequence of non-negative numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty, \text{ and } \phi(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} < \infty \qquad \text{for all} \quad x > 0.$$

Let $\{s_n\}$ and $\{u_n\}$ be sequences of complex numbers. The Abelian summability method $A_\lambda$ (see [6, p. 71]) and the Dirichlet series method $D_{\lambda,a}$ (see [3]) are defined as follows:

$$\sum_{n=1}^{\infty} u_n = l(A_\lambda) \qquad \text{if} \quad \sum_{n=1}^{\infty} u_n e^{-\lambda_n x}$$

is convergent for all $x > 0$ and tends to $l$ as $x \to 0+$;

$$s_n \to l(D_{\lambda,a}) \qquad \text{if} \quad \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}$$

is convergent for all $x > 0$ and

$$\frac{1}{\phi(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \to l \text{ as } x \to 0+.$$

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When $\lambda_n := n$, the method $A_\lambda$ reduces to the Abel method $A$, and the method $D_{\lambda,a}$ reduces to the power series method $J_a$ (as defined in [2], for example).

From now on we assume that $\mu := \{\mu_n\}$, where $\mu_n := \lambda_n^c$, $0 < c < 1$.

The purpose of this note is to prove the following two inclusion theorems for the Abelian and Dirichlet series methods respectively:

THEOREM A. *Suppose that* $\sum_{n=1}^\infty u_n = l(A_\lambda)$, *and that* $\sum_{n=1}^\infty u_n e^{-\mu_n x}$ *is convergent for all* $x > 0$. *Then* $\sum_{n=1}^\infty u_n = l(A_\mu)$.

THEOREM D. *Suppose that* $s_n \to l(D_{\lambda,a})$, *and that* $\sum_{n=1}^\infty a_n e^{-\mu_n x}$ *and* $\sum_{n=1}^\infty a_n s_n e^{-\mu_n x}$ *are convergent for all* $x > 0$. *Then* $s_n \to l(D_{\mu,a})$.

The case $\lambda_n := n^p$, $p > 0$, of Theorem A is due to Cartwright [5]. An alternate proof of this case appears in [6, Appendix V]. The versions of both Theorem A and Theorem D with $\mu_n := \log \lambda_n$, $\lambda_1 \geqslant 1$ are also known ([6, Theorem 28] and [4] respectively).

## 2. A PRELIMINARY RESULT

Let $g(x) := e^{-x^c}$. Then $g(0+) = g(0)$ and, for $x > 0$, $g'(x) = -cx^{c-1}g(x)$, so that

$$g^{(n+1)}(x) = -cx^{c-1}g^{(n)}(x) - c\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1-c)(2-c)\cdots(k-c)\,x^{c-k-1}g^{(n-k)}(x).$$

Since $0 < c < 1$, it follows by induction that

$$(-1)^m g^{(m)}(x) > 0 \qquad \text{for} \quad x > 0, m = 0, 1, 2, ...;$$

i.e., $e^{-x^c}$ is completely monotonic in $[0, \infty)$. Therefore, by Bernstein's theorem [1, p. 56] (see also [7, p. 160]),

$$e^{-x^c} = \int_0^\infty e^{-xt}\,d\alpha(t) \qquad \text{for} \quad x \geqslant 0, \tag{1}$$

and hence

$$e^{-\mu_n x} = \int_0^\infty e^{-\lambda_n x^{1/c} t}\,d\alpha(t) \qquad \text{for} \quad x \geqslant 0, \tag{2}$$



where $\alpha(t)$ is bounded and non-decreasing in $[0, \infty)$. The integrals are normally interpreted in the Riemann–Stieltjes sense, but since it follows from (1) that $e^{-x^c} \geqslant \alpha(0+) - \alpha(0) \geqslant 0$ for all $x > 0$, and hence that $\alpha(0+) = \alpha(0)$, we may assume that $\alpha(t)$ is right-continuous on $[0, \infty)$ and interpret all integrals involving $d\alpha(t)$ in the Lebesgue–Stieltjes sense as follows: For $0 \leqslant a < b$,

$$\int_a^b d\alpha(t) := \int_{(a,b]} d\alpha(t) = \alpha(b+) - \alpha(a+) = \alpha(b) - \alpha(a)$$

and

$$\int_a^b f(t)\,d\alpha(t) := \int_{(a,b]} f(t)\,d\alpha(t),$$

the integrals over $(a, b]$ being Lebesgue–Stieltjes integrals. Further

$$\int_a^\infty f(t)\,d\alpha(t) := \lim_{b\to\infty}\int_a^b f(t)\,d\alpha(t)$$

whenever the limit exists.

## 3. PROOF OF THEOREM A

Suppose that $x > 0$ and define

$$f(x) := \sum_{n=1}^\infty u_n e^{-\lambda_n x}.$$

It follows from (2) that, for $\delta > 0$,

$$\sum_{n=1}^\infty u_n e^{-\mu_n x - \delta\lambda_n} = \int_0^\infty \sum_{n=1}^\infty u_n e^{-\lambda_n x^{1/c} t - \delta\lambda_n}\,d\alpha(t) = \int_0^\infty f(x^{1/c}t + \delta)\,d\alpha(t), \tag{3}$$

the inversion being justified because the series

$$\sum_{n=1}^\infty u_n e^{-\lambda_n x^{1/c} t - \delta\lambda_n}$$

is uniformly convergent for $t > 0$ (see [6, p. 76]) and $\int_0^\infty d\alpha(t) = 1$.

Further, the hypotheses of Theorem A imply that $f(t)$ is bounded on $(0, \infty)$. Letting $\delta \to 0+$ in (3), we obtain, by the Lebesgue–Stieltjes theorem on dominated convergence, that

$$\sum_{n=1}^{\infty} u_n e^{-\mu_n x} = \int_0^{\infty} f(x^{1/c} t)\, d\alpha(t),$$

and hence that

$$\sum_{n=1}^{\infty} u_n e^{-\mu_n x} \to \int_0^{\infty} l\, d\alpha(t) = l \quad \text{as} \quad x \to 0+. \blacksquare$$

## 4. Proof of Theorem D

Suppose that $x > 0$ and define

$$\phi_s(x) := \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}, \qquad \psi(x) := \sum_{n=1}^{\infty} a_n e^{-\mu_n x},$$

$$\psi_s(x) := \sum_{n=1}^{\infty} a_n s_n e^{-\mu_n x}.$$

It follows from (2) that

$$\psi(x) = \int_0^{\infty} \sum_{n=1}^{\infty} a_n e^{-\lambda_n x^{1/c} t}\, d\alpha(t) = \int_0^{\infty} \phi(x^{1/c} t)\, d\alpha(t), \tag{4}$$

the inversion of summation and integration in (4) being justified because all the terms involved are positive. Next, it follows from (2) that, for $\delta > 0$,

$$\sum_{n=1}^{\infty} a_n s_n e^{-\mu_n x - \delta\lambda_n} = \int_0^{\infty} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x^{1/c} t - \delta\lambda_n}\, d\alpha(t)$$

$$= \int_0^{\infty} \phi_s(x^{1/c} t + \delta)\, d\alpha(t), \tag{5}$$

the inversion being justified in this case because the series

$$\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x^{1/c} t - \delta\lambda_n}$$

is uniformly convergent for $t > 0$ (see [6, p. 76]) and $\int_0^{\infty} d\alpha(t) = 1$.



Further, the hypotheses of Theorem D imply that, for fixed positive $x$ and all positive $t$ and $\delta$, $|\phi_s(x^{1/c} t + \delta)| \leqslant M\phi(x^{1/c} t + \delta) + M \leqslant M\phi(x^{1/c} t) + M$, where $M$ is a positive number independent of $t$ and $\delta$. Letting $\delta \to 0+$ in (5), we obtain, by the Lebesgue–Stieltjes theorem on dominated convergence, that

$$\psi_s(x) = \int_0^{\infty} \phi_s(x^{1/c} t)\, d\alpha(t). \tag{6}$$

It follows that

$$\frac{\psi_s(x)}{\psi(x)} = \frac{1}{\psi(x)} \int_0^{\infty} \phi(t)\, \sigma(t)\, d\alpha(x^{-1/c} t), \qquad \text{where} \quad \sigma(t) := \frac{\phi_s(t)}{\phi(t)}.$$

Suppose without loss of generality that $l = 0$, i.e., that $\sigma(t) \to 0$ as $t \to 0+$. Since $\sum_{n=1}^{\infty} a_n = \infty$, we have that $\psi(x) \to \infty$ as $x \to 0+$. Further, $\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n t}$ is uniformly convergent for $t \geqslant \delta > 0$, so that $|\phi_s(t)| \leqslant M_\delta$ for $t \geqslant \delta > 0$, where $M_\delta$ is a positive number independent of $t$. It follows from (4) and (6) that

$$\limsup_{x \to 0+} \left| \frac{\psi_s(x)}{\psi(x)} \right|$$

$$= \limsup_{x \to 0+} \frac{1}{\psi(x)} \left| \int_0^{\delta} \phi(t)\, \sigma(t)\, d\alpha(x^{-1/c} t) + \int_{\delta}^{\infty} \phi_s(t)\, d\alpha(x^{-1/c} t) \right|$$

$$\leqslant \sup_{0 < t \leqslant \delta} |\sigma(t)| + \limsup_{x \to 0+} \frac{M_\delta}{\psi(x)} \int_0^{\infty} d\alpha(t)$$

$$= \sup_{0 < t \leqslant \delta} |\sigma(t)| \to 0 \qquad \text{as} \quad \delta \to 0+,$$

and hence that $\psi_s(x)/\psi(x) \to 0$ as $x \to 0+$. $\blacksquare$

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