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AN INCULSION THEOREM FOR DIRICHLET SERIES

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ABSTRACT. It is shown that under certain conditions the asymptotic relationship

$$\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \sim l \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} \text{ as } x \longrightarrow 0+$$

between two Dirichlet series implies the same relationship with λ_n replaced by $\log \lambda_n$.

1. **Introduction.** Suppose throughout that $\lambda := \{\lambda_n\}$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geqq 0$, and that $a := \{a_n\}$ is a sequence of non-negative numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty, \text{ and } \phi(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} < \infty \text{ for all } x > 0.$$

Let $\{s_n\}$ be a sequence of complex numbers with $s_0 = 0$. The Abelian summability method A_λ (see [3, p. 71]) and the Dirichlet series method $D_{\lambda,a}$ (see [12]) are defined as follows:

$$s_n \longrightarrow l(A_\lambda) \text{ if } \sum_{n=1}^{\infty} (s_n - s_{n-1}) e^{-\lambda_n x}$$

is convergent for all $x > 0$ and tends to l as $x \longrightarrow 0+$;

$$s_n \longrightarrow l(D_{\lambda,a}) \text{ if} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}$$

is convergent for all $x > 0$ and $\dfrac{1}{\phi(x)} \displaystyle\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \longrightarrow l$ as $x \longrightarrow 0+$.

When $\lambda_n := n$, the method A_λ reduces to the Abel method A, and the method $D_{\lambda,a}$ reduces to the power series method J_a (as defined in [1], for example). Denote by

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A^*_λ the method $D_{\lambda,a}$ with $a_1 := \lambda_1, a_n := \lambda_n - \lambda_{n-1}$ for $n \geqq 2$. The method A^*_λ also reduces to A when $\lambda_n := n$. Further, it is known (see [2, Lemma 2]) that, under the additional hypothesis $\lambda_{n+1} \sim \lambda_n$,

$$x\sum_{n=2}^{\infty}(\lambda_n - \lambda_{n-1})e^{-\lambda_n x} \longrightarrow 1 \text{ as } x \longrightarrow 0+.$$

Thus, when $\lambda_{n+1} \sim \lambda_n$,

$$s_n \longrightarrow l(A^*_\lambda) \text{ if and only if } x\sum_{n=2}^{\infty}(\lambda_n - \lambda_{n-1})s_n e^{-\lambda_n x}$$
$$\text{is convergent for all } x > 0 \text{ and tends to } l \text{ as } x \longrightarrow 0+.$$

The exact relationship between A_λ and A^*_λ for general λ remains to be investigated.

From now on we assume that $\lambda_1 \geqq 1$ and that $\mu := \{\mu_n\}$ where $\mu_n := \log \lambda_n$. The following inclusion theorem for Abelian methods is known [3, Theorem 28]:

THEOREM A. *If $s_n \longrightarrow l(A_\lambda)$, and $\sum_{n=1}^{\infty}(s_n - s_{n-1})\lambda_n^{-x}$ is convergent for all $x > 0$, then $s_n \longrightarrow l(A_\mu)$.*

The purpose of this note is to prove the following analogous theorem for Dirichlet series methods:

THEOREM D. *Suppose that $s_n \longrightarrow l(D_{\lambda,a})$, and that $\sum_{n=1}^{\infty} a_n\lambda_n^{-x}$ and $\sum_{n=1}^{\infty} a_n s_n\lambda_n^{-x}$ are convergent for all $x > 0$. Then $s_n \longrightarrow l(D_{\mu,a})$.*

2. **Proof of Theorem D.** Suppose that $x > 0$, and let

$$\phi_s(x) := \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}, \psi(x) := \sum_{n=1}^{\infty} a_n\lambda_n^{-x}, \text{ and } \psi_s(x) := \sum_{n=1}^{\infty} a_n s_n\lambda_n^{-x}.$$

Then the hypotheses of Theorem D imply [3, Theorem 30] that

$$\psi(x) = \frac{1}{\Gamma(x)}\int_0^{\infty} t^{x-1}\phi(t)dt \text{ and } \psi_s(x) = \frac{1}{\Gamma(x)}\int_0^{\infty} t^{x-1}\phi_s(t)dt.$$

Hence

$$\frac{\psi_s(x)}{\psi(x)} = \frac{1}{F(x)}\int_0^{\infty} t^{x-1}\phi(t)\sigma(t)dt,$$

where

$$F(x) := \int_0^{\infty} t^{x-1}\phi(t)dt \text{ and } \sigma(t) := \frac{\phi_s(t)}{\phi(t)}.$$

Suppose without loss of generality that $l = 0$, i.e., that $\sigma(t) \longrightarrow 0$ as $t \longrightarrow 0+$. Since $\sum_{n=1}^{\infty} a_n = \infty$, we have that $\phi(t) \longrightarrow \infty$ as $t \longrightarrow 0+$ and hence that $F(x) \longrightarrow \infty$ as $x \longrightarrow 0+$. Further, $\sum_{n=1}^{\infty} a_n s_n e^{-(\lambda_n - \lambda_1)t}$ is uniformly convergent for $t \geqq \delta > 0$ (see [3, p. 76]); so that $|\phi_s(t)| \leqq H_\delta e^{-\lambda_1 t}$ for $t \geqq \delta > 0$, where H_δ is a positive number independent of t. It follows that

$$\limsup_{x\to 0+}\left|\frac{\psi_s(x)}{\psi(x)}\right| = \limsup_{x\to 0+}\frac{1}{F(x)}\left(\int_0^{\delta} t^{x-1}\phi(t)\sigma(t)dt + \int_{\delta}^{\infty} t^{x-1}\phi_s(t)dt\right)$$
$$\leqq \sup_{0<t<\delta}|\sigma(t)| + \limsup_{x\to 0+}\frac{H_\delta}{\delta^{1-x}F(x)}\int_\delta^{\infty} e^{-\lambda_1 t}dt$$
$$= \sup_{0<t<\delta}|\sigma(t)| \longrightarrow 0 \text{ as } \delta \longrightarrow 0+,$$

and hence that $\psi_s(x)/\psi(x) \longrightarrow 0$ as $x \longrightarrow 0+$. □

EXAMPLE. With $\lambda_n := n, a_n := 1/n$, Theorem D yields the following interesting result concerning the Riemann zeta function:

$$\text{if } \frac{1}{-\log(1-y)}\sum_{n=1}^{\infty}\frac{s_n}{n}y^n \longrightarrow l \text{ as } y \longrightarrow 1-$$
$$\text{and } \sum_{n=1}^{\infty}\frac{s_n}{n^w} \text{ is convergent for all } w > 1, \text{ then } \frac{1}{\zeta(w)}\sum_{n=1}^{\infty}\frac{s_n}{n^w} \longrightarrow l \text{ as } w \longrightarrow 1+.$$

The first of the above hypotheses can be stated as $s_n \longrightarrow l(L)$, where L is the logarithmic power series method of summability; and, because of the familiar result that $(w-1)\zeta(w) \longrightarrow 1$ as $w \longrightarrow 1+$, the conclusion can be simplified to

$$(w-1)\sum_{n=1}^{\infty}\frac{s_n}{n^w} \longrightarrow l \text{ as } w \longrightarrow 1+.$$

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