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ON A SCALE OF ABEL-TYPE SUMMABILITY METHODS

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1. *Introduction.* In this note Abel-type summability methods (A_λ) are defined and some of their properties investigated.

Let
$$\epsilon_n^\lambda = \binom{n+\lambda}{n},$$
and let $\{s_n\}$ be any sequence of numbers. If
$$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n$$
is convergent for all x in the open interval $(0, 1)$ and tends to a finite limit s as $x \to 1$ in $(0, 1)$, we shall say that the sequence is A_λ-convergent to s and write $s_n \to s\,(\mathrm{A}_\lambda)$. The A_0 method is the ordinary Abel method.

It will be convenient to use the notation
$$\sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n.$$
Evidently $s_n \to s\,(\mathrm{A}_\lambda)$ if and only if the series is convergent for all $y > 0$ and $\sigma_\lambda(y) \to s$ as $y \to \infty$.

2. *Regularity and inclusion theorems.* Suppose that $\lambda > -1$ and that m is any positive integer. Then
$$\varlimsup_{y\to\infty} |\sigma_\lambda(y)| \leqslant \varlimsup_{y\to\infty} (1+y)^{-\lambda-1} \sum_{n=m}^{\infty} \epsilon_n^\lambda |s_n| \left(\frac{y}{1+y}\right)^n \leqslant \sup_{n\geqslant m} |s_n|,$$
whence $s_n \to 0\,(\mathrm{A}_\lambda)$ whenever $s_n \to 0$; it follows immediately that $s_n \to s\,(\mathrm{A}_\lambda)$ whenever $s_n \to s$. We have thus proved

THEOREM 1. *A_λ is regular for $\lambda > -1$.*

In order to obtain results about the relative strengths of different A_λ methods we shall use the following two lemmas:

LEMMA 1. *For $\lambda > \mu > -1$, $y > 0$, $n = 0, 1, \ldots$, we have*
$$\frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\,\Gamma(\lambda-\mu)} \epsilon_n^\lambda y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^{\mu+n} (1+t)^{-\lambda-1-n} dt = \epsilon_n^\mu (1+y)^{-\mu-1} \left(\frac{y}{1+y}\right)^n.$$
Proof. Making the substitutions $u = t/(1+t)$, $x = y/(1+y)$, we get
$$\int_0^y (y-t)^{\lambda-\mu-1} t^{\mu+n} (1+t)^{-\lambda-1-n} dt = (1-x)^{1+\mu-\lambda} \int_0^x (x-u)^{\lambda-\mu-1} u^{\mu+n} du$$
$$= x^{\lambda+n} (1-x)^{1+\mu-\lambda} \frac{\Gamma(\lambda-\mu)\,\Gamma(\mu+1+n)}{\Gamma(\lambda+1+n)},$$

from which the lemma follows.

LEMMA 2. *If* $\lambda > \mu > -1$, $y > 0$ *and* $\Sigma \epsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n$ *is convergent for all* $t > 0$, *then*

(i) $\sigma_\mu(y) = \dfrac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} y^{-\lambda} \displaystyle\int_0^y (y-t)^{\lambda-\mu-1} t^\mu \sigma_\lambda(t)\,dt,$

(ii) $\overline{\lim}_{y\to\infty} |\sigma_\mu(y)| \leqslant \overline{\lim}_{y\to\infty} |\sigma_\lambda(y)|.$

Proof. It is easily verified that the convergence of $\Sigma \epsilon_n^\alpha s_n \left(\frac{t}{1+t}\right)^n$ for $\alpha = \lambda$ and all $t > 0$ implies its absolute convergence for all α and all $t > 0$. This justifies the term by term integration by means of which we can deduce (i) from Lemma 1.

To prove (ii) suppose that $y > w > 0$. Then it follows from (i) that

$$|\sigma_\mu(y)| \leqslant \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu)} y(y-w)^{-\mu-2} \int_0^w t^\mu |\sigma_\lambda(t)|\,dt + \sup_{t \geqslant w} |\sigma_\lambda(t)|.$$

and hence that
$$\overline{\lim}_{y\to\infty} |\sigma_\mu(y)| \leqslant \sup_{t\geqslant w} |\sigma_\lambda(t)|.$$

Result (ii) follows.

We deduce from Lemma 2 (ii) that if $\lambda > \mu > -1$ and $\lim_{y\to\infty} \sigma_\lambda(y) = 0$, then

$$\lim_{y\to\infty} \sigma_\mu(y) = 0.$$

Consequently, for $\lambda > \mu > -1$,

$$s_n \to s\ (A_\mu) \quad \text{whenever} \quad s_n \to s\ (A_\lambda).$$

We state this result concisely as follows:

THEOREM 2. $A_\lambda \subseteq A_\mu$ *for* $\lambda > \mu > -1$.

We next prove

THEOREM 3. *For any* $\lambda > -1$, *there is a sequence which is* A_μ*-convergent for every* μ *such that* $\lambda > \mu > -1$, *but not* A_λ*-convergent.*

Proof. Let $\{s_n\}$ be the sequence such that

$$\sum_{n=0}^\infty \epsilon_n^\lambda s_n x^n = (1-x)^{-\lambda-1} \sin\frac{1}{1-x} \quad (0 \leqslant x < 1).$$

The power series is convergent for all x in $(0, 1)$ but, since $\sin(1-x)^{-1}$ oscillates when $x \to 1$ in $(0, 1)$, $\{s_n\}$ is not A_λ-convergent.

For this sequence $\sigma_\lambda(y) = \sin(1+y)$ so that, for $\lambda > \mu > -1$, $y > 0$, we have by Lemma 2 (i),

$$\frac{\Gamma(\mu+1)\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)} \sigma_\mu(y) = y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu \sin(1+t)\,dt$$
$$= \int_0^1 (1-u)^{\lambda-\mu-1} u^\mu \sin(1+yu)\,du,$$

which, by the Riemann–Lebesgue theorem, tends to zero as $y \to \infty$. Consequently $s_n \to 0\ (A_\mu)$ for $\lambda > \mu > -1$; and this completes the proof.

In view of the above two theorems we may write

$$A_\lambda \subset A_\mu \quad (\lambda > \mu > -1); \tag{1}$$

the notation signifying that any sequence which is A_λ-convergent is also A_μ-convergent to the same limit, and that there is at least one A_μ-convergent sequence which is not A_λ-convergent.

3. *Relative strengths of* A_λ *and Cesàro methods.* We shall use the following lemma which is due to Bosanquet ((2), Lemma 6):

LEMMA 3. *If* k *is a positive integer,* $p > -1$, $\lambda > -1-p$, *and if* $\{\gamma_n\}$, $\{s_n\}$ *are sequences such that* (i) $\gamma_n = O(n^\lambda)$, (ii) $\Delta^k \gamma_n = O(n^{\lambda-k})$, (iii) $s_n = o(n^p)$ (C, k), *then*

$$s_n \gamma_n = o(n^{\lambda+p}) \ (\mathrm{C}, k).$$

We have used here the notation

$$\Delta\gamma_n = \Delta^1\gamma_n = \gamma_n - \gamma_{n+1}, \Delta^r\gamma_n = \Delta\Delta^{r-1}\gamma_n \quad (r = 2, 3, \ldots).$$

and understand (iii) to mean

$$\sum_{\nu=0}^n \epsilon_{n-\nu}^{k-1} s_\nu = o(n^{p+k}).$$

Suppose that k is a positive integer and

$$s_n = o(1) \ (\mathrm{C}, k); \tag{2}$$

and let
$$t_n = \frac{1}{\epsilon_n^{\lambda+k}} \sum_{\nu=0}^n \epsilon_{n-\nu}^{k-1} \epsilon_\nu^\lambda s_\nu,$$

where $\lambda > -1$.

Since $\epsilon_n^\alpha = O(n^\alpha)$ and $\Delta\epsilon_n^\alpha = -\epsilon_{n+1}^{\alpha-1}$ for all real α, and, for $\beta > -1$, $\Gamma(\beta+1)n^{-\beta}\epsilon_n^\beta \to 1$, we deduce from (2), by means of Lemma 3 with $p = 0$ and $\gamma_n = \epsilon_n^\lambda$, that

$$t_n \to 0. \tag{3}$$

Further it is well known that (2) implies that $s_n = O(n^k)$. The power series $\Sigma \epsilon_n^\lambda s_n x^n$ is therefore convergent for all x in $(0, 1)$, whence, for such x,

$$(1-x)^{\lambda+1} \sum_{n=0}^\infty \epsilon_n^\lambda s_n x^n = (1-x)^{\lambda+k+1} \sum_{\nu=0}^\infty \epsilon_\nu^{k-1} x^\nu \sum_{n=0}^\infty \epsilon_n^\lambda s_n x^n$$
$$= (1-x)^{\lambda+k+1} \sum_{n=0}^\infty \epsilon_n^{\lambda+k} t_n x^n. \tag{4}$$

Since $A_{\lambda+k}$ is regular, it follows from (3) and (4) that

$$s_n \to 0 \ (A_\lambda).$$

We immediately deduce that $s_n \to s\ (A_\lambda)$ whenever $s_n \to s$ (C, k), that is, whenever

$$\frac{1}{\epsilon_n^k} \sum_{\nu=0}^n \epsilon_{n-\nu}^{k-1} s_\nu \to s.$$

Further, it is familiar that $(\mathrm{C}, \alpha) \subset (\mathrm{C}, k)$ for $k > \alpha > -1$. Consequently we have

THEOREM 4. $(\mathrm{C}, \alpha) \subset A_\lambda$ *for* $\alpha > -1$, $\lambda > -1$.

Remark. It can be shown by use of the theory of Hausdorff means that (2) and (3) are equivalent for all $k > -1$, $\lambda + k > -1$.

4. *Translativity of the* A_λ *methods.* In this section we prove

THEOREM 5. A_λ *is translative for* $\lambda > -1$.

By this we mean that $s_n \to s\ (A_\lambda)$ if and only if $s_{n+1} \to s\ (A_\lambda)$.

We require

LEMMA 4. *If* $\lambda > -1$ *and* a *is real, and if* $\{s_n\}$ *is an* A_λ*-convergent sequence and* $(n+a)\,u_n = s_n$ *for* $n = 0, 1, \ldots$, *then* $u_n \to 0\ (A_\lambda)$.

Proof. Let

$$\phi(x) = \sum_{n=m}^{\infty} \epsilon_n^\lambda s_n x^{n+a-1} \quad (|x| < 1),$$

where $m > |a|+1$. Then $(1-x)^{\lambda+1}\phi(x)$ tends to a finite limit as $x \to 1$ in $(0,1)$ and $\phi(x) \to 0$ as $x \to 0$. Hence, for $0 \leqslant x < 1$,

$$\phi(x) = O\{(1-x)^{-\lambda-1}\},$$

and so

$$(1-x)^{\lambda+1} \sum_{n=m}^{\infty} \epsilon_n^\lambda u_n x^n = (1-x)^{\lambda+1} x^{-a} \int_0^x \phi(t)\,dt$$

$$= O\left\{(1-x)^{\lambda+1} x^{-a} \int_0^x (1-t)^{-\lambda-1}\,dt\right\}$$

$$= o(1) \quad \text{as } x \to 1 \text{ in } (0,1).$$

The lemma follows.

Proof of Theorem 5. Suppose that $\lambda > -1$ and $s_n \to s\ (A_\lambda)$. It is easily verified that, for $0 \leqslant x < 1$,

$$\sum_{n=1}^{\infty} \epsilon_n^\lambda s_{n-1} x^n = x \sum_{n=0}^{\infty} \epsilon_n^\lambda s_n x^n + \lambda x \sum_{n=0}^{\infty} \epsilon_n^\lambda \frac{s_n}{n+1} x^n, \tag{5}$$

$$x \sum_{n=0}^{\infty} \epsilon_n^\lambda s_{n+1} x^n = \sum_{n=1}^{\infty} \epsilon_n^\lambda s_n x^n - \lambda \sum_{n=1}^{\infty} \epsilon_n^\lambda \frac{s_n}{n+\lambda} x^n. \tag{6}$$

Applying Lemma 4, we deduce from (5) that $s_{n-1} \to s\ (A_\lambda)$, and from (6) that $s_{n+1} \to s\ (A_\lambda)$. The theorem follows.

5. *Product of* A_λ *and Hausdorff methods.* We recall the definition of the regular Hausdorff summability method (H_χ):

Throughout this section, suppose that $\chi(t)$ is a real function of bounded variation in the interval $[0,1]$ such that $\chi(0+) = \chi(0) = 0$ and $\chi(1) = 1$, and let

$$h_n = \sum_{\nu=0}^{n} \binom{n}{\nu} s_\nu \int_0^1 t^\nu (1-t)^{n-\nu}\,d\chi(t).$$

If $h_n \to s$ we write $s_n \to s\ (H_\chi)$. The conditions on $\chi(t)$ ensure the regularity of the method (see (3), § 11·8).

The product method $(A_\lambda H_\chi)$ is now defined as follows:

$$s_n \to s\ (A_\lambda H_\chi) \quad \text{if and only if} \quad h_n \to s\ (A_\lambda).$$

The next lemma has been proved by A. Amir ((1), p. 376), the case $\lambda = 0$ having first been established by O. Szász (4). The proof given below is considerably shorter than that given by Amir.

LEMMA 5. *If* $\lambda > -1$ *and* $\Sigma \epsilon_n^\lambda s_n x^n$ *is convergent for* $0 \leqslant x < 1$, *then, for* $y > 0$,

$$(1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \epsilon_n^\lambda h_n \left(\frac{y}{1+y}\right)^n = \int_0^1 \sigma_\lambda(yt)\,d\chi(t).$$

Proof. For $y > 0$,

$$\sum_{n=0}^{\infty} \frac{y^n}{(1+y)^{\lambda+1+n}} \epsilon_n^\lambda \sum_{\nu=0}^{n} \binom{n}{\nu} s_\nu \int_0^1 t^\nu (1-t)^{n-\nu}\,d\chi(t)$$

$$= \int_0^1 d\chi(t) \sum_{\nu=0}^{\infty} \epsilon_\nu^\lambda s_\nu \frac{(yt)^\nu}{(1+y)^{\lambda+1+\nu}} \sum_{n=\nu}^{\infty} \epsilon_{n-\nu}^{\lambda+\nu} \left(\frac{y-yt}{1+y}\right)^{n-\nu}$$

$$= \int_0^1 d\chi(t) \sum_{\nu=0}^{\infty} \epsilon_\nu^\lambda s_\nu \frac{(yt)^\nu}{(1+yt)^{\lambda+1+\nu}};$$

all the inversions being justified since $\int_0^1 |d\chi(t)| < \infty$, and, for $0 \leqslant t \leqslant 1$, $y > 0$,

$$\sum_{\nu=0}^{\infty} \epsilon_\nu^\lambda |s_\nu| \frac{(yt)^\nu}{(1+yt)^{\lambda+1+\nu}} \leqslant \sum_{\nu=0}^{\infty} \epsilon_\nu^\lambda |s_\nu| \left(\frac{y}{1+y}\right)^\nu,$$

which is finite and independent of t. The lemma follows.

If $\lambda > -1$ and $\sigma_\lambda(t)$ tends to a finite limit s as $t \to \infty$ then $\sigma_\lambda(t)$ is continuous for $t \geqslant 0$, and the conditions on $\chi(t)$ are such that $\int_0^1 \sigma_\lambda(yt)\,d\chi(t) \to s$ as $y \to \infty$ (see (3) § 11·18). Thus a consequence of Lemma 5 is

$$A_\lambda \subseteq A_\lambda H_\chi \quad (\lambda > -1). \tag{7}$$

The case $\lambda = 0$ of this result was proved by Szász (4) and the general case by Amir (1). The new result we shall prove is:

THEOREM 6. *If* $\chi(t)$ *is absolutely continuous in* $[0,1]$, $\chi(1)-\chi(0) = 1$, *and* $\lambda > -1$, *then* $A_\lambda \subset A_\lambda H_\chi$.

Proof. The sequence given in the proof of Theorem 3 is not A_λ-convergent. However, in view of Lemma 5, it is $A_\lambda H_\chi$-convergent to zero, since, for the sequence in question, we have

$$\int_0^1 \sigma_\lambda(yt)\,d\chi(t) = \int_0^1 \sin(1+yt)\,\chi'(t)\,dt,$$

which, by the Riemann–Lebesgue theorem, tends to zero as $y \to \infty$. This together with (7) yields the required result.

REFERENCES

(1) AMIR (JAKIMOVSKI), A. Some relations between the methods of summability of Abel, Borel, Cesàro, Holder and Hausdorff. *J. Anal. Math.* 3 (1954), 346–81.
(2) BOSANQUET, L. S. Note on convergence and summability factors (II). *Proc. Lond. Math. Soc.* (2), 50 (1948), 295–304.
(3) HARDY, G. H. *Divergent series* (Oxford, 1949).
(4) SZÁSZ, O. On the product of two summability methods. *Ann. Polon. Math.* 25 (1952), 75–84.

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