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ON STRONG GENERALIZED HAUSDORFF SUMMABILITY

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Introduction

For a series $\sum_0^\infty a_k$, let $s_n = \sum_{k=0}^n a_k$. Let $Q = \{q_{n,k}\}$ $(n, k = 0, 1, \ldots)$ be a matrix and let

$$\sigma_n = Q(s_n) = \sum_{k=0}^\infty q_{n,k} s_k.$$

The series $\sum_0^\infty a_k$ is said to be summable Q to s if σ_n exists for $n=0, 1, \ldots$ and tends to s as n tends to infinity. In this case we write $s_n \to s(Q)$. The symbol P is reserved for matrices $\{p_{n,k}\}$ with $p_{n,k} \geqq 0$, and I denotes the identity matrix. We now recall the definition of strong summability introduced by Borwein [1].

Strong summability. A series $\sum_0^\infty a_k$ is said to be summable $[P, Q]_\beta$ $(\beta > 0)$ to s if $\sum_{k=0}^\infty p_{n,k} |\sigma_k - s|^\beta$ exists for $n=0, 1, \ldots$ and tends to zero as n tends to infinity. In this case we write $s_n \to s[P, Q]_\beta$.

For summability methods V and W, the notation $V \subseteqq W$ means that any series summable V to s is also summable W to s. The notation $V \simeq W$ means that both $V \subseteqq W$ and $W \subseteqq V$.

Generalized Hausdorff matrices. Suppose throughout that $\lambda = \{\lambda_n\}$ is a sequence of real numbers with

$$\lambda_0 \geqq 0, \quad \inf_{n \geqq 1} \lambda_n > 0 \quad \text{and} \quad \sum_{n=0}^\infty 1/\lambda_n = \infty.$$

Let Ω be a simply connected region that contains every positive λ_n, and suppose, for $n=0, 1, \ldots$, that Γ_n is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_k \in \Omega$ with $0 \leqq k \leqq n$. Suppose that f is holomorphic in Ω and that $f(\lambda_0)$ is defined even when $\lambda_0 \notin \Omega$. Define

$$(1) \qquad \lambda_{n,k} = \begin{cases} -\lambda_{k+1} \ldots \lambda_n \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_n} \dfrac{f(z)\,dz}{(\lambda_k - z) \ldots (\lambda_n - z)} + \delta_k & \text{for} \quad 0 \leqq k \leqq n \\ 0 \quad \text{for} \quad k > n \end{cases}$$

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where $\delta_k = f(\lambda_0)$ if $k=0$ and $\lambda_0 \notin \Omega$, and $\delta_k=0$ otherwise. Here and elsewhere we observe the convention that products like $\lambda_{k+1} \dots \lambda_n = 1$ when $k=n$. Denote the triangular matrix $\{\lambda_{n,k}\}$ by $(\lambda; f)$. This is called a generalized Hausdorff matrix. The set of all such matrices is denoted by $\mathscr{H}_\lambda$.

For α any real number, the generalized Hausdorff matrix H_α is defined to be the matrix $(\lambda; f)$ with $f(z)=(z+1)^{-\alpha}$. For $\alpha > -1$, the generalized Cesàro matrix C_α is defined to be the matrix $(\lambda; f)$ with

$$f(z) = \frac{\Gamma(\alpha+1)\Gamma(z+1)}{\Gamma(\alpha+z+1)}.$$

These reduce to the standard Hölder and Cesàro matrices when $\lambda_n = n$. (See [1].)

Preliminary results

For $0 < t \leqq 1$, let $\lambda_{n,k}(t)$ denote the value of $\lambda_{n,k}$ obtained from (1) with $f(z)=t^z$, and let $\lambda_{n,k}(0)=\lambda_{n,k}(0+)$. Let

$$D_0 = (1+\lambda_0)\, d_0 = 1;$$

$$D_n = \left(1+\frac{1}{\lambda_1}\right) \dots \left(1+\frac{1}{\lambda_n}\right) = (1+\lambda_n)\, d_n \quad \text{for} \quad n \geqq 1.$$

Then, (see [3]),

$$\int_0^1 \lambda_{n,k}(t)\, dt = \frac{d_k}{D_n} \quad \text{for} \quad 0 \leqq k \leqq n.$$

If

(2) $$f(z) = \int_0^1 t^z\, d\chi(t) \quad \text{with} \quad \chi \in \mathrm{BV}$$

where BV is the space of functions of bounded variation on the closed interval [0, 1], then

$$\lambda_{n,k} = \int_0^1 \lambda_{n,k}(t)\, d\chi(t).$$

It follows that

$$C_1(s_n) = \frac{1}{D_n} \sum_{k=0}^{n} d_k s_k$$

so that

(3) $$s_n - C_1(s_n) = C_1(\lambda_n a_n).$$

If f satisfies (2), $\chi(1)-\chi(0)=1$ and $\chi(0+)=\chi(0)$, then $X=(\lambda; f)$ is regular, i.e. $s_n \to s(X)$ whenever $s_n \to s$. (See [2; Theorem 1].)

Lemma 2 of [2] shows that if g and h are holomorphic in Ω and defined at λ_0, $X=(\lambda; g)$ and $Y=(\lambda; h)$, then

(4) $$XY = YX = (\lambda; gh).$$

Lemma 1 of [3] shows that if $X=(\lambda;f)$ with f satisfying (2), $\tilde{X}=(\lambda;\tilde{f})$ with $\tilde{f}(z)=\int_0^1 t^z |d\chi(t)|$, and $\beta\geqq 1$, then, for any sequence $\{w_n\}$,

$$|X(w_n)|^\beta \leqq M^{\beta-1}\tilde{X}(|w_n|^\beta) \tag{5}$$

where $M=\int_0^1 |d\chi(t)|$.

From (4) it can be seen that $H_\alpha H_\delta=H_{\alpha+\delta}$ for all real α, δ. Theorem 2 of [2] shows that

$$C_\alpha \simeq H_\alpha \quad \text{for} \quad \alpha>-1. \tag{6}$$

(See also [5] and [6].) Thus

$$C_\alpha C_\delta \simeq C_{\alpha+\delta} \quad \text{for} \quad \alpha>-1, \quad \delta>-1, \quad \alpha+\delta>-1. \tag{7}$$

Some theorems on strong summability

The first theorem generalizes Theorem 5 in [1].

THEOREM 1. *Suppose Q is a matrix, P is a regular matrix in $\mathscr{H}_\lambda$, and $X=(\lambda;f)$ where $f(z)=\int_0^1 t^z\,d\chi(t)$ with $\chi\in \mathrm{BV}$, $\chi(1)-\chi(0)=1$ and $\chi(0+)=\chi(0)$. Then, for $\beta\geqq 1$, $[P, Q]_\beta\subseteqq[P, XQ]_\beta$.*

PROOF. Let $\tilde{X}=\{\tilde{x}_{n,k}\}=(\lambda;\tilde{f})$ where $\tilde{f}(z)=\int_0^1 t^z|d\chi(t)|$. Since $\chi\in\mathrm{BV}$ and $\chi(0+)=\chi(0)$, it follows that $\lim_{n\to\infty}\tilde{x}_{n,k}=0$ for $k=0, 1, \dots$ and $\sup_{n\geqq 0}\sum_{k=0}^{n}|\tilde{x}_{n,k}|<\infty$. (See [2, Theorem 1].) Hence $\tilde{X}(u_n)\to 0$ whenever $u_n\to 0$. (See [4, Theorem 4].)

Let $\{s_n\}$ be a sequence, $\sigma_n=X(s_n)$ and $w_n=s_n-s$. In view of the regularity of X we have $\sigma_n-s=X(w_n)+\varepsilon_n$ where $\varepsilon_n\to 0$. From (4) and (5) it follows that

$$P(|X(w_n)|^\beta)\leqq M^{\beta-1}P\tilde{X}(|w_n|^\beta) = M^{\beta-1}\tilde{X}P(|s_n-s|^\beta) \tag{8}$$

where $M=\int_0^1 |d\chi(t)|$. Next, by Minkowski's inequality,

$$\big(P(|X(w_n)+\varepsilon_n|^\beta)\big)^{1/\beta}\leqq \big(P(|X(w_n)|^\beta)\big)^{1/\beta}+\big(P(|\varepsilon_n|^\beta)\big)^{1/\beta}. \tag{9}$$

Suppose now that $P(|s_n-s|^\beta)\to 0$. Then, by (8), $P(|X(w_n)|^\beta)\to 0$ so that, by (9), $P(|\sigma_n-s|^\beta)\to 0$. Hence $[P, I]_\beta\subseteqq[P, X]_\beta$, from which it follows that $[P, Q]_\beta\subseteqq[P, XQ]_\beta$. □

The next two theorems generalize corollaries to Theorem 7 in [1].

THEOREM 2. *If $X\in\mathcal{H}_\lambda$ and $\beta\geqq 1$, then necessary and sufficient conditions for a series $\sum_0^\infty a_n$ to be summable $[C_1, X]_\beta$ to s are that it be summable C_1X to s and that $\lambda_n a_n \to 0[C_1, C_1X]_\beta$.*

PROOF. It follows from Theorem 1 in [1] that $\sum_0^\infty a_n$ is summable $[C_1, X]_\beta$ to s if and only if it is summable C_1X to s and summable $[C_1, (I-C_1)X]_\beta$ to 0. Further, by (3) and (4),

$$(I-C_1)X(s_n) = X(s_n - C_1(s_n)) = C_1X(\lambda_n a_n).$$

The result follows. □

In conformity with notation introduced earlier (see [1; p. 123]), the generalized strong Cesàro method $[C_1, C_{\alpha-1}]_\beta$ will be denoted by $[C, \alpha]_\beta$ and the generalized strong Hölder method $[H_1, H_{\alpha-1}]_\beta$ by $[H, \alpha]_\beta$. We require the following known result (see [8]).

LEMMA 1. *Let*

$$g(z) = \frac{\Gamma(\delta+1)\Gamma(z+1)(z+1)^\delta}{\Gamma(\delta+z+1)}, \quad \delta > -1.$$

Then both $g(z)$ and $1/g(z)$ can be expressed as Mellin transforms of the form $\int_0^1 t^z\, d\chi(t)$ with $\chi\in\mathrm{BV}$, $\chi(1)-\chi(0)=1$ and $\chi(0+)=\chi(0)$.

THEOREM 3. *If $\alpha\geqq 0$ and $\beta\geqq 1$, then necessary and sufficient conditions for a series $\sum_0^\infty a_n$ to be summable $[C, \alpha]_\beta$ to s are that it be summable C_α to s and that $\lambda_n a_n \to 0[C, \alpha+1]_\beta$.*

PROOF. It follows from Theorem 2 that $\sum_0^\infty a_n$ is summable $[C, \alpha]_\beta$ to s if and only if it is summable $C_1C_{\alpha-1}$ to s and $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$. Next, it follows from Lemma 1 with $\delta=\alpha-1$ and Theorem 1 that $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$ if and only if $\lambda_n a_n \to 0[C_1, H_\alpha]_\beta$. Applying Lemma 1 and Theorem 1 again, we see that $\lambda_n a_n \to 0[C_1, C_1C_{\alpha-1}]_\beta$ if and only if $\lambda_n a_n \to 0[C_1, C_\alpha]_\beta$. This together with (7) yields the result. □

The above theorem suggests the following extension of the definition of $[C, \alpha]_\beta$ to the case $\alpha=0$: $\sum_0^\infty a_n$ is summable $[C, 0]_\beta$ to s if the series is convergent with sum s and $\sum_{k=0}^n d_k |\lambda_k a_k|^\beta = o(D_n)$. When $\lambda_n = n$, this definition reduces to the one given by Hyslop [7].

The next theorem is an analogue of the equivalence relation (6) for strong summability.

THEOREM 4. *For $\alpha\geqq 0$, $\beta\geqq 1$, $[C, \alpha]_\beta \simeq [H, \alpha]_\beta$.*

PROOF. The case $\alpha=0$ follows from Theorem 2 and the definition of $[C,0]_\beta$. Suppose therefore that $\alpha>0$. By Theorem 3, $\sum_0^\infty a_n=s[C,\alpha]$ if and only if $\sum_0^\infty a_n=s(C_\alpha)$ and $\lambda_n a_n\to 0[C_1, C_\alpha]_\beta$. Further, by Theorem 2, $\sum_0^\infty a_n=s[H,\alpha]_\beta$ if and only if $\sum_0^\infty a_n=s(H_\alpha)$ and $\lambda_n a_n\to 0[C_1, H_\alpha]_\beta$. The result now follows from (6), Lemma 1 and Theorem 1. □

Generalized Hausdorff matrices associated with L^p functions

Let L^p denote the function space $L^p(0,1)$. In this section we deal with Hausdorff matrices $(\lambda;f)$ with $f(z)=\int_0^1 t^z\varphi(t)\,dt$ where $\varphi\in L^p$ for some $p>1$. An ordinary Hausdorff matrix $\{x_{n,k}\}$ satisfies these conditions if and only if $\sum_{k=0}^n |x_{n,k}|^p<M(n+1)^{1-p}$ for $n=0,1,\dots$ where M is independent of n. (See [4, Theorem 215].)

The following lemma is needed for the proof of Theorem 5.

LEMMA 2. *Let $\varphi\in L^p$ with $p>1$. Let $X=(\lambda;f)$ and $X^{(p)}=(\lambda;f^{(p)})$ where $f(z)=\int_0^1 t^z\varphi(t)\,dt$ and $f^{(p)}(z)=\int_0^1 t^z|\varphi(t)|^p\,dt$. If $\mu>\beta\geqq 1$ and $1/p=1/\mu-1/\beta$, then for any sequence $\{w_n\}$,*

$$|X(w_n)|^\mu \leqq M^{\mu(1-1/\beta)}(C_1(|w_n|^\beta))^{\mu/\beta-1}X^{(p)}(|w_n|^\beta)$$

where $M=\int_0^1 |\varphi(t)|^p\,dt.$

PROOF. Let $f_n(t)=\sum_{k=0}^n \lambda_{n,k}(t)w_k$ where $0\leqq t\leqq 1$. Then, by Hölder's inequality,

$$|f_n(t)|^\beta \leqq \sum_{k=0}^n \lambda_{n,k}(t)\,|w_k|^\beta.$$

(See [3, (8)].) Hence

$$\int_0^1 |f_n(t)|^\beta\,dt \leqq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t)\,dt=\frac{1}{D_n}\sum_{k=0}^n d_k|w_k|^\beta=C_1(|w_n|^\beta) \tag{10}$$

and

$$\int_0^1 |\varphi(t)|^p|f_n(t)|^\beta\,dt \leqq \sum_{k=0}^n |w_k|^\beta \int_0^1 \lambda_{n,k}(t)\,|\varphi(t)|^p\,dt=X^{(p)}(|w_n|^\beta).$$

It follows, by Hölder's inequality, that

$$|X(w_n)| = \left|\int_0^1 \varphi(t) f_n(t)\,dt\right| \leqq$$

$$\leqq \left(\int_0^1 |\varphi(t)|^p\,dt\right)^{1-1/\beta} \left(\int_0^1 |f_n(t)|^\beta\,dt\right)^{1/\beta-1/\mu} \left(\int_0^1 |\varphi(t)|^p |f_n(t)|\,dt\right)^{1/\mu} \leqq$$

$$\leqq M^{1-1/\beta}\left(C_1(|w_n|^\beta)^{1/\beta-1/\mu} X^{(p)}(|w_n|^\beta)\right)^{1/\mu}. \quad \square$$

The following theorem generalizes Theorem 10 in [1].

THEOREM 5. *Let* $\mu>\beta\geqq 1$, $1/p=1+1/\mu-1/\beta$. *Let* $X=(\lambda;f)$ *where* $f(z)=\int_0^1 t^z\varphi(t)\,dt$ *with* $\varphi\in L^p$ *and* $\int_0^1 \varphi(t)\,dt=1$. *Then, for any matrix* Q, $[C_1, Q]_\beta\subseteqq[C_1, XQ]_\mu$.

The theorem remains valid when $\mu=\infty$ (with $1/p=1-1/\beta$ if $\lambda>1$ and $p=\infty$ if $\beta=1$) provided $[C_1, XQ]_\infty$ is interpreted to mean XQ.

PROOF. We use the notation introduced in Lemma 2, and note that X is regular and $X^{(p)}(v_n)\to 0$ whenever $v_n\to 0$. Suppose that $s_n\to s[C_1, Q]_\beta$, and let $\sigma_n=Q(s_n)$, $w_n=\sigma_n-s$, and $v_n=C_1(|w_n|^p)$.

(i) Suppose μ is finite. By hypothesis, $v_n\to 0$ and hence, by Lemma 2,

$$C_1(|X(w_n)|^\mu) \leqq KC_1X^{(p)}(|w_n|^\beta) = KX^{(p)}(v_n) \to 0$$

where $K=M_n^{\mu(1-1/\beta)}\sup v_n^{\mu/\beta-1}$. Also, by the regularity of X, we have $X(\sigma_n)-s=X(w_n)+\varepsilon_n$ where $\varepsilon_n\to 0$. Thus, by Minkowski's inequality,

$$\left(C_1(|X(\sigma_n)-s|^\mu)\right)^{1/\mu} \leqq \left(C_1(|X(w_n)|^\mu)\right)^{1/\mu}+\left(C_1(|\varepsilon_n|^\mu)\right)^{1/\mu} \to 0,$$

i.e. $s_n\to s[C_1, XQ]_\mu$.

(ii) Suppose now that $\mu=\infty$. By Hölder's inequality,

$$|X(w_n)|^\beta = \left|\int_0^1 f_n(t)\varphi(t)\,dt\right|^\beta \leqq m\int_0^1 |f_n(t)|^\beta\,dt$$

where $m=M^{\beta-1}$ if $\beta>1$ and $m=\operatorname*{ess\,sup}_{0<t<1}|\varphi(t)|$ if $\beta=1$. Since (10) holds under the operative hypotheses, it follows that

$$|X(w_n)|^\beta \leqq mC_1(|w_n|^\beta) = mv_n \to 0,$$

and hence that $s_n\to s(XQ)$. $\square$

THEOREM 6. *Let* $\varrho>1/\beta-1/\mu$ *where* $\mu\geqq\beta\geqq 1$. *Then, for any matrix* Q, $[C_1, Q]_\beta\subseteqq[C_1, C_\varrho Q]_\mu$.

PROOF. When $\mu=\beta$, the result follows from Theorem 1. Suppose that $\mu>\beta$ and let $1/p=1+1/\mu-1/\beta$. Then $C_\varrho=(\lambda;f)$, where $f(z)=\int_0^1 t^z\varphi(t)\,dt$ with $\varphi(t)=\varrho(1-t)^{\varrho-1}$. Since $\varphi\in L^p$, Theorem 5 now yields the result. □

THEOREM 7. *Let* $\gamma>\alpha+1/\beta-1/\mu$ *where* $\mu\geqq\beta\geqq 1$ *and* α *is any real number. Then* $[H,\alpha]_\beta\subseteqq[H,\gamma]_\mu$.

PROOF. Applying first Theorem 6 and then Theorem 1 together with Lemma 1, we get

$$[H,\alpha]_\beta=[H_1,H_{\alpha-1}]_\beta\subseteqq[H_1,C_{\gamma-\alpha}H_{\alpha-1}]_\mu\simeq[H_1,H_{\gamma-\alpha}H_{\alpha-1}]_\mu=$$
$$=[H_1,H_{\gamma-1}]_\mu=[H,\gamma]_\mu. \quad \square$$

REMARK. It is known that, in the special case $\lambda_n=n$, Theorem 6 also holds when $\varrho=1/\beta-1/\mu$ and Theorem 7 when $\gamma=\alpha+1/\beta-1/\mu$. (See [1] and the references there given.) Whether the same is true for more general λ_n is an open question.

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