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A TAUBERIAN THEOREM CONCERNING BOREL-TYPE AND CESÀRO METHODS OF SUMMABILITY

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1. Introduction. Suppose throughout that $r \geqq 0$, $\alpha > 0$, $\alpha q + \beta > 0$ where q is a non-negative integer. Let $\{s_n\}$ be a sequence of real numbers,

$$c_n(x) := \frac{\alpha e^{-x} x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)} \quad \text{and} \quad b(x) := \sum_{n=q}^{\infty} c_n(x) s_n.$$

The Borel-type summability method (B, α, β) is defined as follows:

$$s_n \to l(B, \alpha, \beta) \text{ if } b(x) \to l \text{ as } x \to \infty.$$

The method (B, α, β) is regular **[5]**; and $(B, 1, 1)$ is the standard Borel exponential method B. For a real sequence $\{s_n\}$ we consider the slowly decreasing-type Tauberian condition

$$(\mathrm{T}_r): \quad \lim_{\delta \to 0+} \liminf_{n \to \infty} \min_{n \leqq m \leqq n+\delta\sqrt{n}} \frac{s_m - s_n}{n^r} \geqq 0.$$

We shall also be concerned with the Cesàro summability method $C_p (p > -1)$, the Valiron method V_α, and the Meyer-König method S_a $(0 < a < 1)$ defined as follows:

$s_n \to l(C_p)$ if

$$\frac{1}{\binom{n+p}{p}} \sum_{k=0}^{n} s_k \binom{n-k+p-1}{n-k} \to l \quad \text{as } n \to \infty;$$

$s_n \to l(V_\alpha)$ if

$$\left(\frac{\alpha}{2\pi n}\right)^{1/2} \sum_{k=0}^{\infty} s_k \exp\left\{-\frac{\alpha(n-k)^2}{2n}\right\} \to l \quad \text{as } n \to \infty;$$

$s_n \to l(S_a)$ if

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$$(1-a)^{n+1} \sum_{k=0}^{\infty} s_k \binom{n+k}{k} a^k \to l \quad \text{as } n \to \infty.$$

Our main result is

THEOREM 1. *If $s_n \to l(B, \alpha, \beta)$ and (T_r), then $s_n \to l(C_{2r})$.*

Now suppose that

$$s_n = \sum_{k=0}^{n} a_k$$

and note that if

$$(\mathrm{L}_r): a_n > -Hn^{r-1/2} \quad \text{for } n = 1, 2, \ldots$$

then, for $n \leqq m \leqq n + \delta\sqrt{n}$,

$$\frac{s_m - s_n}{n^r} = \frac{1}{n^r} \sum_{j=n+1}^{m} a_j > \frac{-H}{n^r} \sum_{j=n+1}^{m} j^{r-1/2}$$

$$> \frac{-H(m-n)}{\sqrt{n+1}} \left(\frac{m}{n}\right)^r \geqq -H\delta\left(1 + \frac{\delta}{\sqrt{n}}\right)^r$$

so that

$$\lim_{\delta \to 0+} \liminf_{n \to \infty} \min_{n \leqq m \leqq n+\delta\sqrt{n}} \frac{s_m - s_n}{n^r}$$

$$\geqq \lim_{\delta \to 0+} \liminf_{n \to \infty} \left\{-H\delta\left(1 + \frac{\delta}{\sqrt{n}}\right)^r\right\} = 0.$$

Thus (L_r) implies (T_r).

The special case $\alpha = \beta = 1$, $r = 0$ of Theorem 1 with (T_0) replaced by $a_n = O(n^{-1/2})$ is the original O-Tauberian theorem for Borel summability due to Hardy and Littlewood **[10]**. The Borel summability case $\alpha = \beta = 1$ of Theorem 1 has been proved by Rajagopal **[13]**, and the corresponding theorem for Meyer-König summability S_a by Sitaraman **[14]**. More recently Bingham **[3]** proved the theorem for summability methods of the random walk-type of which B and S_a are special cases. For the general (B, α, β) method, the case $r \geqq 0$ of Theorem 1 with (T_r) replaced by $a_n = o(n^{r-1/2})$ is due to Borwein **[6]**, and the case $r = 0$ with (T_0) replaced by $a_n = O(n^{-1/2})$ is due to Borwein and Robinson **[7]**. The most general result to-date for the (B, α, β) method is due to Kwee **[12]** who proved the case of Theorem 1 with (T_r) replaced by $a_n = O(n^{r-1/2})$.

Theorem 1 remains true if the hypothesis $s_n \to l(B, \alpha, \beta)$ is replaced by $s_n \to l(B', \alpha, \beta)$, by which it is meant that, as $y \to \infty$,

$$\int_0^\infty e^{-x}dx \sum_{n=q}^{\infty} a_n \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \to l - s_{q-1} \quad (s_{-1} = 0).$$

This is a consequence of the following known result due to Borwein ([4], Theorem 2) that $s_n \to l(B, \alpha, \beta + 1)$ if and only if $s_n \to l(B', \alpha, \beta)$.

Borwein [5] also proved:

If

$$J(z) = \sum_{n=q}^{\infty} \frac{z^n}{h(n)}$$

is a holomorphic function of $z = x + iy$ *in the half-plane* $x > x_0$, *such that*

(i) *when* $x > x_0$ *and* $|z|$ *is large*

$$h(z) = z^{\alpha z+\beta} e^{\gamma z}\left\{C + O\left(\frac{1}{|z|}\right)\right\}$$

where $C > 0$, $\alpha > 0$, β *and* γ *are real, and*

(ii) $h(x)$ *is real and positive for* $x \geqq q > x_0$, *then* $s_n \to l(J)$

$$\left(i.e., \frac{1}{J(x)} \sum_{n=q}^{\infty} \frac{s_n x^n}{h(n)} \to l \quad as\ x \to \infty\right)$$

if and only if

$$s_n \to l(B, \alpha, \beta + 1/2).$$

In particular, taking

$$J(z) = \sum_{n=q}^{\infty} \frac{z^n}{\{\Gamma(\alpha n+\beta)\}^c (n+p)^{sn+t}}$$

where c, p, s, t are real and $\alpha c + s > 0$, we have

$$s_n \to l(J)$$

if and only if

$$s_n \to l(B, \alpha c + s, \beta c + t - c/2 + 1/2).$$

Thus Theorem 1 is in fact a Tauberian theorem for a wide class of power series methods of summability [9].

Since the actual choice of q is immaterial, it is convenient to assume in all that follows that $\alpha q + \beta - r > 0$.

2. Preliminary results.

Lemma 1 ([6], Lemma 2). *Let* $h_n = n - x/\alpha$, $1/2 < \xi < 2/3$, *and* $0 < \eta < 2\xi - 1$. *Then*

(i) $$\sum_{|h_n|>x^\xi} c_n(x) = O(e^{-x^\eta});$$

(ii) $$c_n(x) = \frac{\alpha}{\sqrt{2\pi x}} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\{1 + O(x^{3\xi-2})\}$$

when $|h_n| \leqq x^\xi$.

Lemma 2 ([5], Result (I); [4], Lemma 4). *If* $\alpha > \gamma > 0$ *and for any non-negative integer* $M > -\delta/\gamma$,

$$\sum_{n=M}^{\infty} a_n \frac{x^n}{\Gamma(\gamma n+\delta)}$$

is convergent for all x, *then* $s_n \to l(B, \alpha, \beta)$ *implies*

$$s_n \to l(B, \gamma, \delta).$$

The next result follows from Stirling's formula (see [1], p. 47).

Lemma 3.

$$\frac{(\alpha n)^r}{\Gamma(\alpha n+\beta)} \sim \frac{1}{\Gamma(\alpha n+\beta-r)} \quad as\ n \to \infty.$$

Lemma 4. *Let* $1/2 < \xi < 2/3$, *then as* $x \to \infty$

(i) $$\sum_{q \leqq n < x/\alpha - x^\xi} n^r c_n(x) = o(1),$$

(ii) $$\sum_{n > x/\alpha + x^\xi} n^r c_n(x) = o(1).$$

Proof. For (i) we have, by Lemmas 3 and 1 (i), that, as $x \to \infty$,

$$\sum_{q \leqq n < x/\alpha - x^\xi} n^r c_n(x) = O\left\{x^r e^{-x} \sum_{q \leqq n < x/\alpha - x^\xi} \frac{x^{\alpha n+\beta-r-1}}{\Gamma(\alpha n+\beta-r)}\right\}$$

$$= O\{x^r e^{-x^\eta}\} = o(1).$$

The proof of (ii) is similar.

Lemma 5 ([13], Lemma 1). *If* $\{s_n\}$ *satisfies* (T_r), *then there exist positive constants* K, K' *such that, for* $m \geqq n \geqq 1$,

$$s_m - s_n > -Km^r(m^{1/2} - n^{1/2}) - K'n^r,$$

$$s_m - s_n \geqq -K(m^{r+1/2} - n^{r+1/2}) - K'n^r.$$

The next lemma is essentially due to Hyslop ([11], Lemma 1).

LEMMA 6. *Let* $h_n = n - x/\alpha$, $p \geqq 0$, *and* $1/2 < \xi < 1$, *then, as* $x \to \infty$,

(i) $$\sum_{n > x/\alpha + x^\xi} n^r h_n^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1),$$

(ii) $$\sum_{0 \leqq n < x/\alpha - x^\xi} n^r |h_n|^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1),$$

(iii) $$\sum_{n=0}^{\infty} n^r |h_n|^p \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = O\{x^{r+(p+1)/2}\}.$$

LEMMA 7 (cf. [14], Lemma 5 and [3], Theorem 5). *Let M and N be any positive integers such that*

$$M > x/\alpha + t\sqrt{x/\alpha},\ q < N < x/\alpha - t\sqrt{x/\alpha}.$$

Then, as t, x $\to \infty$,

(i) $$\sum_{n=q}^{N} n^r c_n(x) = o(x^r),$$

(ii) $$\sum_{n=M}^{\infty} n^r c_n(x) = o(x^r),$$

(iii) $$\sum_{n=N}^{M} n^r c_n(x) \sim (x/\alpha)^r,$$

(iv) $$\sum_{n=M}^{\infty} (n^{r+1/2} - M^{r+1/2}) c_n(x) = o(x^r).$$

(The precise meaning of part (iii), for example, is that for every $\epsilon > 0$ there is a X_0 such that

$$\left| x^{-r} \sum_{n=N}^{M} n^r c_n(x) - \alpha^{-r} \right| < \epsilon \quad \text{whenever } x > X_0,\ t > X_0,$$

$q < N < x/\alpha - t\sqrt{x/\alpha}$, and $M > x/\alpha + t\sqrt{x/\alpha}$. The meanings of the other parts are similar.)

Proof. Part (i). For $1/2 < \xi < 2/3$ we have

$$0 \leqq S := \sum_{n=q}^{N} c_n(x) \leqq \sum_{q \leqq n \leqq x/\alpha - t\sqrt{x/\alpha}} c_n(x)$$
$$= \left(\sum_{q \leqq n \leqq x/\alpha - x^\xi} + \sum_{x/\alpha - x^\xi < n \leqq x/\alpha - t\sqrt{x/\alpha}} \right) c_n(x)$$
$$=: S_1 + S_2.$$

By Lemma 4 (i), we have $S_1 = o(1)$ as $x \to \infty$. Further, by Lemma 1 (ii), as t, $x \to \infty$

$$S_2 = O\left\{ x^{-1/2} \sum_{x/\alpha - x^\xi < n \leqq x/\alpha - t\sqrt{x/\alpha}} \exp\left(-\frac{\alpha^2 (x/\alpha - n)^2}{2x}\right) \right\}$$
$$= o(1) + O\left\{ x^{-1/2} \int_{t\sqrt{x/\alpha}}^{x^\xi} \exp\left(-\frac{\alpha^2 y^2}{2x}\right) dy \right\}$$
$$= o(1) + O\left\{ \int_{t\sqrt{\alpha/2}}^{\infty} \exp(-u^2)\, du \right\}$$
$$= o(1).$$

It follows that, as t, $x \to \infty$, $S = o(1)$, and hence

$$0 \leqq \sum_{n=q}^{N} n^r c_n(x) \leqq (x/\alpha)^r \sum_{n=q}^{N} c_n(x) = o(x^r).$$

Part (ii). For $1/2 < \xi < 2/3$, we have

$$S := x^{-r} \sum_{n=M}^{\infty} n^r c_n(x)$$
$$= x^{-r} \left\{ \sum_{M \leqq n \leqq x/\alpha + x^\xi} + \sum_{n > x/\alpha + x^\xi} \right\} n^r c_n(x)$$
$$=: S_1 + S_2.$$

By Lemma 4 (ii), we have $S_2 = o(1)$ as $x \to \infty$. Furthermore, it follows from Lemmas 3 and 1 (ii) that

$$S_1 = O\left\{ x^{-r} \sum_{x/\alpha + t\sqrt{x/\alpha} < n \leqq x/\alpha + x^\xi} n^r c_n(x) \right\}$$
$$= O\left\{ e^{-x} \sum_{x/\alpha + t\sqrt{x/\alpha} < n \leqq x/\alpha + x^\xi} \frac{x^{\alpha n + \beta - r - 1}}{\Gamma(\alpha n + \beta - r)} \right\}$$
$$= O\left\{ x^{-1/2} \sum_{x/\alpha + t\sqrt{x/\alpha} < n \leqq x/\alpha + x^\xi} \exp\left(-\frac{\alpha^2 (n - x/\alpha)^2}{2x}\right) \right\}.$$

Now exactly as in the proof of part (i) we find that, as t, $x \to \infty$, $S_1 = o(1)$. The conclusion is now immediate.

Part (iii). The case $r = 0$ follows from parts (i) and (ii) with $r = 0$ and the known result that

$$\sum_{n=q}^{\infty} c_n(x) \to 1 \quad \text{as } x \to \infty$$

(see [5], p. 130).

To prove the result for $r > 0$, observe that it is equivalent to proving the following assertion:

$$\sum_{n=N_i}^{M_i} n^r c_n(x) \sim (x_i/\alpha)^r \quad \text{as } i \to \infty,$$

whenever $\{M_i\}$, $\{N_i\}$, $\{t_i\}$, $\{x_i\}$ are sequences such that $t_i \to \infty$, $x_i \to \infty$, and

$$M_i > x_i/\alpha + t_i\sqrt{x_i/\alpha}, \qquad q < N_i < x_i/\alpha - t_i\sqrt{x_i/\alpha}.$$

Suppose therefore that $\{M_i\}$, $\{N_i\}$, $\{t_i\}$, $\{x_i\}$ are sequences satisfying the above conditions, and let

$$w_i = \min\{(x_i)^{1/4}, t_i\}$$

so that

$$0 \leqq w_i \leqq t_i, \quad w_i \to \infty, \quad \text{and} \quad w_i/\sqrt{x_i} \to 0.$$

Now choose sequences of positive integers $\{M_i'\}$, $\{N_i'\}$ such that

$$M_i' - 1 \leqq x_i/\alpha + w_i\sqrt{x_i/\alpha} < M_i' \leqq M_i,$$
$$N_i \leqq N_i' < x_i/\alpha - w_i\sqrt{x_i/\alpha} \leqq N_i' + 1.$$

Then

$$\sum_{n=N_i}^{M_i} n^r c_n(x_i) = \left(\sum_{n=N_i}^{N_i'-1} + \sum_{n=N_i'}^{M_i'} + \sum_{n=M_i'+1}^{M_i}\right) n^r c_n(x_i). \tag{2.1}$$

(The first series on the right side of (2.1) is defined to be zero if $N_i' = N_i$ as is the last series if $M_i' = M_i$.)

Since

$$(N_i')^r \sum_{n=N_i'}^{M_i'} c_n(x_i) \leqq \sum_{n=N_i'}^{M_i'} n^r c_n(x_i) \leqq (M_i')^r \sum_{n=N_i'}^{M_i'} c_n(x_i)$$

and

$$(N_i')^r \sim (x_i/\alpha)^r, \; (M_i')^r \sim (x_i/\alpha)^r \quad \text{as } i \to \infty,$$

if follows that

$$\sum_{n=N_i'}^{M_i'} n^r c_n(x_i) \sim (x_i/\alpha)^r \sum_{n=N_i'}^{M_i'} c_n(x_i)$$
$$= (x_i/\alpha)^r \left(\sum_{n=q}^{\infty} - \sum_{n=q}^{N_i'-1} - \sum_{n=M_i'+1}^{\infty}\right) c_n(x_i) \quad \text{as } i \to \infty.$$

Since

$$\sum_{n=q}^{\infty} c_n(x_i) \to 1 \quad \text{as } i \to \infty$$

we have, by parts (i) and (ii) with $r = 0$, that

$$\sum_{n=N_i'}^{M_i'} n^r c_n(x_i) \sim (x_i/\alpha)^r \quad \text{as } i \to \infty. \tag{2.2}$$

Further, from (2.1), (2.2), and parts (i) and (ii), we obtain

$$\sum_{n=N_i}^{M_i} n^r c_n(x_i) \sim (x_i/\alpha)^r \quad \text{as } i \to \infty,$$

as required.

Part (iv). An application of the mean value theorem shows that in order to prove the desired result it suffices to show that

$$S := x^{-r} \sum_{n=M}^{\infty} (\sqrt{n} - \sqrt{M}) n^r c_n(x) = o(1) \quad \text{as } t, x \to \infty.$$

To prove this observe that since $M > x/\alpha$ we have

$$\sqrt{\alpha/x}(n - M)/2 \geqq \sqrt{n} - \sqrt{M}$$

and hence

$$0 \leqq S \leqq \sqrt{\alpha/2}x^{-r-1/2} \sum_{n=M}^{\infty} (n - M) n^r c_n(x)$$
$$= \sqrt{\alpha/2}x^{-r-1/2} \left\{ \sum_{M \leqq n \leqq x/\alpha + x^{\xi}} + \sum_{n > x/\alpha + x^{\xi} (\geqq M)} \right\} (n - M) n^r c_n(x)$$
$$=: S_1 + S_2$$

where $1/2 < \xi < 2/3$.

Since

$$M > x/\alpha + t\sqrt{x/\alpha} \quad \text{and} \quad n - M < n - x/\alpha,$$

it follows from Lemmas 3 and 1 (ii) that

$$S_1 = O\left\{ x^{-1/2} e^{-x} \sum_{x/\alpha + t\sqrt{x/\alpha} \leqq n \leqq x/\alpha + x^{\xi}} (n - x/\alpha) n^r \frac{x^{\alpha n + \beta - r - 1}}{\Gamma(\alpha n + \beta)} \right\}$$

$$= O\left\{x^{-1/2}e^{-x} \sum_{x/\alpha+t\sqrt{x/\alpha} \leqq n \leqq x/\alpha+x^{\xi}} (n - x/\alpha)\frac{x^{\alpha n+\beta-r-1}}{\Gamma(\alpha n + \beta - r)}\right\}$$

$$= O\left\{x^{-1} \sum_{x/\alpha+t\sqrt{x/\alpha} \leqq n \leqq x/\alpha+x^{\xi}} (n - x/\alpha)\exp\left(-\frac{\alpha^2(n - x/\alpha)^2}{2x}\right)\right\}$$

$$= o(1) + O\left\{x^{-1}\int_{t\sqrt{x/\alpha}}^{\infty} y \exp\left(-\frac{\alpha^2 y^2}{2x}\right)dy\right\}$$

$$= o(1) + O\left\{\int_{t\sqrt{\alpha/2}}^{\infty} u \exp(-u^2)du\right\} = o(1) \text{ as } t, x \to \infty.$$

Next, by Lemmas 3 and 1 (i), we have that, as $x \to \infty$,

$$S_2 = O\left\{x^{1/2}e^{-x} \sum_{n>x/\alpha+x^{\xi}} n^{r+1}\frac{x^{\alpha n+\beta-r-2}}{\Gamma(\alpha n + \beta)}\right\}$$

$$= O\left\{x^{1/2}e^{-x} \sum_{n>x/\alpha+x^{\xi}} \frac{x^{\alpha n+\beta-r-2}}{\Gamma(\alpha n + \beta - r - 1)}\right\}$$

$$= O\{x^{1/2}e^{-x^{\eta}}\} = o(1).$$

If follows that $S = o(1)$ as $t, x \to \infty$.

THEOREM 2. *Suppose that* $\{s_n\}$ *is a sequence such that* (T_r) *holds and*

$$b(x) = O(x^r) \quad \text{as } x \to \infty.$$

Then $s_n = O(n^r)$.

Proof. Following Sitaraman ([**14**], proof of Theorem 1) define

$$\sigma_n := n^{-r}s_n,\ \sigma_1(n) := \max_{\nu \leqq n} \sigma_\nu, \text{ and } \sigma_2(n) := \max_{\nu \leqq n} (-\sigma_\nu).$$

We assume that $\{\sigma_n\}$ is unbounded and show that this leads to a contradiction.

There are two logical possibilities:

Case (A). $\sigma_1(n) \geqq \sigma_2(n)$ for infinitely many values of n.

Case (B). $\sigma_1(n) < \sigma_2(n)$ for all n sufficiently large.

First, suppose that Case (A) holds. Then in view of our assumption we conclude that $\sigma_1(n) \to \infty$. Now write

$$(2.3) \quad b(x) = \left(\sum_{n=q}^{N-1} + \sum_{n=N}^{M-1} + \sum_{n=M}^{\infty}\right)c_n(x)s_n$$

$$=: T_1(x) + T_2(x) + T_3(x)$$

where first N and then M are chosen as follows. Corresponding to any positive $H > \sigma_1(q)$ there exist integers $N = N(H) > q$ such that

$$(2.4) \quad \sigma_N = \sigma_1(N) > 2H, \quad \sigma_1(N) \geqq \sigma_2(N).$$

Take the least value of N and then the least $M = M(H) > N$ such that

$$(2.5) \quad \sigma_M \leqq \frac{1}{2}\sigma_N.$$

There are such M's when H is large, for otherwise $\sigma_n \to \infty$, and then Lemma 3 and the total regularity of the $(B, \alpha, \beta - r)$ method ([**9**], Theorem 9) would imply that

$$x^{-r}b(x) \to \infty \quad \text{as } x \to \infty,$$

contradicting the hypothesis $b(x) = O(x^r)$.

In view of Lemma 5, and the choice of M and N in (2.4) and (2.5), we have that

$$K(M^{1/2} - N^{1/2}) > \sigma_1(N)\left\{\left(\frac{N}{M}\right)^r - \frac{1}{2}\right\} - K',$$

where K and K' are positive constants (cf. [**14**], proof of Theorem 1). Now we have either

$$\left(\frac{N}{M}\right)^r > \frac{3}{4} \quad \text{or} \quad \left(\frac{M}{N}\right)^r \geqq \frac{4}{3}.$$

In the first case,

$$K(M^{1/2} - N^{1/2}) > \frac{1}{4}\sigma_1(N) - K',$$

while in the second case

$$M^{1/2} - N^{1/2} \geqq N^{1/2}\left\{\left(\frac{4}{3}\right)^{1/(2r)} - 1\right\}.$$

Hence

$$(2.6) \quad t := t(H) = \frac{1}{2}(M^{1/2} - N^{1/2}) \to \infty \quad \text{as } N \to \infty \text{ (or } H \to \infty).$$

Next, let

$$(2.7) \quad x := x(H) = \frac{\alpha}{4}(M^{1/2} + N^{1/2})^2$$

so that $x \to \infty$ as $H \to \infty$, since $M > N \to \infty$ as $H \to \infty$. It follows from (2.6) and (2.7) that

$$\text{(2.8)} \quad \begin{cases} M > x/\alpha + t\sqrt{x/\alpha}, \\ q < N < x/\alpha - t\sqrt{x/\alpha}, \end{cases}$$

where t, $x \to \infty$ as $H \to \infty$.

In the analysis which follows, suppose that N, M and x are chosen as in (2.4), (2.5) and (2.7) and consequently satisfy (2.8). Therefore t, $x \to \infty$ as $H \to \infty$ and the properties (i), (ii), (iii) and (iv) of Lemma 7 hold. With reference to (2.3), we see, that as $H \to \infty$

$$\text{(2.9)} \quad T_1(x) \geqq -\sigma_2(N) \sum_{n=q}^{N-1} n^r c_n(x)$$

$$\geqq -\sigma_1(N) \sum_{n=q}^{N} n^r c_n(x) = -\sigma_1(N) o(1),$$

by Lemma 7 (i). Further, since M is the least integer greater than N which satisfies (2.5), we have

$$\text{(2.10)} \quad \sigma_n > \frac{1}{2}\sigma_N = \frac{1}{2}\sigma_1(N) \quad \text{for } N \leqq n \leqq M-1.$$

Thus, as $H \to \infty$,

$$\text{(2.11)} \quad T_2(x) > \frac{1}{2}\sigma_1(N) \sum_{n=N}^{M-1} n^r c_n(x) \sim \frac{1}{2}\sigma_1(N)(x/\alpha)^r,$$

by Lemma 7 (iii).

Next, by Lemma 5, there are positive constants K and K' such that

$$s_n - s_{M-1} \geqq -K(n^{r+1/2} - (M-1)^{r+1/2}) - K'(M-1)^r$$

for $n \geqq M$. Thus

$$\text{(2.12)} \quad s_n > s_{M-1} - K(n^{r+1/2} - (M-1)^{r+1/2}) - O(M^{r-1/2}) - K'M^r$$
$$> -K(n^{r+1/2} - M^{r+1/2}) - O(M^r)$$

for $n \geqq M$, since, by (2.10) and (2.4),

$$s_{M-1} = \sigma_{M-1}(M-1)^r > \frac{1}{2}\sigma_N(M-1)^r > H(M-1)^r > 0.$$

By (2.12) and Lemma 7 (ii) and (iv), we have

$$\text{(2.13)} \quad T_3(x) \geqq -K \sum_{n=M}^{\infty} (n^{r+1/2} - M^{r+1/2})c_n(x) - O(1) \sum_{n=M}^{\infty} n^r c_n(x)$$

$$\geqq -o(x^r) \quad \text{as } H \to \infty.$$

Substituting (2.9), (2.11) and (2.13) in (2.3), we get

$$x^{-r}b(x) \geqq \sigma_1(N)\left(\frac{1}{2}\alpha^{-r} - o(1)\right) - o(1) \to \infty \quad \text{as } H \to \infty,$$

since $\sigma_1(N) \to \infty$ as $N \to \infty$ (or $H \to \infty$). This implies that $x^{-r}b(x)$ is unbounded above, contradicting the hypothesis $b(x) = O(x^r)$.

Next, suppose that Case (B) holds (i.e., there exists an M_0 such that $\sigma_2(n) > \sigma_1(n)$ for $n \geqq M_0$). Then in view of our underlying assumption we have $\sigma_2(n) \to \infty$. Now write

$$\text{(2.14)} \quad b(x) = \left(\sum_{n=q}^{N} + \sum_{n=N+1}^{M} + \sum_{n=M+1}^{\infty}\right) c_n(x) s_n$$

$$=: T_1(x) + T_2(x) + T_3(x)$$

where first M and then N are chosen as follows. Corresponding to any positive $H > \sigma_2(M_0)$ choose the least $M = M(H)$ such that

$$\text{(2.15)} \quad \sigma_2(n) > \sigma_1(n) \quad \text{for } n \geqq M, \ \sigma_M = -\sigma_2(M) < -2H.$$

Then choose the largest $N = N(H) \in (q, M)$ for which

$$\text{(2.16)} \quad \sigma_N \geqq \frac{1}{2}\sigma_M = -\frac{1}{2}\sigma_2(M).$$

There are such N's when H is large, for otherwise $\sigma_n \to -\infty$ and then Lemma 3 and the total regularity of the $(B, \alpha, \beta - r)$ method would imply that

$$x^{-r}b(x) \to -\infty \quad \text{as } x \to \infty,$$

contradicting the hypothesis $b(x) = O(x^r)$.

The choice of M and N in (2.14) and (2.15), and Lemma 5 imply that there are positive constants K, K' for which

$$K(M^{1/2} - N^{1/2}) \geqq \sigma_2(M)\left\{1 - \frac{1}{2}\left(\frac{N}{M}\right)^r\right\} - K'\left(\frac{N}{M}\right)^r$$

$$\geqq \frac{1}{2}\sigma_2(M) - K' \to \infty$$

as $H \to \infty$ (cf. [**14**], proof of Theorem 1). Hence defining $t = t(H)$ and $x = x(H)$ as in (2.6) and (2.7) we see that t, $x \to \infty$ as $H \to \infty$, and that (2.8) holds. Consequently, as $H \to \infty$, the properties (i), (ii), (iii) and (iv) of Lemma 7 hold. The rest of the proof of Case (B) is exactly as given in ([**14**], case (ii) of Theorem 1) with the roles of N and M interchanged. This rules out the possibility of Case (B) holding.

LEMMA 8. (cf. [8], Hilfssatz 5). *Suppose* $h_n = n - x/\alpha$, $0 < H < 1$, $(1 - H)x/\alpha \leqq n \leqq (1 + H)x/\alpha$, *and* k *is any integer* $\geqq 2$. *Then, as* $x \to \infty$

$$c_n(x) = \frac{\alpha}{\sqrt{2\pi x}} \exp\left(-\frac{\alpha^2 h_n^2}{2x} + g_k + R_k\right),$$

where

$$R_k = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\}, \quad g_k = \sum_{i=1}^{k}\sum_{j=0}^{i+1} b_{i,j}\frac{h_n^j}{x^i},$$

and the $b_{i,j}$*'s are constants with* $b_{1,2} = b_{k,k+1} = 0$.

(Note: In particular, the result is true for all n such that $|h_n| \leqq x^{\xi}$, $1/2 < \xi < 2/3$.)

Proof. Since

$$\alpha n = \alpha h_n + x \quad \text{and} \quad 0 < 1 - H \leqq \frac{\alpha h_n}{x} + 1 \leqq 1 + H$$

if follows from a form of Stirling's formula ([1], p. 48, equation 12) that, as $x \to \infty$,

$$\begin{aligned}(2.17)\quad &\log \Gamma(\alpha n + \beta)\\ &= (\alpha h_n + x + \beta - 1/2)\log x - \alpha h_n - x + (1/2)\log 2\pi\\ &+ (\alpha h_n + x + \beta - 1/2)\log\left(\frac{\alpha h_n}{x} + 1\right)\\ &+ \sum_{r=1}^{k}\frac{(-1)^{r+1}B_{r+1}(\beta)}{r(r+1)x^r}\left(\frac{\alpha h_n}{x} + 1\right)^{-r} + O\left(\frac{1}{x^{k+1}}\right),\end{aligned}$$

where $k \geqq 1$ and each $B_{r+1}(\beta)$ is a Bernoulli polynomial. Since

$$\left|\frac{\alpha h_n}{x}\right| \leqq H < 1$$

we have

$$(2.18)\quad \left(\frac{\alpha h_n}{x} + 1\right)^{-r} = \sum_{j=0}^{k-r}\binom{-r}{j}\left(\frac{\alpha h_n}{x}\right)^j + O\left\{\left(\frac{|h_n|}{x}\right)^{k-r+1}\right\},$$

and

$$(2.19)\quad \log\left(\frac{\alpha h_n}{x} + 1\right) = \sum_{j=1}^{k}\frac{(-1)^{j-1}}{j}\left(\frac{\alpha h_n}{x}\right)^j + O\left\{\left(\frac{|h_n|}{x}\right)^{k+1}\right\}.$$

It follows from (2.18) that

$$\begin{aligned}(2.20)\quad &\sum_{r=1}^{k}\frac{(-1)^{r+1}B_{r+1}(\beta)}{r(r+1)x^r}\left(\frac{\alpha h_n}{x} + 1\right)^{-r}\\ &= \sum_{r=1}^{k}\sum_{j=0}^{k-r} d_{r,j}\frac{h_n^j}{x^{r+j}} + \sum_{r=1}^{k}\frac{1}{x^r}O\left\{\left(\frac{|h_n|}{x}\right)^{k-r+1}\right\}\\ &= \sum_{i=1}^{k}\sum_{j=0}^{i-1} d_{i-j,j}\frac{h_n^j}{x^i} + O\left\{\frac{|h_n|^{k+1} + 1}{x^{k+1}}\right\},\end{aligned}$$

where the $d_{r,j}$'s are constants.

If we denote the double sum on the right side of (2.20) by t_k and then substitute (2.19) and (2.20) in (2.16) we obtain, after some simplification,

$$\begin{aligned}(2.21)\quad &\log c_n(x)\\ &= \log\alpha - x + (\alpha h_n + x + \beta - 1)\log x - \log\Gamma(\alpha n + \beta)\\ &= \log\frac{\alpha}{\sqrt{2\pi x}} + \alpha h_n\\ &+ (\alpha h_n + x + \beta - 1/2)\sum_{j=1}^{k}\frac{(-1)^j}{j}\left(\frac{\alpha h_n}{x}\right)^j - t_k\\ &+ O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\} \quad \text{as } x \to \infty.\end{aligned}$$

We now combine the O-term with the term

$$\frac{(-1)^k(\alpha h_n)^{k+1}}{kx^k}$$

on the right side of (2.20) into R_k to get, after a further simplification,

$$\log c_n(x) = \log\frac{\alpha}{\sqrt{2\pi x}} - \frac{\alpha^2 h_n^2}{2x} + g_k + R_k,$$

where

$$R_k = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\} \quad \text{and}$$

$$g_k = \sum_{i=1}^{k}\sum_{j=0}^{i+1} b_{i,j}\frac{h_n^j}{x^i}$$

with $b_{1,2} = b_{k,k+1} = 0$.

3. An equivalence theorem.

LEMMA 9 ([**11**], Lemma 3 or [**8**], Hilfssatz 3). *Suppose that* $s_n = O(n^r)$, *and that*

$$\sum_{n=0}^{\infty} s_n \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(x^{1/2+b})$$

as $x \to \infty$ *where* $b \geqq 0$. *Then, for each integer* $j \geqq 0$ *and each* $\epsilon > 0$,

$$\sum_{n=0}^{\infty} s_n(n-x)^j \exp\left\{-\frac{\alpha(n-x)^2}{2x}\right\} = o(x^{(j+1)/2+b+\epsilon})$$

as $x \to \infty$.

LEMMA 10 ([**11**], Theorem 2 or [**8**], Hilfssatz 4 with $q = 0$). *Suppose that* $s_n = O(n^r)$, $h_n = n - x/\alpha$, *and that*

$$\sum_{k=0}^{\infty} s_k \exp\left\{-\frac{\alpha(n-k)^2}{2n}\right\} = o(n^{1/2}) \quad \text{as } n \to \infty.$$

Then

$$\sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2}) \quad \text{as } x \to \infty.$$

THEOREM 3. (cf. [**11**], Theorems 3 and 6). *Suppose that* $s_n = O(n^r)$. *Then* $s_n \to l(B, \alpha, \beta)$ *if and only if* $s_n \to l(V_\alpha)$.

Proof. Let

$$1/2 < \xi < 2/3, \quad h_n = n - x/\alpha,$$

$$\bar{b}(x) := \sum_{|h_n| \leqq x^\xi} c_n(x) s_n \quad \text{and}$$

$$\bar{t}(x) := \frac{\alpha}{\sqrt{2\pi x}} \sum_{|h_n| \leqq x^\xi} \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) s_n.$$

We first prove that $s_n \to l(V_\alpha)$ implies $s_n \to l(B, \alpha, \beta)$. Because of the regularity of both methods it suffices to prove this result for $l = 0$. Suppose therefore that $s_n \to 0(V_\alpha)$. In order to show that $s_n \to 0(B, \alpha, \beta)$ it is enough, by Lemma 4, to prove that $\bar{b}(x) = o(1)$ as $x \to \infty$. By Lemma 8, for x sufficiently large and an integer $k > 2r + 1$, we have

$$(3.1) \quad \bar{b}(x) - \bar{t}(x)$$

$$= \sum_{|h_n| \leqq x^\xi} s_n \left\{c_n(x) - \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\}$$

$$= \frac{\alpha}{\sqrt{2\pi x}} \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{\infty} \frac{(g_k + R_k)^\mu}{\mu!}$$

$$= \frac{\alpha}{\sqrt{2\pi x}} (A_1(x) + A_2(x) + A_3(x)),$$

where

$$(3.2) \quad A_1(x) := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{g_k^\mu}{\mu!},$$

$$(3.3) \quad A_2(x) := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=1}^{2s} \frac{(g_k + R_k)^\mu - g_k^\mu}{\mu!},$$

$$(3.4) \quad A_3(x) := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=2s+1}^{\infty} \frac{(g_k + R_k)^\mu}{\mu!},$$

and the integer $s > r - 1/2$.

We proceed to show that each of the above is $o(x^{1/2})$ as $x \to \infty$.

To see that $A_1(x) = o(x^{1/2})$ as $x \to \infty$ consider, for $1 \leqq \mu \leqq 2s$,

$$v(x) := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{g_k^\mu}{\mu!}.$$

The expansion of g_k given in Lemma 8 shows that g_k^μ is a finite combination of terms of the form $x^{-i} h_n^j$, where (i) $0 \leqq j \leqq \mu$ for $i = \mu$ and (ii) $0 \leqq j \leqq i + \mu$ for $i \geqq \mu + 1$. Hence, if we can show that

$$v_{i,j} := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{h_n^j}{x^i}$$

$$= o(x^{1/2}) \quad \text{as } x \to \infty$$

for the i's and j's in (i) and (ii) it will follow that

$$v(x) = o(x^{1/2})$$

and hence that

$$A_1(x) = o(x^{1/2}) \quad \text{as } x \to \infty.$$

Now our hypotheses together with Lemma 10, and Lemma 9 with $b = 0$, $\epsilon = 1/4$, imply that, for each integer $j \geqq 0$,

$$\sum_{n=0}^{\infty} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{(j+1)/2+1/4}) \quad \text{as } x \to \infty.$$

An application of Lemma 6 shows that, for each integer $j \geqq 0$,

$$v_{i,j} = o(x^{-i+j/2+3/4}) \quad \text{as } x \to \infty.$$

From this it is clear that, in both cases (i) and (ii), $v_{i,j} = o(x^{1/2})$ and hence that

$$A_1(x) = o(x^{1/2}) \quad \text{as } x \to \infty.$$

To prove that $A_2(x) = o(x^{1/2})$ as $x \to \infty$, it suffices to show that, for $1 \leqq \mu \leqq 2s$,

$$u(x) := \sum_{|h_n| \leqq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) |(g_k + R_k)^\mu - g_k^\mu|$$

$$= o(x^{1/2}) \quad \text{as } x \to \infty.$$

Since $k \geqq 2$, $1/2 < \xi < 2/3$, and $|h_n| \leqq x^\xi$ we have, by Lemma 8, that, as $x \to \infty$,

$$R_k = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\} = O(1) \quad \text{and} \quad g_k = O(1).$$

Hence,

$$|(g_k + R_k)^\mu - g_k^\mu| \leqq \sum_{j=1}^{\mu} \binom{\mu}{j} |R_k|^j |g_k|^{\mu-j}$$

$$= O(|R_k|) = O\left\{\frac{|h_n|^{k+1} + 1}{x^k}\right\},$$

and so,

$$u(x) = O\left\{x^{-k} \sum_{|h_n| \leqq x^\xi} n^r (1 + |h_n|^{k+1}) \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right)\right\}$$

as $x \to \infty$.

By Lemma 6, since $k > 2r+1$,

$$u(x) = O(x^{r-k+1/2}) + O(x^{r-k/2+1})$$

$$= o(x^{1/2}) \quad \text{as } x \to \infty.$$

Finally, to show that $A_3(x) = o(x^{1/2})$ as $x \to \infty$, we observe that, since $1/2 < \xi < 2/3$, and $|h_n| \leqq x^\xi$, we have, by Lemma 8,

$$(3.5) \qquad g_k + R_k = g_2 + R_2 = O\left\{\frac{|h_n| + 1}{x} + \frac{|h_n|^3}{x^2}\right\}.$$

In particular, $g_k + R_k = o(1)$ as $x \to \infty$ and hence

$$\sum_{\mu=2s+1}^{\infty} \frac{|g_k + R_k|^\mu}{\mu!} = O\left(|g_k + R_k|^{2s+1}\right) \text{ as } x \to \infty.$$

Thus, from this and (3.5), we obtain

$$A_3(x) = O\left\{\sum_{|h_n| \leqq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \sum_{\mu=2s+1}^{\infty} \frac{|g_k + R_k|^\mu}{\mu!}\right\}$$

$$= O\left\{\sum_{|h_n| \leqq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) |g_k + R_k|^{2s+1}\right\}$$

$$= O\left\{\sum_{|h_n| \leqq x^\xi} n^r \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \left(\frac{1 + |h_n|^{2s+1}}{x^{2s+1}} + \frac{|h_n|^{6s+3}}{x^{4s+2}}\right)\right\}.$$

Hence, by Lemma 6, since $s > r - 1/2$,

$$A_3(x) = O(x^{-2s+r-1/2}) + O(x^{-s+r}) + O(x^{-s+r})$$

$$= o(x^{1/2}) \quad \text{as } x \to \infty.$$

Consequently, it follows from (3.1) that

$$\bar{b}(x) - \bar{t}(x) = o(1) \quad \text{as } x \to \infty.$$

Next, by our hypotheses, Lemma 10, and Lemma 6 with $p = 0$, we have that $\bar{t}(x) = o(1)$ as $x \to \infty$. Therefore $\bar{b}(x) = o(1)$ as $x \to \infty$. This completes the proof of the first part of the theorem.

We now prove that $s_n \to l(B, \alpha, \beta)$ implies $s_n \to l(V_\alpha)$. Again it is enough to prove the result for $l = 0$ and we do this by following ([**8**], Satz II). Suppose that $s_n \to 0(B, \alpha, \beta)$. Then by Lemmas 4 and 8 we have

$$\frac{\alpha}{\sqrt{2\pi x}} \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x} + g_k + R_k\right)$$

$$= o(1) \quad \text{as } x \to \infty,$$

i.e.,

$$\bar{t}(x) + \frac{\alpha}{\sqrt{2\pi x}}(A_1(x) + A_2(x) + A_3(x))$$

$$= o(1) \quad \text{as } x \to \infty,$$

where A_1, A_2, A_3 are defined by (3.2), (3.3), (3.4) respectively with $k > 2r+1$ and the integer $s > r - 1/2$.

Observe that in the proof of the first part of the theorem we only required the hypothesis $s_n = O(n^r)$ to establish that A_2 and A_3 were $o(\sqrt{x})$. Since the hypothesis is still operative we now have

$$(3.6) \qquad \bar{t}(x) + \frac{\alpha}{\sqrt{2\pi x}} A_1(x) = o(1) \quad \text{as } x \to \infty.$$

Further, by Lemma 6 (iii) with $p = 0$, we have $\bar{t}(x) = O(x^r)$. Let

$$\gamma := \inf\{\delta : \bar{t}(x) = O(x^\delta)\}.$$

Then either $\gamma < 0$ or $0 \leqq \gamma \leqq r$. We wish to show that $\bar{t}(x) = o(1)$ as $x \to \infty$ in either case. This is evidently so when $\gamma < 0$. Suppose therefore that $0 \leqq \gamma \leqq r$. Consider $A_1(x)$, and for $1 \leqq \mu \leqq 2s$, let

$$p(x) := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{g_k^\mu}{\mu!},$$

where g_k^μ is a finite combination of terms of the form $x^{-i} h_n^j$ with (i) $0 \leqq j \leqq \mu$ for $i = \mu$ and (ii) $0 \leqq j \leqq i + \mu$ for $i \geqq \mu + 1$. For the i's and j's in (i) and (ii) let

$$p_{i,j} := \sum_{|h_n| \leqq x^\xi} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) \frac{h_n^j}{x^i}.$$

Since $\bar{t}(x) = o(x^{\gamma + 1/8})$ as $x \to \infty$ it follows, by Lemma 6 (i) and (ii) with $p = 0$, that

$$\sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{1/2 + \gamma + 1/8}) \quad \text{as } x \to \infty.$$

Next, it follows from Lemma 9 with $b = \gamma + 1/8$ and $\epsilon = 1/8$ that, for each integer $j \geqq 0$,

$$\sum_{n=0}^{\infty} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{j/2 + \gamma + 3/4}) \quad \text{as } x \to \infty.$$

Lemma 6 implies that, for each integer $j \geqq 0$,

$$\sum_{|h_n| \leqq x^\xi} s_n h_n^j \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(x^{j/2 + \gamma + 3/4}) \quad \text{as } x \to \infty.$$

Thus,

$$\begin{aligned} p_{i,j} &= o(x^{-i + j/2 + \gamma + 3/4}) \\ &= o(x^{\gamma + 1/4}) \quad \text{as } x \to \infty, \end{aligned}$$

in both cases (i) and (ii). It follows that

$$A_1(x) = o(x^{\gamma + 1/4}) \quad \text{as } x \to \infty,$$

and hence, by (3.6), that

$$\bar{t}(x) = o(x^{\gamma - 1/4}) + o(1) \quad \text{as } x \to \infty.$$

Now if $\gamma > 1/4$, then

$$\bar{t}(x) = o(x^{\gamma - 1/4}),$$

and this contradicts the definition of γ. Hence $\gamma \leqq 1/4$ and so

$$\bar{t}(x) = o(1) \quad \text{as } x \to \infty.$$

If follows, by Lemma 6 (i) and (ii) with $p = 0$, that

$$\frac{\alpha}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} s_n \exp\left(-\frac{\alpha^2 h_n^2}{2x}\right) = o(1) \quad \text{as } x \to \infty,$$

so that $s_n \to l(V_\alpha)$.

4. Proof of theorem 1. The hypothesis $s_n \to l(B, \alpha, \beta)$ implies that $b(x) = O(x^r)$ as $x \to \infty$ and hence, by Theorem 2, that $s_n = O(n^r)$. Theorem 3 now shows that $s_n \to l(V_\alpha)$ while Lemma 2 shows that there is no loss in generality in making the restriction $0 < \alpha < 1$. It follows by a result due to Faulhaber [**8**] or Bingham [**2**] that $s_n \to l(S_{1-\alpha})$ and hence, by a result due to Sitaraman ([**14**], Theorem 2), that $s_n \to l(C_{2r})$.

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