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Tauberian and other theorems concerning Dirichlet's series with non-negative coefficients

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Abstract

The paper is concerned with properties of the Dirichlet series $a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$, where $\{\lambda_n\}$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geqslant 0$. One of the main Tauberian results proved is that if $a_1 > 0$, $a_n \geqslant 0$ for $n = 2, 3, \ldots$, $a(x) < \infty$ for all $x > 0$, $A_n := a_1 + a_2 + \ldots + a_n \to \infty$, $a_n \lambda_n = o((\lambda_{n+1} - \lambda_n) A_n)$, $a_n \lambda_n s_n \geqslant -H(\lambda_{n+1} - \lambda_n) A_n$ and $\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \sim s a(x)$ as $x \to 0+$, then $\sum_{k=1}^{n} a_k s_k \sim s A_n$. A new summability method $D_{\lambda,a}$ based on the Dirichlet series $a(x)$ is defined and its relationship to the weighted mean method M_a investigated.

1. *Introduction*

Suppose throughout that $\lambda := \{\lambda_n\}$ is a strictly increasing unbounded sequence of real numbers with $\lambda_1 \geqslant 0$, and that $a := \{a_n\}$ is a sequence of real numbers. Let

$$A_n := \sum_{k=1}^{n} a_k \quad \text{and} \quad a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}.$$

The same system of notation will be used with letters other than a, A. Except in §6 and §7, we shall suppose that

$$a_1 > 0 \quad \text{and} \quad a_n \geqslant 0 \quad \text{for} \quad n = 2, 3, \ldots,$$

and that the Dirichlet series $a(x)$ is convergent for all $x > 0$. Let $\{s_n\}$ be a sequence of real numbers,

$$t_n := \frac{1}{A_n} \sum_{k=1}^{n} a_k s_k \quad \text{and} \quad \sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x}.$$

We shall be concerned with the weighted mean summability method M_a and the Dirichlet series method $D_{\lambda,a}$, the latter method being new. These methods are defined as follows:

$s_n \to s(M_a)$ if $t_n \to s$;

$s_n \to s(D_{\lambda,a})$ if $\sigma(x)$ exists for all $x > 0$ and $\sigma(x) \to s$ as $x \to 0+$.

When $\lambda_n := n$ the method $D_{\lambda,a}$ reduces to the power series method J_a (as defined in [1] for example). It is familiar that the method M_a is regular (i.e. $s_n \to s(M_a)$ whenever $s_n \to s$) if and only if $A_n \to \infty$ (see [5], theorems 2 and 12), and, since

$a(x) \to \infty$ as $x \to 0+$ if and only if $A_n \to \infty$ (see [5], theorem 27), it is easily proved that $D_{\lambda,a}$ is also regular if and only if $A_n \to \infty$.

The primary purpose of this paper is to prove the following four theorems, the latter three being Tauberian in character:

THEOREM 1. *If* $A_n \to \infty$ *and* $s_n \to s(M_a)$, *then* $s_n \to s(D_{\lambda,a})$.

THEOREM 2. *Let* $s_n \to s(D_{\lambda,a})$, *let* $s_n > -H$ *for* $n = 1, 2, \ldots,$ *where H is a positive constant, and let* $a(x)$ *satisfy*

$$\lim_{x\to 0+} \frac{a(mx)}{a(x)} = \alpha_m > 0 \quad \textit{for} \quad m = 2 \quad \textit{and} \quad m = 3. \tag{1}$$

Then $s_n \to s(M_a)$.

THEOREM 3. *Suppose that* $\lambda_{n+1} \sim \lambda_n$, $A_n \to \infty$ *and*

$$a_n \lambda_n = o((\lambda_{n+1} - \lambda_n) A_n). \tag{2}$$

Suppose also that $s_n \to s(D_{\lambda,a})$ *and*

$$a_n \lambda_n s_n \geqslant -H(\lambda_{n+1} - \lambda_n) A_n, \tag{3}$$

where H is a positive constant. Then $s_n \to s(M_a)$.

THEOREM 4. *If* $a(x)$ *satisfies* (1), *then*

$$a(x) = x^{-\rho} L\left(\frac{1}{x}\right) \quad \textit{for} \quad x > 0,$$

where $\rho = -\log_2 \alpha_2 \geqslant 0$ *and* $L(x)$ *is a function (defined for* $x > 0$*) satisfying*

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \textit{for all} \quad t > 0,$$

and

$$A_n \sim \frac{1}{\Gamma(\rho+1)} a\left(\frac{1}{\lambda_n}\right) = \frac{\lambda_n^\rho}{\Gamma(\rho+1)} L(\lambda_n).$$

With regard to condition (1), it should be noted that the condition

$$\lim_{x\to 0+} \frac{a(2x)}{a(x)} = 1 \tag{4}$$

implies that

$$\lim_{x\to 0+} \frac{a(mx)}{a(x)} = 1 \quad \text{for} \quad m = 2, 3, \ldots.$$

The special case $\lambda_n := n$ of Theorem 1 is due to Ishiguro [6]; and the same case of Theorem 2 to Borwein and Meir [2], and of Theorem 3 to Borwein [1]. Theorem 4 generalizes a theorem due to Hardy and Littlewood ([3], theorem D), which has in place of condition (1) the stronger condition

$$a(x) \sim Ax^{-\rho} \quad \text{as} \quad x \to 0+,$$

with $A > 0$ and $\rho \geqslant 0$. Theorem 4 is in fact a corollary of Karamata's Tauberian theorem and a known result about regularly varying functions (see [9], theorems 2·3

and 1·8), and Theorem 2 can be deduced from Theorem 4. Our proofs of Theorems 2 and 4, however, are more direct and more elementary in that they do not involve the extended continuity theorem for Laplace–Stieltjes transforms on which the proof of Karamata's theorem is based. We indicate the scope of Theorem 3 by means of two examples at the end of §4. In §5 we express a slight extension of Theorem 2 in a different form. In §6 we show how to generate Dirichlet series $a(x)$ that satisfy (1). In §7 we show that Theorem 4 remains valid when the condition $a_n \geqslant 0$ is relaxed.

2. *Proof of Theorem* 1

Suppose that $x > 0$. Then, for $0 < \epsilon < x$,

$$0 \leqslant A_n e^{-\lambda_n x} \leqslant e^{-\lambda_n \epsilon} \sum_{k=1}^{n} a_k e^{-\lambda_k (x-\epsilon)} \leqslant e^{-\lambda_n \epsilon} a(x-\epsilon) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, by hypothesis, $t_n \to s$, and so

$$\begin{aligned} \sum_{k=1}^{n} a_k s_k e^{-\lambda_k x} &= \sum_{k=1}^{n} (A_k t_k - A_{k-1} t_{k-1}) e^{-\lambda_k x} \quad (A_0 t_0 := 0) \\ &= \sum_{k=1}^{n-1} A_k t_k (e^{-\lambda_k x} - e^{-\lambda_{k+1} x}) + t_n A_n e^{-\lambda_n x} \\ &\to \sum_{k=1}^{\infty} A_k t_k (e^{-\lambda_k x} - e^{-\lambda_{k+1} x}) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Consequently
$$\sigma(x) = \frac{1}{a(x)} \sum_{k=1}^{\infty} A_k t_k (e^{-\lambda_k x} - e^{-\lambda_{k+1} x}).$$

Since
$$\frac{1}{a(x)} \sum_{k=1}^{\infty} A_k (e^{-\lambda_k x} - e^{-\lambda_{k+1} x}) = 1,$$

and, for $k = 1, 2, \ldots,$

$$0 < \frac{1}{a(x)} A_k (e^{-\lambda_k x} - e^{-\lambda_{k+1} x}) \to 0 \quad \text{as} \quad x \to 0+,$$

it follows, by a standard result ([5], theorem 2), that

$$\sigma(x) \to s \quad \text{as} \quad x \to 0+. \quad ∎$$

3. *Proofs of Theorems* 2 *and* 4

We require the following known result ([9], theorem 1·8):

LEMMA 1. *If* $a(x)$ *satisfies* (1), *then*

$$\lim_{x\to 0+} \frac{a(tx)}{a(x)} = t^{-\rho} \quad \textit{for all} \quad t > 0,$$

where $\rho = -\log_2 \alpha_2 \geqslant 0$, *and*

$$a(x) = x^{-\rho} L\left(\frac{1}{x}\right) \quad \textit{for} \quad x > 0,$$

where
$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \textit{for all} \quad t > 0.$$

Proof of Theorem 2. (Cf. the proof of theorem 1 in [1].) It follows from (1), by Lemma 1, that

$$\lim_{x\to 0+}\frac{a(mx)}{a(x)} = m^{-\rho} \quad \text{for} \quad m = 1, 2, \ldots.$$

Further, for $m = 0, 1, \ldots$,

$$(m+1)^{-\rho} = \int_0^1 t^m \, d\chi(t),$$

where $$\chi(t) = \frac{1}{\Gamma(\rho)}\int_0^t (-\log u)^{\rho-1} du \quad \text{when} \quad \rho > 0,$$

and, when $\rho = 0$, $$\chi(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t < 1, \\ 1 & \text{if} \quad t = 1. \end{cases}$$

Suppose without loss of generality that $H = 0$, i.e. that $s_n \geqslant 0$ for $n = 1, 2, \ldots$. Define

$$\phi(x) := \begin{cases} \dfrac{1}{x} & \text{for} \quad c \leqslant x \leqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < c < 1$, and

$$\psi(x) := \frac{1}{a(x)}\sum_{k=1}^{\infty} a_k s_k e^{-\lambda_k x}\phi(e^{-\lambda_k x}).$$

Then, for $m = 0, 1, \ldots$,

$$\frac{1}{a(x)}\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} e^{-\lambda_n x m} = \sigma(x+xm)\frac{a(x+xm)}{a(x)}$$
$$\to s(m+1)^{-\rho} \quad \text{as} \quad x \to 0+;$$

and so, for any polynomial $p(t) = p_0 + p_1 t + p_2 t^2 + \ldots + p_m t^m$,

$$\frac{1}{a(x)}\sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} p(e^{-\lambda_n x}) \to s\sum_{k=0}^{m} p_k (k+1)^{-\rho} = s\int_0^1 p(t)\, d\chi(t) \quad \text{as} \quad x \to 0+.$$

Since χ is continuous at c it is readily demonstrated that, given $\epsilon > 0$, there are polynomials $p(t)$, $q(t)$ such that

$$p(t) \leqslant \phi(t) \leqslant q(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad \int_0^1 (q(t)-p(t))\, d\chi(t) < \epsilon.$$

It follows that

$$\lim_{x\to 0+}\psi(x) = s\int_0^1 \phi(t)\, d\chi(t) = s\int_c^1 t^{-1}\, d\chi(t) = \frac{s(-\log c)^\rho}{\Gamma(\rho+1)}.$$

Hence $$\psi\left(\frac{-\log c}{\lambda_n}\right) = \frac{1}{a\left(\dfrac{-\log c}{\lambda_n}\right)}\sum_{k=1}^{n} a_k s_k \to \frac{s(-\log c)^\rho}{\Gamma(\rho+1)}.$$

Taking $s_k = 1$ for $k = 1, 2, \ldots$, we obtain

$$\frac{A_n}{a\left(\dfrac{-\log c}{\lambda_n}\right)} \to \frac{(-\log c)^\rho}{\Gamma(\rho+1)}. \tag{5}$$

Consequently $$t_n = \frac{1}{A_n}\sum_{k=1}^{n} a_k s_k \to s. \quad \blacksquare$$

Note that the case in which (4) holds can be proved somewhat more simply along the lines of the proof of Case 1 of theorem 1 in [1].

Proof of Theorem 4. The theorem follows immediately from (5) with $c = 1/e$ and Lemma 1. ∎

4. *Proof of Theorem* 3

(Cf. the proof of theorem 2 in [1].) Let $x > 0$. By Cauchy's mean value theorem,

$$\frac{e^{-\lambda_n x} - e^{-2\lambda_n x}}{e^{-\lambda_n x} - e^{-\lambda_{n+1} x}} = \frac{1 - e^{-\lambda_n x}}{1 - e^{-(\lambda_{n+1}-\lambda_n)x}}$$
$$= \frac{\lambda_n}{\lambda_{n+1}-\lambda_n}\frac{e^{-\lambda_n \xi}}{e^{-(\lambda_{n+1}-\lambda_n)\xi}} \quad (0 < \xi < x)$$
$$= \frac{\lambda_n}{\lambda_{n+1}-\lambda_n} e^{(\lambda_{n+1}-2\lambda_n)\xi}$$
$$< \frac{\lambda_n}{\lambda_{n+1}-\lambda_n} \quad \text{for} \quad n \geqslant N, \tag{6}$$

where N is a sufficiently large positive integer. Since $a(x) \to \infty$ as $x \to 0+$, it follows from (2) and (6) that

$$0 < a(x) - a(2x) = \sum_{n=1}^{\infty} a_n (e^{-\lambda_n x} - e^{-2\lambda_n x}) = \sum_{n=1}^{N-1} + \sum_{n=N}^{\infty}$$
$$= o(a(x)) + \sum_{n=1}^{\infty} o(A_n)(e^{-\lambda_n x} - e^{-\lambda_{n+1} x})$$
$$= o(a(x)) \quad \text{as} \quad x \to 0+$$

as in the proof of Theorem 1. Hence

$$\lim_{x\to 0+}\frac{a(2x)}{a(x)} = 1,$$

and this implies that

$$\lim_{x\to 0+}\frac{a(tx)}{a(x)} = 1 \quad \text{for all} \quad t > 0. \tag{7}$$

Now let $\{\gamma_n\}$ be a sequence of positive numbers such that

$$a_n \lambda_n \gamma_n = H(\lambda_{n+1}-\lambda_n) A_n,$$

so that, by (3), $s_n+\gamma_n \geqslant 0$. Next, for $\phi(t)$ defined as in the proof of Theorem 2, we have

$$-\frac{c}{1-c}+\frac{t}{1-c} \leqslant \phi(t) \leqslant 1+\frac{1}{c}-\frac{t}{c} \quad \text{for} \quad 0 \leqslant t \leqslant 1,$$

and hence, by (6),

$$\psi(x) = \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n(s_n+\gamma_n) e^{-\lambda_n x}\phi(e^{-\lambda_n x}) - \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n \gamma_n e^{-\lambda_n x}\phi(e^{-\lambda_n x})$$

$$\leqslant \left(1+\frac{1}{c}\right)\sigma(x) - \frac{\sigma(2x)\,a(2x)}{ca(x)} + \frac{1}{c(1-c)\,a(x)} \sum_{n=1}^{\infty} a_n \gamma_n (e^{-\lambda_n x}-e^{-2\lambda_n x})$$

$$\leqslant \left(1+\frac{1}{c}\right)\sigma(x) - \frac{\sigma(2x)\,a(2x)}{ca(x)} + \frac{M}{a(x)}\left(\sum_{n=1}^{\infty} A_n(e^{-\lambda_n x}-e^{-\lambda_{n+1} x}) + \sum_{n=1}^{N-1} a_n \gamma_n\right),$$

where M is a positive constant. Therefore

$$\limsup_{x\to 0+} \psi(x) \leqslant \left(1+\frac{1}{c}\right)s - \frac{s}{c} + M = s+M < \infty.$$

Similarly

$$\liminf_{x\to 0+} \psi(x) > -\infty;$$

and hence $\psi(x) = O(1)$ as $x \to 0+$. It follows that

$$\psi\left(\frac{-\log c}{\lambda_n}\right) = \frac{1}{a\left(\dfrac{-\log c}{\lambda_n}\right)} \sum_{k=1}^{n} a_k s_k = O(1).$$

Further, since (5) is a consequence of (7),

$$A_n \sim a\left(\frac{-\log c}{\lambda_n}\right),$$

and therefore

$$t_n = \frac{1}{A_n} \sum_{k=1}^{n} a_k s_k = O(1). \tag{8}$$

Now let

$$b_n := A_n(\lambda_{n+1}-\lambda_n) \quad \text{for} \quad n = 1, 2, \ldots,$$

and

$$\tau(x) := \frac{1}{b(x)} \sum_{n=1}^{\infty} b_n t_n e^{-\lambda_n x}.$$

Then

$$xb(x) \geqslant \sum_{n=1}^{\infty} A_n x \int_{\lambda_n}^{\lambda_{n+1}} e^{-tx}\,dt = a(x).$$

Also, given $\gamma \in (0, 1)$, there is a positive integer r such that $\lambda_n > \gamma\lambda_{n+1}$ for $n > r$, and so

$$\gamma x b(x) \leqslant \sum_{n=1}^{\infty} A_n \gamma x \int_{\lambda_n}^{\lambda_{n+1}} e^{-t\gamma x}\,dt + \gamma x B_r = a(\gamma x) + \gamma x B_r.$$

Consequently, by (7),

$$1 \leqslant \liminf_{x\to 0+} \frac{xb(x)}{a(x)} \leqslant \limsup_{x\to 0+} \frac{xb(x)}{a(x)} \leqslant \lim_{x\to 0+} \frac{a(\gamma x)+\gamma x B_r}{\gamma a(x)} = \frac{1}{\gamma},$$

and therefore

$$xb(x) \sim a(x) \quad \text{as} \quad x \to 0+. \tag{9}$$

Further, $t_n + K > 0$ for $n = 1, 2, \ldots$ and some positive constant K, and, as in the proof of Theorem 1,

$$(\sigma(x)+K)\,a(x) = \sum_{n=1}^{\infty} (t_n+K) A_n(e^{-\lambda_n x}-e^{-\lambda_{n+1} x})$$

$$\sim (s+K)\,a(x) \quad \text{as} \quad x \to 0+.$$

Hence, as in the proof of (9),

$$x\tau(x)\,b(x) + Kxb(x) = x \sum_{n=1}^{\infty} (t_n+K) A_n(\lambda_{n+1}-\lambda_n) e^{-\lambda_n x}$$

$$\sim (s+K)\,a(x) \quad \text{as} \quad x \to 0+,$$

and so, by (9),

$$\tau(x) \to s \quad \text{as} \quad x \to 0+.$$

Since, by (7) and (9),

$$\lim_{x\to 0+} \frac{b(tx)}{b(x)} = \frac{1}{t} \quad \text{for all} \quad t > 0,$$

it follows, by (8) and Theorem 2, that

$$u_n := \frac{1}{B_n} \sum_{k=1}^{n} b_k t_k \to s. \tag{10}$$

Further, by (2), (3) and (8), we have that, for $n > 1$,

$$t_n - t_{n-1} = s_n \frac{a_n}{A_n} - t_{n-1}\frac{a_n}{A_n} \geqslant -\frac{\mu(\lambda_{n+1}-\lambda_n)}{\lambda_n}$$

for some positive constant μ. Thus, since $\lambda_{n+1} \sim \lambda_n$ and $\lambda_n \to \infty$,

$$t_m - t_n \geqslant -\mu \sum_{k=n+1}^{m} \frac{(\lambda_{k+1}-\lambda_k)}{\lambda_k} \sim -\mu \log \frac{\lambda_m}{\lambda_n} \quad \text{when} \quad m > n \to \infty$$

(see [7], p. 292), and so

$$\liminf\,(t_m - t_n) \geqslant 0 \quad \text{when} \quad m > n \to \infty \quad \text{and} \quad \frac{\lambda_m}{\lambda_n} \to 1. \tag{11}$$

Now, by (2),

$$\lambda_{n+1}A_n - \lambda_n A_{n-1} = b_n + \lambda_n a_n \sim b_n,$$

and hence, since $B_n \geqslant A_1(\lambda_{n+1}-\lambda_1) \to \infty$,

$$\lambda_{n+1}A_n \sim B_n.$$

It follows that, provided $\lambda_{m+1} > (1+\delta)\lambda_{n+1}, \delta > 0$,

$$\frac{B_m}{B_n} = 1 + \frac{1}{B_n}\sum_{k=n+1}^{m} A_k(\lambda_{k+1}-\lambda_k)$$

$$\geqslant 1 + \frac{A_n}{B_n}(\lambda_{m+1}-\lambda_{n+1}) \geqslant 1 + \delta\frac{A_n}{B_n}\lambda_{n+1} \to 1+\delta \quad \text{as} \quad n\to\infty. \tag{12}$$

Suppose now without loss of generality that $s = 0$, i.e. $u_n \to 0$. It follows from (11) that, given $\epsilon > 0$, there are positive numbers p, δ such that $t_m - t_n > -\epsilon$ when $m > n > p$ and $\lambda_{m+1} < (1+2\delta)\lambda_{n+1}$. Consequently if m, n satisfy these conditions we have, by (10), that

$$(t_n - \epsilon)\sum_{k=n+1}^{m} b_k \leqslant \sum_{k=n+1}^{m} t_k b_k = u_m B_m - u_n B_n \leqslant (t_m+\epsilon)\sum_{k=n+1}^{m} b_k,$$

and hence that

$$t_n - \epsilon \leqslant \frac{u_m B_m - u_n B_n}{B_m - B_n} = u_m + \frac{u_m - u_n}{(B_m/B_n)-1} \leqslant t_m + \epsilon. \tag{13}$$

Letting m, $n \to \infty$ subject to $1+\delta < \lambda_{m+1}/\lambda_{n+1} < 1+2\delta$, it follows from (12) that

$$\frac{1}{(B_m/B_n)-1} = O(1),$$

and hence from (13) that

$$\limsup t_n \leqslant \epsilon \quad \text{and} \quad \liminf t_m \geqslant -\epsilon.$$

Therefore $t_n \to 0$. ∎

Example 1. Since $\lambda_n := n$, $a_n := 1/n$ satisfy the conditions of Theorem 3, we get as a corollary of that theorem a result due to Kochanovski [8], namely if

$$\frac{1}{-\log(1-y)}\sum_{n=1}^{\infty}\frac{s_n}{n}y^n \to s \quad \text{as} \quad y \to 1- \quad \text{and} \quad s_n \geqslant -H\log n \quad \text{for} \quad n \geqslant 1,$$

then

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{s_k}{k} \to s.$$

Example 2. Let $\lambda_n := \log n$; $a_1 := 1$, $a_n := 1/(n\log n)$ $(n \geqslant 2)$. These sequences satisfy the conditions of Theorem 3. Further, for $x > 0$,

$$a'(x) = -\sum_{n=2}^{\infty}\frac{1}{n^{1+x}} = 1 - \zeta(1+x) \sim -\frac{1}{x} \quad \text{as} \quad x \to 0+,$$

by a known property of the Riemann zeta function or by Lemma 2 (below) with $\lambda_n := \log n$. Consequently, as $x \to 0+$,

$$a(x) = a(1) - \int_x^1 a'(t)\,dt \sim \int_x^1 \frac{1}{t}\,dt = -\log x.$$

Thus Theorem 3 yields the following result: if

$$\frac{1}{-\log x}\sum_{n=2}^{\infty}\frac{s_n}{n^{x+1}\log n} \to s \quad \text{as} \quad x \to 0+ \quad \text{and} \quad s_n \geqslant -H\log\log n \quad \text{for} \quad n \geqslant 3,$$

then

$$\frac{1}{\log\log n}\sum_{k=2}^{n}\frac{s_k}{k\log k} \to s.$$

5. *An alternative version of Theorem 2*

THEOREM 5. *Suppose that* $b_n \geqslant -Ha_n$ *for* $n = 1, 2, \ldots$ *where* H *is a positive constant and that the Dirichlet series* $b(x)$ *is convergent for all* $x > 0$. *If* $a(x)$ *satisfies* (1) *and* $b(x)/a(x) \to s$ *as* $x \to 0+$, *then* $B_n/A_n \to s$.

Proof. Since (see [5], theorem 27)

$$\lim_{n\to\infty} A_n = \lim_{x\to 0+} a(x),$$

the proof is immediate when $\{A_n\}$ is convergent. Suppose therefore that

$$A_n \to \infty.$$

Case 1. $a_n > 0$ for $n = 1, 2, \ldots$. This case follows immediately from Theorem 2.

Case 2. $a_n \geqslant 0$ for $n = 1, 2, \ldots$. Let

$$\epsilon(x) := \sum_{k=1}^{\infty}\epsilon_k e^{-\lambda_k x},$$

where $\epsilon_k > 0$ for $k = 1, 2, \ldots$, and $\sum_{k=1}^{\infty}\epsilon_k < \infty$. Then (see [5], theorem 27)

$$\epsilon(x) \to \epsilon(0) \quad \text{as} \quad x \to 0+.$$

Let

$$a^*(x) := a(x) + \epsilon(x), \quad b^*(x) = b(x) + \epsilon(x),$$

and define a_n^*, A_n^*, b_n^*, B_n^* in the obvious way. Then $a_n^* > 0$ and $b_n^* \geqslant -Ha_n^*$ for $n = 1, 2, \ldots$, and, since $a(x) \to \infty$ as $x \to 0+$, (1) is satisfied with $a^*(x)$ in place of $a(x)$. Further,

$$\frac{b^*(x)}{a^*(x)} \to s \quad \text{as} \quad x \to 0+ \quad \text{if and only if} \quad \frac{b(x)}{a(x)} \to s \quad \text{as} \quad x \to 0+,$$

and

$$\frac{A_n^*}{B_n^*} \to s \quad \text{if and only if} \quad \frac{B_n}{A_n} \to s.$$

Case 2 now follows from Case 1. ∎

In the rest of the paper we no longer presuppose that $a_n \geqslant 0$.

6. *Auxiliary results*

The following two lemmas show how to generate Dirichlet series $a(x)$ that satisfy (1). The conditions in the lemmas on $L(x)$ are satisfied when $L(x)$ is (for large x) a logarithmico-exponential function (see [4]) in the range

$$x^{-\delta} < L(x) < x^{\delta}.$$

Examples of such functions are given by

$$L(x) := (\log x)^{c_1} (\log\log x)^{c_2} \ldots,$$

where $c_1, c_2, \ldots$ are real numbers.

LEMMA 2. *Suppose that $\rho > \delta > 0$, $\epsilon > 0$, $c > 0$ and that $L(x)$ is a positive continuous function on $(0, \infty)$ such that $x^{\delta}L(x)$ is increasing and $x^{-\epsilon}L(x)$ is decreasing on (c, ∞), and*

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \textit{for all} \quad t > 0.$$

Suppose also that $\lambda_{n+1} \sim \lambda_n$ and

$$a_n \sim (\lambda_n - \lambda_{n-1}) \lambda_n^{\rho-1} L(\lambda_n).$$

Then

$$a(x) \sim \Gamma(\rho)\, x^{-\rho} L\left(\frac{1}{x}\right) \quad as \quad x \to 0+,$$

and

$$A_n \sim \frac{1}{\Gamma(\rho+1)} a\left(\frac{1}{\lambda_n}\right) \sim \frac{\lambda_n^{\rho}}{\rho} L(\lambda_n).$$

Proof. Suppose without loss of generality that $\lambda_1 > \lambda_0 = 0$, and let

$$b_n := (\lambda_n - \lambda_{n-1}) \lambda_n^{\rho-1} L(\lambda_n) \quad \text{for} \quad n = 1, 2, \ldots.$$

Given $\gamma > 1$, there is a positive integer N such that $\lambda_{N-1} > c$ and $\lambda_n < \gamma\lambda_{n-1}$ for $n \geqslant N$. Hence, for $n \geqslant N$ and $\lambda_{n-1} \leqslant t \leqslant \lambda_n$, we have $t \leqslant \lambda_n \leqslant \gamma t$ and so

$$t^{\delta} L(t)\, t^{\rho-\delta} (\gamma t)^{-1} e^{-\gamma t x} \leqslant \lambda_n^{\rho-1} L(\lambda_n)\, e^{-\lambda_n x} \leqslant (\gamma t)^{\delta} L(\gamma t)\, (\gamma t)^{\rho-\delta} t^{-1} e^{-tx}$$

for $x > 0$. It follows that

$$\gamma^{-\rho} \int_{\lambda_N}^{\infty} (\gamma t)^{\rho-1} L(t)\, e^{-\gamma t x}\, dt \leqslant \sum_{n=N+1}^{\infty} (\lambda_n - \lambda_{n-1}) \lambda_n^{\rho-1} L(\lambda_n)\, e^{-\lambda_n x}$$

$$\leqslant \gamma^{\rho} \int_{\lambda_N}^{\infty} t^{\rho-1} L(\gamma t)\, e^{-tx}\, dt,$$

and hence that

$$\gamma^{-\rho-1} \int_{\gamma x \lambda_N}^{\infty} u^{\rho-1} e^{-u} \frac{L\left(\frac{u}{\gamma x}\right)}{L\left(\frac{1}{x}\right)} du \leqslant \frac{x^{\rho}}{L\left(\frac{1}{x}\right)} \sum_{n=N+1}^{\infty} b_n e^{-\lambda_n x}$$

$$\leqslant \gamma^{\rho} \int_{x\lambda_N}^{\infty} u^{\rho-1} e^{-u} \frac{L\left(\frac{\gamma u}{x}\right)}{L\left(\frac{1}{x}\right)} du.$$

Observe next that, for $y \geqslant c$ and $t \geqslant c/y$,

$$\frac{L(ty)}{L(y)} = \frac{(ty)^{\delta} L(ty)}{y^{\delta} L(y)} t^{-\delta} \leqslant t^{-\delta} \quad \text{when} \quad t \leqslant 1,$$

and

$$\frac{L(ty)}{L(y)} = \frac{(ty)^{-\epsilon} L(ty)}{y^{-\epsilon} L(y)} t^{\epsilon} \leqslant t^{\epsilon} \quad \text{when} \quad t > 1;$$

so that

$$\frac{L(ty)}{L(y)} \leqslant g(t),$$

where

$$g(t) = \begin{cases} t^{-\delta} & \text{if} \quad 0 < t \leqslant 1, \\ t^{\epsilon} & \text{if} \quad t > 1. \end{cases}$$

It follows that, for $0 < x \leqslant 1/c$ and $u \geqslant \gamma x \lambda_N$,

$$\frac{L\left(\frac{u}{\gamma x}\right)}{L\left(\frac{1}{x}\right)} \leqslant g\left(\frac{u}{\gamma}\right),$$

and hence, by Lebesgue's theorem on dominated convergence, that

$$\lim_{x\to 0+} \int_{\gamma x \lambda_N}^{\infty} u^{\rho-1} e^{-u} \frac{L\left(\frac{u}{\gamma x}\right)}{L\left(\frac{1}{x}\right)} du = \int_0^{\infty} u^{\rho-1} e^{-u}\, du = \Gamma(\rho),$$

since

$$\lim_{x\to 0+} \frac{L\left(\frac{u}{\gamma x}\right)}{L\left(\frac{1}{x}\right)} = 1 \quad \text{and} \quad \int_0^{\infty} u^{\rho-1} e^{-u} g\left(\frac{u}{\gamma}\right) du < \infty.$$

Likewise

$$\lim_{x\to 0+} \int_{x\lambda_N}^{\infty} u^{\rho-1} e^{-u} \frac{L\left(\frac{\gamma u}{x}\right)}{L\left(\frac{1}{x}\right)} du = \Gamma(\rho),$$

and consequently, since

$$\lim_{x\to 0+} \frac{x^{\rho}}{L\left(\frac{1}{x}\right)} = 0,$$

$$\gamma^{-\rho-1}\Gamma(\rho) \leqslant \liminf_{x\to 0+} \frac{x^{\rho}}{L\left(\frac{1}{x}\right)} b(x) \leqslant \limsup_{x\to 0+} \frac{x^{\rho}}{L\left(\frac{1}{x}\right)} b(x) \leqslant \gamma^{\rho}\Gamma(\rho).$$

Since $\gamma^{-\rho-1} \to 1$ and $\gamma^{\rho} \to 1$ as $\gamma \to 1-$, it follows that

$$\lim_{x\to 0+} \frac{x^{\rho}}{L\left(\frac{1}{x}\right)} b(x) = \Gamma(\rho).$$

Therefore (1) is satisfied with $b(x)$ in place of $a(x)$ and so, by Theorem 4,

$$B_n \sim \frac{1}{\Gamma(\rho+1)} b\left(\frac{1}{\lambda_n}\right) \sim \frac{\lambda_n^{\rho}}{\rho} L(\lambda_n).$$

Finally, since $B_n \to \infty$, the methods M_b and $D_{\lambda,b}$ are regular; and hence, since $a_n \sim b_n$, we have

$$a(x) \sim b(x) \sim \Gamma(\rho) x^{-\rho} L\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0+,$$

and

$$A_n \sim B_n \sim \frac{1}{\Gamma(\rho+1)} a\left(\frac{1}{\lambda_n}\right) \sim \frac{\lambda_n^\rho}{\rho} L(\lambda_n). \quad ▮$$

LEMMA 3. *If the function $L(x)$ and the sequence $\{\lambda_n\}$ satisfy the conditions of Lemma 2 and*

$$A_n \sim \frac{\lambda_n^\rho}{\rho} L(\lambda_n),$$

then

$$a(x) \sim \Gamma(\rho) x^{-\rho} L\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0+.$$

Proof. Let $x > 0$. Suppose without loss of generality that $\lambda_1 > 0$, and let

$$B_n = \sum_{k=1}^{n} b_k := \lambda_n^\rho L(\lambda_n) \quad \text{for} \quad n = 1, 2, \ldots.$$

Then $B_n e^{-\lambda_n x} = O(\lambda_n^{\rho+\delta} e^{-\lambda_n x}) = o(1)$ as $n \to \infty$, and hence, as in the proof of Theorem 1,

$$b(x) = \sum_{n=1}^{\infty} B_n(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}).$$

Likewise

$$a(x) = \sum_{n=1}^{\infty} A_n(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}),$$

and, since $\rho A_n \sim B_n$, it follows, again as in the proof of Theorem 1, that $\rho a(x) \sim b(x)$ as $x \to 0+$. Thus it suffices to prove that

$$b(x) \sim \Gamma(\rho+1) x^{-\rho} L\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0+.$$

Given $\gamma > 1$, there is a positive integer N such that $\lambda_N > c$ and $\lambda_{n+1} < \gamma\lambda_n$ for $n \geqslant N$. Hence, for $n \geqslant N$ and $\lambda_n \leqslant t \leqslant \lambda_{n+1}$, we have $t < \gamma\lambda_n$ and so

$$\begin{aligned}\gamma^{-\rho-\epsilon} t^\rho L(t) &= \gamma^{-\rho-\epsilon} t^{\rho+\epsilon} t^{-\epsilon} L(t) \leqslant \lambda_n^{\rho+\epsilon} \lambda_n^{-\epsilon} L(\lambda_n) = \lambda_n^\rho L(\lambda_n)\\ &= B_n = \lambda_n^{\rho-\delta} \lambda_n^\delta L(\lambda_n) \leqslant t^{\rho-\delta} t^\delta L(t) = t^\rho L(t).\end{aligned}$$

It follows that, for $n \geqslant N$,

$$\gamma^{-\rho-\epsilon} \int_{\lambda_n}^{\lambda_{n+1}} t^\rho L(t) e^{-tx}\, dt \leqslant B_n x^{-1}(e^{-\lambda_n x} - e^{-\lambda_{n+1} x}) \leqslant \int_{\lambda_n}^{\lambda_{n+1}} t^\rho L(t) e^{-tx}\, dt,$$

and hence that

$$\begin{aligned}\gamma^{-\rho-\epsilon} \int_{x\lambda_N}^{\infty} u^\rho e^{-u} \frac{L\left(\frac{u}{x}\right)}{L\left(\frac{1}{x}\right)} du &\leqslant \frac{x^\rho}{L\left(\frac{1}{x}\right)} \sum_{n=N}^{\infty} B_n(e^{-\lambda_n x} - e^{-\lambda_{n+1} x})\\ &\leqslant \int_{x\lambda_N}^{\infty} u^\rho e^{-u} \frac{L\left(\frac{u}{x}\right)}{L\left(\frac{1}{x}\right)} du.\end{aligned}$$

Now, as in the proof of Lemma 2,

$$\lim_{x \to 0+} \int_{x\lambda_N}^{\infty} u^\rho e^{-u} \frac{L\left(\frac{u}{x}\right)}{L\left(\frac{1}{x}\right)} du = \Gamma(\rho+1),$$

and consequently, since

$$\lim_{x \to 0+} \frac{x^\rho}{L\left(\frac{1}{x}\right)} = 0,$$

$$\gamma^{-\rho-\epsilon} \Gamma(\rho+1) \leqslant \liminf_{x \to 0+} \frac{x^\rho}{L\left(\frac{1}{x}\right)} b(x) \leqslant \limsup_{x \to 0+} \frac{x^\rho}{L\left(\frac{1}{x}\right)} b(x) \leqslant \Gamma(\rho+1).$$

Since $\gamma^{-\rho-\epsilon} \to 1$ as $\gamma \to 1-$, it follows that

$$\lim_{x \to 0+} \frac{x^\rho}{L\left(\frac{1}{x}\right)} b(x) = \Gamma(\rho+1). \quad ▮$$

Example 3. Suppose $A_n \sim n(\log n)^{\rho-1}$ where $\rho > 0$. Then it is easily seen that

$$\sum_{k=1}^{n} \frac{a_k}{k} = \sum_{k=1}^{n} \frac{A_k}{k(k+1)} + \frac{A_n}{n+1} \sim \frac{(\log n)^\rho}{\rho}.$$

Hence, by Lemma 3 with $\lambda_n := \log n$,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{1+x}} \sim \Gamma(\rho) x^{-\rho} \quad \text{as} \quad x \to 0+.$$

The case $\rho = 1, 2, \ldots$ of this example appears as example 20 on page 317 in [**1**].

7. *An extension of Theorem 4*

THEOREM 6. *Suppose that $b_1 > 0$, $b_n \geqslant 0$ and $a_n \geqslant -Hb_n$ for $n = 1, 2, \ldots$ where H is a positive constant, and that the Dirichlet series $a(x)$ and $b(x)$ are convergent for all $x > 0$. Suppose also that*

$$\liminf_{x \to 0+} \frac{a(x)}{b(x)} > 0,$$

and that

$$\lim_{x \to 0+} \frac{b(mx)}{b(x)} = \lim_{x \to 0+} \frac{a(mx)}{a(x)} = \alpha_m > 0 \quad \textit{for} \quad m = 2 \quad \textit{and} \quad m = 3. \tag{14}$$

Then

$$a(x) = x^{-\rho} L\left(\frac{1}{x}\right) \quad \textit{for} \quad x > 0,$$

where $\rho = -\log_2 \alpha_2 \geqslant 0$ and $L(x)$ is a function (defined for $x > 0$) satisfying

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 \quad \textit{for all} \quad t > 0,$$

and

$$A_n \sim \frac{1}{\Gamma(\rho+1)} a\left(\frac{1}{\lambda_n}\right) = \frac{\lambda_n^\rho}{\Gamma(\rho+1)} L(\lambda_n).$$

Proof. Let
$$c(x) := a(x) + Hb(x).$$
Then, for $m = 1, 2$,
$$\frac{c(mx)}{c(x)} = \frac{a(mx)}{a(x)} + H\left(\frac{b(mx)}{b(x)} - \frac{a(mx)}{a(x)}\right)\frac{b(x)}{c(x)} \to \alpha_m \quad \text{as} \quad x \to 0+,$$
since $b(x) = O(c(x))$ as $x \to 0+$. Further, $c_n \geqslant 0$ for $n = 1, 2, \ldots$. It follows, by Lemma 1, that, for all $t > 0$,
$$\lim_{x\to 0+} \frac{b(tx)}{b(x)} = \lim_{x \to 0+} \frac{c(tx)}{c(x)} = t^{-\rho},$$
where $\rho = -\log_2 \alpha_2$, and so
$$\frac{a(tx)}{a(x)} = \frac{c(tx)}{c(x)} - H\left(\frac{b(tx)}{b(x)} - \frac{c(tx)}{c(x)}\right)\frac{b(x)}{a(x)} \to t^{-\rho} \quad \text{as} \quad x \to 0+,$$
since $b(x) = O(a(x))$ as $x \to 0+$. If we now define
$$L(x) := x^{-\rho} a\left(\frac{1}{x}\right),$$
we see that
$$\frac{L(tx)}{L(x)} = t^{-\rho} \frac{a\left(\frac{1}{tx}\right)}{a\left(\frac{1}{x}\right)} \to 1 \quad \text{as} \quad x \to \infty.$$

Finally, we have, by Theorem 4, that
$$B_n \sim \frac{1}{\Gamma(\rho+1)} b\left(\frac{1}{\lambda_n}\right) \quad \text{and} \quad c_n \sim \frac{1}{\Gamma(\rho+1)} c\left(\frac{1}{\lambda_n}\right),$$
and so
$$\frac{A_n}{a(1/\lambda_n)} = \frac{C_n}{c(1/\lambda_n)} - H\left(\frac{B_n}{b(1/\lambda_n)} - \frac{C_n}{c(1/\lambda_n)}\right)\frac{b(1/\lambda_n)}{a(1/\lambda_n)} \to \frac{1}{\Gamma(\rho+1)},$$
since $b(x) = O(a(x))$ as $x \to 0+$. ∎

The following corollary of Theorem 6 generalizes Hardy and Littlewood's theorem E in [3].

COROLLARY. *Suppose that $\rho > \delta > 0$, $c > 0$, $\epsilon > 0$ and that $K(x)$ is a positive continuous function on $(0, \infty)$ such that $x^{\delta}K(x)$ is increasing and $x^{-\epsilon}K(x)$ is decreasing on (c, ∞), and*
$$\lim_{x\to\infty} \frac{K(tx)}{K(x)} = 1 \quad \textit{for all} \quad t > 0.$$
Suppose that $\lambda_{n+1} \sim \lambda_n$ and
$$a_n \geqslant -H(\lambda_n - \lambda_{n-1})\lambda_n^{\rho-1}K(\lambda_n) \quad \textit{for} \quad n = 2, 3, \ldots,$$

where H is a positive constant, and that the Dirichlet series $a(x)$ is convergent for all $x > 0$. Suppose also that
$$\liminf_{x\to 0+} \frac{x^{\rho}}{K\left(\frac{1}{x}\right)} a(x) > 0,$$
and that
$$\lim_{x\to 0+} \frac{a(mx)}{a(x)} = m^{-\rho} \quad \textit{for} \quad m = 2 \quad \textit{and} \quad m = 3. \tag{15}$$
Then
$$a(x) = x^{-\rho} L\left(\frac{1}{x}\right) \quad \textit{for} \quad x > 0,$$
where $L(x)$ is a function (defined for $x > 0$) satisfying
$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \textit{for all} \quad t > 0,$$
and
$$A_n \sim \frac{1}{\Gamma(\rho+1)} a\left(\frac{1}{\lambda_n}\right) = \frac{\lambda_n^{\rho}}{\Gamma(\rho+1)} L(\lambda_n).$$

Proof. Suppose without loss of generality that $b_1 := 1$ and $a_1 := -H$. Let
$$b_n := (\lambda_n - \lambda_{n-1})\lambda_n^{\rho-1}K(\lambda_n) \quad \text{for} \quad n = 2, 3, \ldots.$$
Then, by Lemma 2, the Dirichlet series $b(x)$ is convergent for all $x > 0$, and
$$b(x) \sim \Gamma(\rho)x^{-\rho}K\left(\frac{1}{x}\right) \quad \text{as} \quad x \to 0+.$$

The result now follows from Theorem 6. ∎

Remarks. In view of theorem 1·8 in [9], the integers 2, 3 in (1), (14) and (15) can be replaced by any pair of positive numbers p, $q \neq 1$ such that $\log_q p$ is irrational.

It is known (see [9], p. 49) that the hypothesis

(*h*) *$L(x)$ is a positive continuous function on $(0, \infty)$ such that, for some $c > 0$ and every $\delta > 0$, $x^{\delta}L(x)$ is increasing and $x^{-\delta}L(x)$ is decreasing on (c, ∞)*

implies that
$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for all} \quad t > 0.$$

Thus the hypotheses on $L(x)$ in Lemmas 2 and 3 can be replaced by (*h*), and the hypotheses on $K(x)$ in the corollary of Theorem 6 can be replaced by (*h*) with $K(x)$ in place of $L(x)$.

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REFERENCES

[1] D. BORWEIN. Tauberian conditions for the equivalence of weighted mean and power series methods of summability. *Canad. Math. Bull.* (3), **24** (1981), 309–316.

[2] D. BORWEIN and A. MEIR. A Tauberian theorem concerning weighted means and power series. *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 283–286.

[3] G. H. HARDY and J. E. LITTLEWOOD. Some theorems concerning Dirichlet's series. *Messenger of Math.* **43** (1914), 134–147.

[4] G. H. HARDY. *Orders of Infinity*. Camb. Math. Tract No. 12 (Cambridge University Press, 1910).

[5] G. H. HARDY. *Divergent Series* (Oxford University Press, 1949).

[6] K. ISHIGURO. A Tauberian theorem for (J, p_n) summability. *Proc. Japan Acad.* **40** (1964), 807–812.

[7] K. KNOPP. *Theory and Application of Infinite Series* (Blackie, 1928).

[8] A. P. KOCHANOVSKI. A condition for the equivalence of logarithmic methods of summability. *Ukranian Math. J.* **27** (1975), 229–234.

[9] E. SENETA. *Regularly Varying Functions*. Lecture Notes in Math., Vol. 508 (Springer-Verlag, 1976).

[10] E. C. TITCHMARSH. *The Theory of Functions* (Oxford University Press, second edition, 1939).