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A Tauberian theorem concerning weighted means and power series

By DAVID BORWEIN
Department of Mathematics, University of Western Ontario, London, Ont. N6A 5B7, Canada

AND AMRAM MEIR
Department of Mathematics, University of Alberta, Edmonton, Alta. T6G 2G1, Canada

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1. *Introduction.* Suppose throughout that $\{p_n\}$ is a sequence of non-negative numbers with $p_0 > 0$, that

$$P_n = \sum_{k=0}^{n} p_k \to \infty$$

and that

$$p(x) = \sum_{k=0}^{\infty} p_k x^k < \infty \quad \text{for} \quad 0 < x < 1.$$

Let $\{s_n\}$ be a sequence of real numbers.

The weighted mean summability method M_p and the power series method J_p are defined as follows:

$$s_n \to s\,(M_p) \quad \text{if} \quad \frac{1}{P_n}\sum_{k=0}^{n} p_k s_k \to s;$$

$$s_n \to s\,(J_p) \quad \text{if} \quad \sum_{k=0}^{\infty} p_k s_k x^k \text{ is convergent for } 0 < x < 1,$$

and

$$\frac{1}{p(x)}\sum_{k=0}^{\infty} p_k s_k x^k \to s \quad \text{as} \quad x \to 1-.$$

It is known (see [**4**]) that $s_n \to s\,(M_p)$ implies $s_n \to s\,(J_p)$. The primary object of this paper is to establish the following Tauberian theorem concerning the reverse implication:

THEOREM. *If $s_n \to s\,(J_p)$ and $s_n > -H$ for $n = 0, 1, \ldots,$ where H is a constant, and if*

$$\lim_{x\to 1-} \frac{p(x^m)}{p(x)} = \lambda_m > 0 \quad \textit{for} \quad m = 2 \quad \textit{and} \quad m = 3, \tag{1}$$

then $s_n \to s\,(M_p)$.

A version of this theorem with (1) replaced by the more cumbersome condition

$$\lim_{x\to 1-} \frac{p(x^m)}{p(x)} = \mu_{m-1} > 0 \quad \textit{for} \quad m = 1, 2, \ldots, \quad \textit{where } \{\mu_m\} \textit{ is totally monotone} \tag{2}$$

was proved in [**1**]. We shall show that conditions (1) and (2) are in fact equivalent, and that the sequence $\{\mu_m\}$ in (2) must necessarily have the form $\mu_m = (m+1)^{-\rho}$ for some $\rho \geqslant 0$. It will also emerge that the integers 2, 3 in (1) can be replaced by any two integers $m_1, m_2 \geqslant 2$ such that no power of the one is a power of the other (i.e. $\log m_1/\log m_2$ is irrational). It is natural to ask whether

$$\lim_{x\to 1-} \frac{p(x^2)}{p(x)} = \lambda > 0 \qquad (3)$$

alone is equivalent to (1). It follows from the inequalities

$$\frac{p(x^2)}{p(x)} \geqslant \frac{p(x^3)}{p(x)} \geqslant \frac{p(x^4)}{p(x^2)}\frac{p(x^2)}{p(x)} \quad \text{for} \quad 0<x<1,$$

that (1) and (3) are indeed equivalent in the special case when $\lambda = 1$. We shall, however, show that (1) and (3) are *not* equivalent when $0<\lambda<1$ by constructing a function $p(x)$ satisfying (3) for which $p(x^3)/p(x)$ does not tend to a limit as $x\to 1-$.

2. *Proof of the theorem.* We shall show that (1) implies (2) with $\mu_m = (m+1)^{-\rho}$ for some $\rho \geqslant 0$. Suppose (1) holds. For $t>0$, define

$$\lambda(t) = \lim_{x\to 1-} \frac{p(x^t)}{p(x)}$$

whenever the limit exists. It is readily verified that if $\lambda(t)$ and $\lambda(u)$ are defined and positive, then $\lambda(tu) = \lambda(t)\lambda(u)$ and $\lambda(t/u) = \lambda(t)/\lambda(u)$. It therefore follows from (1) that $\lambda(t)$ is defined throughout the set A of numbers t of the form $t = 2^j 3^k$ with $j, k = 0, \pm 1, \pm 2, \ldots$. Since $\log 2/\log 3$ is irrational, it follows from Kronecker's theorem ([**3**], theorem 439) that the sequence $\{n \log 2/\log 3\}$ is dense modulo 1 in the interval (0, 1) and hence that A is dense in $(0, \infty)$. Consequently, by lemma 1 on page 275 of [**2**], there is a finite ρ such that

$$\lambda(t) = t^{-\rho} \quad \text{for} \quad t \in A.$$

Since $\lambda(t)$ is non-increasing in A we must have $\rho \geqslant 0$. Further, if $u \leqslant t \leqslant v$ where $u, v \in A$, then $p(x^v) \leqslant p(x^t) \leqslant p(x^u)$ for $0<x<1$ and so

$$v^{-\rho} \leqslant \liminf_{x\to 1-} \frac{p(x^t)}{p(x)} \leqslant \limsup_{x\to 1-} \frac{p(x^t)}{p(x)} \leqslant u^{-\rho}.$$

This implies that

$$\lambda(t) = t^{-\rho} \quad \text{for all} \quad t>0.$$

Hence (2) holds with $\mu_m = (m+1)^{-\rho}$ for $m = 0, 1, \ldots$. Our theorem thus follows from the second part of theorem 1 in [1], the proof of which can now be simplified since, for $\rho > 0$, $m = 0, 1, \ldots,$

$$(m+1)^{-\rho} = \int_0^1 t^m \, d\chi(t) \quad \text{with} \quad \chi(t) = \frac{1}{\Gamma(\rho)} \int_0^t (-\log u)^{\rho-1} \, du,$$

and c in the proof can be taken to be any number in (0, 1).

It is evident that the above proof works, *mutatis mutandis*, if the integers 2, 3 in (1) are replaced by positive integers m_1, m_2 such that $\log m_1/\log m_2$ is irrational.

3. *Construction.* We shall construct a function $p(x)$ satisfying (3) for which $p(x^3)/p(x)$ does not tend to a limit as $x\to 1-$. We require the following lemma which was proved in essence in [**1**].

LEMMA. *If $P_{n+1} \sim P_n$ and $\lim_{n\to\infty}(P_n/P_{nm}) = \lambda > 0$ where m is a positive integer, then $\lim_{x\to 1-}[p(x^m)/p(x)] = \lambda$.*

We first construct a sequence $\{a_n\}$ of positive numbers converging to 0 such that

$$\lim_{n\to\infty} \sum_{k=n}^{2n} a_k = 1$$

for which $\sum_{k=n}^{3n} a_k$ does not tend to a limit as $n\to\infty$. The construction is due to E. F. Smet. Define $a_0 = 1$,

$$a_k = 2^{-m} \quad \text{for} \quad 2^m \leqslant k < 2^{m+1} \quad (m = 0, 1, \ldots).$$

Then for any positive integer n we have $n = 2^m + r$ where $0 \leqslant r < 2^m$, so that $2n = 2^{m+1} + 2r < 2^{m+2}$ and hence

$$\sum_{k=n}^{2n} a_k = (2^{m+1} - 2^m - r)\,2^{-m} + (2r+1)\,2^{-m-1} \to 1 \quad \text{as} \quad n\to\infty.$$

On the other hand, for $n = 2^m$ we have

$$\sum_{k=2n}^{3n} a_k = (2^m+1)\,2^{-m-1} \to \tfrac{1}{2} \quad \text{as} \quad m\to\infty,$$

whereas for $n = 3\cdot 2^{m-1}$ we have

$$\sum_{k=2n}^{3n} a_k = (2^{m+2} - 3\cdot 2^m)\,2^{-m-1} + (9\cdot 2^{m-1} - 2^{m+2} + 1)\,2^{-m-2} \to \tfrac{5}{8} \quad \text{as} \quad m\to\infty.$$

Thus $\sum_{k=2n}^{3n} a_k$ does not tend to a limit as $n\to\infty$ and consequently neither does $\sum_{k=n}^{3n} a_k$.

Now define a sequence $\{P_n\}$ by setting $P_0 = 1$ and

$$P_{n+1} = P_n 2^{a_n} \quad \text{for} \quad n = 0, 1, \ldots.$$

Then $P_{n+1} > P_n > 0$, for $n = 0, 1, \ldots$, $P_n \to \infty$ and $P_{n+1} \sim P_n$. Further

$$\frac{P_{2n}}{P_n} = 2^{a_n + a_{n+1} + \ldots + a_{2n-1}} \to 2,$$

whereas

$$\frac{P_{3n}}{P_n} = 2^{a_n + a_{n+1} + \ldots + a_{3n-1}}$$

does not tend to a limit as $n\to\infty$.

Setting $p_0 = 1$, $p_n = P_n - P_{n-1}$ for $n = 1, 2, \ldots$, we see from the lemma that

$$\lim_{x\to 1-} \frac{p(x^2)}{p(x)} = \frac{1}{2}.$$

Suppose, contrary to what we wish to establish, that

$$\lim_{x\to 1-} \frac{p(x^3)}{p(x)} = \kappa.$$

Since

$$\frac{p(x^3)}{p(x)} \geqslant \frac{p(x^4)}{p(x^2)}\frac{p(x^2)}{p(x)} \to \frac{1}{4} \quad \text{as} \quad x \to 1-,$$

it follows that $\kappa > 0$ and hence, by what was proved in §2 above,

$$\lim_{x\to 1-} \frac{p(x^{m+1})}{p(x)} = (m+1)^{-\rho} \quad \text{for some} \quad \rho > 0 \quad \text{and} \quad m = 0, 1, \ldots.$$

It was shown on page 312 in [1] that this implies that, for $0 < c < 1$,

$$\frac{P_n}{p(c^{1/n})} \to \gamma > 0 \quad \text{as} \quad n \to \infty.$$

Therefore

$$\frac{P_{3n}}{P_n} = \frac{P_{3n}}{p(c^{1/3n})}\frac{p(c^{1/n})}{P_n}\frac{p(c^{1/3n})}{p(c^{1/n})} \to \frac{1}{\kappa} \quad \text{as} \quad n \to \infty.$$

This is a contradiction, and so the constructed $p(x)$ satisfies (3) but $p(x^3)/p(x)$ does not tend to a limit as $x \to 1-$.

Remarks. We have shown that (1) implies that

$$\lim_{x\to 1-} \frac{p(x^t)}{p(x)} = t^{-\rho} \quad \text{for some} \quad \rho \geqslant 0 \quad \text{and all} \quad t > 0.$$

It follows from the theorem on page 276 in [2] that such a $p(x)$ must necessarily be of the form

$$p(x) = (1-x)^{-\rho} L(x),$$

where $L(x)$ is slowly varying in the sense that

$$\lim_{x\to 1-} \frac{L(x^t)}{L(x)} = 1 \quad \text{for all} \quad t > 0.$$

A typical example of a slowly varying function is given by

$$L(x) = \left(\log \frac{1}{1-x}\right)^{\beta} \quad \text{with} \quad \beta \geqslant 0.$$

Though we have shown that (3) does not imply (1), it is an open question whether or not the theorem remains valid when (1) is replaced by (3) or even when (1) is omitted altogether.

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A non-causal triangle function

BY T. B. M. MCMASTER
Pure Mathematics Department, Queen's University, Belfast BT7 1NN

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Schweizer and Sklar, in their book *Probabilistic Metric Spaces*, raise the question of whether every triangle function is causal. This note provides a negative answer, by observing certain properties of a suitable partial quaternary operation on the unit interval.

1. *Introduction*

The following notation and terminology are taken from [1].

A *distance distribution function* is a non-decreasing function f from $[0, \infty]$ to $[0, 1]$ which is left-continuous at each point of $(0, \infty)$ and satisfies the conditions $f(0) = 0$ and $f(\infty) = 1$; the set of all distance distribution functions is denoted by Δ^+. For example, given any $a \in [0, \infty)$ the function ϵ_a defined by

$$\epsilon_a(x) = \begin{cases} 0 & \text{for} \quad x \leqslant a, \\ 1 & \text{for} \quad x > a \end{cases}$$

is a distance distribution function. A *triangle function* is a commutative, associative binary operation τ on Δ^+ which, for all elements f, g, h of Δ^+, satisfies the conditions

$$\tau(f, \epsilon_0) = f \quad \text{and} \quad f \leqslant g \text{ implies } \tau(f, h) \leqslant \tau(g, h)$$

where Δ^+ is ordered by writing $f \leqslant g$ whenever $f(x) \leqslant g(x)$ for every $x \in [0, \infty]$. All the naturally occurring examples of triangle functions are *causal* in the sense that, for any f_1, f_2, g_1, g_2 in Δ^+, and $a \in [0, \infty]$,

$$f_1(x) \leqslant f_2(x) \quad \text{and} \quad g_1(x) \leqslant g_2(x) \quad \text{for all} \quad x \leqslant a$$

together imply

$$\tau(f_1, g_1)(x) \leqslant \tau(f_2, g_2)(x) \quad \text{for all} \quad x \leqslant a.$$

In particular, this is evidently the case for any triangle function τ in which the value of $\tau(f, g)(x)$ depends solely on the values of $f(x)$ and $g(x)$; we seek a non-causal example by allowing it to depend also on the values of f and g at some point x^* greater than (or equal to) x, so that $\tau(f, g)(x)$ is a function of the four numbers

$$f(x), f(x^*), g(x), g(x^*).$$

This, together with the observation that $f(x) \leqslant f(x^*)$ and $g(x) \leqslant g(x^*)$, motivates the choice of domain of the operation θ in the next paragraph.