A THEOREM ON RIESZ SUMMABILITY

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1. Suppose throughout that a is positive and, unless otherwise stated, that κ is a positive integer. Suppose further that the functions $\phi(w)$,

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$\psi(w)$ are defined in $[0, \infty)$, that $\phi(w)$ is non-negative and unboundedly increasing in this range and that both functions have absolutely continuous κ-th derivatives in every interval $[a, W]$.

I shall be concerned with obtaining sufficient conditions to ensure the truth of the proposition

P. $\sum_{n=1}^{\infty} a_n \psi(\lambda_n)$ *is summable* $(R, \phi(\lambda_n), \kappa)$ *whenever* (λ_n) *is an unboundedly increasing sequence of positive numbers and* $\sum_{n=1}^{\infty} a_n$ *is summable* (R, λ_n, κ).

The following theorems are known.

T_1. *If* $\phi(w) = e^w$, $\psi(w) = w^{-\kappa}$, *then* P.

T_2. *If* (i) $\psi(w) = 1$, (ii) $\int_a^w t^n |\phi^{(n+1)}(t)| \, dt = O\{\phi(w)\}$ $(n = 1, 2, \ldots, \kappa;$ $w \geqslant a)$, *then* P.

T_3. *If* (i) $\phi(w) = w$, (ii) $\psi(w) = O(1)$ $(w \geqslant a)$, $\int_a^{\infty} t^{\kappa} |\psi^{(\kappa+1)}(t)| \, dt < \infty$, *then* P.

T_4. *If* $\phi(w)$ *is an L-function* such that* $w\phi'(w)/\phi(w) \succcurlyeq 1$ *and* $\psi(w) = \{w\phi'(w)/\phi(w)\}^{-\kappa}$, *then* P.

T_1 (for all $\kappa \geqslant 0$) is due to Hardy and Riesz [5], T_2 to Hirst† [8] and T_3 essentially to Hardy‡ [4]. T_4 (for all $\kappa \geqslant 0$) has recently been obtained by Guha§; it includes T_1 as a special case. Kuttner|| [9] has proved that T_2 (i) and P imply T_2 (ii). He also showed that the truth of T_2 (ii) for $n = 1, 2, \ldots, \kappa-1$ $(\kappa \geqslant 2)$ is a consequence of its truth for $n = \kappa$. I have proved [1] that T_3 (i) and P imply T_3 (ii)¶.

I shall prove the following theorem.

T. *If* (i) $\gamma(w)$ *is positive and absolutely continuous in every interval* $[a, W]$ *and* $\gamma'(w) = O(1)$ *for* $w \geqslant a$,

(ii) $w^n \psi^{(n)}(w) = O\{(\gamma(w)/w)^{\kappa-n}\}$ $(n = 0, 1, \ldots, \kappa;$ $w \geqslant a)$,

(iii) $\int_a^{\infty} t^{\kappa} |\psi^{(\kappa+1)}(t)| \, dt < \infty$,

(iv) $\int_a^w \{\gamma(t)\}^n |\phi^{(n+1)}(t)| \, dt = O\{\phi(w)\}$ $(n = 1, 2, \ldots, \kappa;$ $w \geqslant a)$,

then P.

* For the definition and properties of L-functions (logarithmico-exponential functions) see [6].

† He also obtains a version of T_2 involving fractional values of κ.

‡ Hardy's result has been extended to fractional orders of summability by Cossar [3]. See also Theorem V in [2].

§ "Convergence factors for Riesz summability". [This *Journal*, 31 (1956), 311-319 (preceding paper).]

|| See also his paper [10].

¶ See also [11].

It is evident that T_1 and T_2 are special cases of T, and T_3 can immediately be derived from T and the known result (see [3]) that T (ii) with $\gamma(w) = w$ is a consequence of T_3 (ii). Further, it can readily be shown that the hypotheses of T are satisfied when ϕ, ψ are as in T_4 and $\gamma(w) = \phi(w)/\phi'(w)$.

2. Some lemmas are required.

LEMMA* 1. *When m, n are positive integers, the n-th derivative of* $\{g(t)\}^m$ *is the sum of a number of terms of the form*

$$A\{g(t)\}^{m-r} \prod_{s=1}^{p} \{g^{(s)}(t)\}^{c_s} \quad (r = 1, 2, \ldots, m)$$

where A is a constant, the c's are non-negative integers and

$$1 \leqslant p \leqslant n, \quad c_p \geqslant 1, \quad \sum_{s=1}^{p} c_s = r, \quad \sum_{s=1}^{p} s c_s = n.$$

The proof is elementary.

LEMMA 2. *If* T (i) *and* T (iv), *then*

$$\{\gamma(w)\}^n \phi^{(n)}(w) = O\{\phi(w)\} \quad (n = 1, 2, \ldots, \kappa;\ w \geqslant a).$$

For $n = 1, 2, \ldots, \kappa$, $w \geqslant a$,

$$\{\gamma(w)\}^n \phi^{(n)}(w) = \{\gamma(a)\}^n \phi^{(n)}(a) + \int_a^w \{\gamma(t)\}^n \phi^{(n+1)}(t)\, dt + n\int_a^w \gamma'(t) \{\gamma(t)\}^{n-1} \phi^{(n)}(t)\, dt = O\{\phi(w)\},$$

since $\gamma'(t) = O(1)$.

LEMMA 3. *If* $\overline{\lim}_{w\to\infty} \int_a^{\infty} |f(w, t)| \, dt < \infty$ *and* $\lim_{w\to\infty} \int_a^{y} |f(w, t)| \, dt = 0$, *for every finite* $y > a$, *and if* $s(t)$ *is a bounded measurable function in* (a, ∞) *which tends to zero as* $t \to \infty$, *then*

$$\lim_{w\to\infty} \int_a^{\infty} f(w, t)\, s(t)\, dt = 0.$$

The proof of this result† has been given by Hardy ([7], 50). Because of its brevity it is reproduced here.

* This is Lemma A in [8]. A more general result due to Faa di Bruno is given in de la Vallée Poussin's "Cours d'analyse infinitésimale", I, pp. 89-90 in ed. 7.

† *Cf.* Theorem VI in [2].

For $y > a$,

$$\overline{\lim_{w\to\infty}} \left| \int_a^\infty f(w,t)\,s(t)\,dt \right| \leqslant \overline{\lim_{w\to\infty}} \int_a^y |f(w,t)\,s(t)|\,dt + \overline{\lim_{w\to\infty}} \int_y^\infty |f(w,t)\,s(t)|\,dt$$

$$\leqslant \overline{\operatorname{bound}}_{t\geqslant y} |s(t)| \,.\, \overline{\lim_{w\to\infty}} \int_a^\infty |f(w,t)|\,dt.$$

Since the final expression tends to zero as $y\to\infty$, the result follows.

3. *Proof of* T. Let (λ_n) be an unboundedly increasing sequence of positive numbers. Write, for $w \geqslant 0$,

$$A(w) = \sum_{\lambda_n \leqslant w} a_n, \quad A_m(w) = \frac{1}{m!}\int_0^w (w-t)^m\,dA(t) \quad (m = 0, 1, \ldots),$$

$$F(w) = w^{-\kappa} \sum_{\phi(\lambda_n)\leqslant w} \{w-\phi(\lambda_n)\}^\kappa \psi(\lambda_n)\,a_n, \quad G(w) = F\{\phi(w)\}.$$

We have now to prove that *if*

$(\alpha)'$ *$w^{-\kappa}A_\kappa(w)$ tends to a finite limit as $w\to\infty$,*

then

$(\beta)'$ *$F(w)$ tends to a finite limit as $w\to\infty$.*

There is clearly no loss in generality if we prove this result with $(\alpha)'$ replaced by

(α) *$A(w) = 0$ for $0\leqslant w \leqslant a$ and $w^{-\kappa}A_\kappa(w)\to 0$ as $w\to\infty$.*

Suppose therefore that (α) holds and note that $(\beta)'$ is equivalent to

(β) *$G(w)$ tends to a finite limit as $w\to\infty$.*

Now, for $w\geqslant a$,

$$G(w) = \{\phi(w)\}^{-\kappa} \sum_{\phi(\lambda_n)\leqslant\phi(w)} \{\phi(w)-\phi(\lambda_n)\}^\kappa \psi(\lambda_n)\,a_n$$

$$= \{\phi(w)\}^{-\kappa}\int_a^w \{\phi(w)-\phi(t)\}^\kappa \psi(t)\,dA(t).$$

Integrate $(\kappa+1)$ times by parts and use (α), T (ii) $(n=0)$ and Lemma 2 $(n=1)$ to get

$$(-1)^\kappa G(w) + \{\phi(w)\}^{-\kappa}\int_a^w A_\kappa(t)\left(\frac{\partial}{\partial t}\right)^{\kappa+1}\left(\{\phi(w)-\phi(t)\}^\kappa\psi(t)\right)dt$$

$$= \{\phi(w)\}^{-\kappa}A_\kappa(w)\left(\frac{\partial}{\partial t}\right)^\kappa\left(\{\phi(w)-\phi(t)\}^\kappa\psi(t)\right)_{t=w}$$

$$= (-1)^\kappa\kappa!\{\phi'(w)/\phi(w)\}^\kappa\psi(w)\,A_\kappa(w)$$

$$= o(1) \text{ as } w\to\infty; \tag{1}$$

and observe that, for $w > t > a$,

$$\{\phi(w)\}^{-\kappa}\left(\frac{\partial}{\partial t}\right)^{\kappa+1}\left(\{\phi(w)-\phi(t)\}^\kappa\psi(t)\right)$$

$$= \left\{1-\frac{\phi(t)}{\phi(w)}\right\}^\kappa \psi^{(\kappa+1)}(t)$$

$$+\{\phi(w)\}^{-\kappa}\sum_{n=1}^{\kappa+1}\binom{\kappa+1}{n}\psi^{(\kappa+1-n)}(t)\left(\frac{\partial}{\partial t}\right)^n\{\phi(w)-\phi(t)\}^\kappa. \tag{2}$$

In virtue of T (iii) and (α), $\int_a^\infty |\psi^{(\kappa+1)}(t)\,A_\kappa(t)|\,dt < \infty$, and hence, by Lebesgue's theorem on dominated convergence,

$$\int_a^w \left\{1-\frac{\phi(t)}{\phi(w)}\right\}^\kappa \psi^{(\kappa+1)}(t)\,A_\kappa(t)\,dt \to \int_a^\infty \psi^{(\kappa+1)}(t)\,A_\kappa(t)\,dt \text{ as } w\to\infty. \tag{3}$$

Further, by Lemma 1,

$$\{\phi(w)\}^{-\kappa}t^\kappa\psi^{(\kappa+1-n)}(t)\left(\frac{\partial}{\partial t}\right)^n\{\phi(w)-\phi(t)\}^\kappa \quad (n=1, 2, \ldots, \kappa+1;\ w>t>a)$$

is a linear combination of terms like

$$f(w,t) = \{\phi(w)\}^{-\kappa}t^\kappa\psi^{(\kappa+1-n)}(t)\{\phi(w)-\phi(t)\}^{\kappa-r}\prod_{s=1}^p\{\phi^{(s)}(t)\}^{c_s}, \tag{4}$$

where the c's are non-negative integers and $1\leqslant r\leqslant\kappa$, $1\leqslant p\leqslant n$, $c_p\geqslant 1$,

$$\sum_{s=1}^p c_s = r, \quad \sum_{s=1}^p s\,c_s = n.$$

Since $c_p = 1$ if $p=\kappa+1$, we use T (ii) and Lemma 2 to deduce that there are constants M, N such that, for $w>t>a$,

$$|f(w,t)| < M\{\phi(w)\}^{-r}t^\kappa|\psi^{(\kappa+1-n)}(t)\,\phi^{(p)}(t)|\,\{\phi(t)\}^{-1}\{\gamma(t)\}^p\prod_{s=1}^p\left(\{\gamma(t)\}^{-s}\phi(t)\right)^{c_s}$$

$$= M\{\phi(w)\}^{-r}\{\phi(t)\}^{r-1}t^\kappa|\psi^{(\kappa+1-n)}(t)|\,\{\gamma(t)\}^{1-n}\{\gamma(t)\}^{p-1}|\phi^{(p)}(t)|$$

$$< \frac{N}{\phi(w)}\{\gamma(t)\}^{p-1}|\phi^{(p)}(t)|.$$

Hence, for fixed $y>a$,

$$\int_a^y |f(w,t)|\,dt \to 0 \text{ as } w\to\infty,$$

and, by T (iv),

$$\int_a^w |f(w,t)|\,dt = O(1) \text{ for } w\geqslant a.$$

Consequently, by Lemma 3,

$$\int_a^w f(w,\, t)\, t^{-\kappa} A_\kappa(t)\, dt \to 0 \text{ as } w \to \infty,$$

and this, together with (1), (2), (3) and (4) enables us to deduce (β) as required.

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