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On absolute generalized Hausdorff summability

By

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Introduction. Hausdorff matrices have played an important role in summability theory and are intimately linked with the moment problem for a finite interval. The matrices of such standard methods of summability as the Cesàro, the Hölder, the Euler and the weighted mean methods are all Hausdorff or generalized Hausdorff matrices (see [3], [4], [5] and [6]). In this paper we define the notion of absolute summability appropriate to generalized Hausdorff matrices and extend known results for ordinary Hausdorff matrices. In particular we establish relationships between generalized Cesàro and generalized Hölder absolute summability methods.

Absolute summability. Let $Q = (q_{n,k})$ $(n, k = 0, 1, \ldots)$ be a matrix. Given a series $\sum_{n=0}^{\infty} a_n$, let

$$s_n = \sum_{k=0}^{n} a_k \quad \text{and} \quad \sigma_n = Q(s_n) = \sum_{k=0}^{\infty} q_{n,k} s_k.$$

Let

$$U_n = 1 - u_0 + \sum_{k=0}^{n} u_k \quad \text{where} \quad u_k > 0 \quad \text{for} \quad k = 0, 1, \ldots,$$

and suppose that γ is real and $\beta > 0$. We define absolute summability $|Q, u_n, \gamma|_\beta$ as follows: $\sum_{n=0}^{\infty} a_n$ is summable $|Q, u_n, \gamma|_\beta$ if

$$\sum_{n=1}^{\infty} U_n^{\gamma\beta+\beta-1} u_n^{1-\beta} |\sigma_n - \sigma_{n-1}|^\beta < \infty. \tag{1}$$

If $u_n = 1$ for $n = 0, 1, \ldots$, then (1) is equivalent to

$$\sum_{n=1}^{\infty} n^{\gamma\beta+\beta-1} |\sigma_n - \sigma_{n-1}|^\beta < \infty$$

which is the defining inequality in the definition of absolute summability given by Borwein [1]. Given absolute summability methods V and W, the notation

$$V \Rightarrow W$$

is used to mean that every series summable V is also summable W.

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Generalized Hausdorff matrices. Suppose in all that follows that $\lambda = \{\lambda_n\}$ is a sequence of real numbers with

$$\lambda_0 \geqq 0 \quad \text{and} \quad \inf_{n \geqq 1} \lambda_n > 0. \tag{2}$$

Let Ω be a simply connected region that contains every positive λ_n, and suppose that, for $n = 0, 1, \ldots, \Gamma_n$ is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_k \in \Omega$ with $0 \leqq k \leqq n$. Suppose that f is holomorphic in Ω and that $f(\lambda_0)$ is defined even when $\lambda_0 \notin \Omega$. Define

$$\lambda_{n,k} = \begin{cases} -\lambda_{k+1} \cdots \lambda_n \dfrac{1}{2\pi i} \displaystyle\int_{\Gamma_n} \dfrac{f(z)\,dz}{(\lambda_k - z) \cdots (\lambda_n - z)} + \delta_k & \text{for } 0 \leqq k \leqq n, \\ 0 & \text{for } k > n, \end{cases} \tag{3}$$

where $\delta_k = f(\lambda_0)$ if $k = 0$ and $\lambda_0 \notin \Omega$, and $\delta_k = 0$ otherwise. Here and elsewhere we observe the convention that products like $\lambda_{k+1} \cdots \lambda_n = 1$ when $k = n$. Denote the triangular matrix $(\lambda_{n,k})$ by $(\lambda; f)$. This is called a generalized Hausdorff matrix. The set of all generalized Hausdorff matrices associated with λ is denoted by $\mathscr{H}_\lambda$.

For α real, the generalized Hölder matrix H_α is defined to be the matrix $(\lambda; f)$ with

$$f(z) = (z + 1)^{-\alpha}.$$

For $\alpha > -1$, the generalized Cesàro matrix C_α is defined to be the matrix $(\lambda; f)$ with

$$f(z) = \frac{\Gamma(\alpha + 1)\,\Gamma(z + 1)}{\Gamma(z + \alpha + 1)}.$$

These reduce to the standard Hölder and Cesàro matrices when $\lambda_n = n$.

For $0 < t \leqq 1$, let $\lambda_{n,k}(t)$ denote the value of $\lambda_{n,k}$ obtained from (3) with $f(z) = t^z$, and let $\lambda_{n,k}(0) = \lambda_{n,k}(0+)$.

Let

$$D_0 = (1 + \lambda_0)\,d_0 = 1; \; D_n = \left(1 + \frac{1}{\lambda_1}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n)\,d_n \text{ for } n \geqq 1. \tag{4}$$

Then, for $n \geqq 0$,

$$D_n = 1 - d_0 + \sum_{k=0}^{n} d_k. \tag{5}$$

It is easily seen that if $\lambda_j + \alpha > 0$ for $k \leqq j \leqq n$ and Γ is a positively sensed circle enclosing $\lambda_k, \ldots, \lambda_n$ and lying to the right of $-\alpha$, then

$$\int_0^1 t^{\alpha-1}\,dt \frac{1}{2\pi i} \int_\Gamma \frac{t^z\,dz}{(\lambda_k - z) \cdots (\lambda_n - z)}$$
$$= \frac{1}{2\pi i} \int_\Gamma \frac{dz}{(\alpha + z)(\lambda_k - z) \cdots (\lambda_n - z)} = -\frac{1}{(\lambda_k + \alpha) \cdots (\lambda_n + \alpha)}.$$

It follows that if $\lambda_j + \alpha > 0$ for $k \leqq j \leqq n$, then

$$\int_0^1 t^{\alpha-1} \lambda_{n,k}(t)\, dt = \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + \alpha) \cdots (\lambda_n + \alpha)} \quad \text{for } 0 \leqq k \leqq n, \tag{6}$$

and hence that

$$\int_0^1 \lambda_{n,k}(t)\, dt = \frac{d_k}{D_n} \quad \text{for } 0 \leqq k \leqq n. \tag{7}$$

Further, it is known (see [3]) that

$$0 \leqq \lambda_{n,j}(t) \leqq \sum_{k=0}^{n} \lambda_{n,k}(t) \leqq 1 \quad \text{for } 0 \leqq t \leqq 1,\ 0 \leqq j \leqq n. \tag{8}$$

Also it is evident that if

$$f(z) = \int_0^1 t^z\, d\chi(t)$$

where $\chi \in BV$, the space of functions of bounded variation on [0, 1], then

$$\lambda_{n,k} = \int_0^1 \lambda_{n,k}(t)\, d\chi(t). \tag{9}$$

For $X \in \mathscr{H}_\lambda$, we write $|X, \gamma|_\beta$ for $|X, d_n, \gamma|_\beta$ where d_n is given by (4). Lemma 2 in [3] shows that if $X = (\lambda; g)$ and $Y = (\lambda; h)$ where g and h are holomorphic in Ω and defined at λ_0, then

$$XY = (\lambda; gh). \tag{10}$$

It follows from (10) that $C_1^{-1} \in \mathscr{H}_\lambda$ and hence that C_1^{-1} commutes with any matrix in $\mathscr{H}_\lambda$. Further, since

$$\frac{1}{z+1} = \int_0^1 t^z\, dt,$$

it follows from (7) and (9) that

$$C_1(s_n) = \frac{1}{D_n} \sum_{k=0}^{n} d_k s_k,$$

and hence that

$$C_1^{-1}(s_n) = s_n + \lambda_n a_n$$

where $s_n = \sum_{k=0}^{n} a_k$. It is now easy to show (as in [1], p. 126) that if $X \in \mathscr{H}_\lambda$, then

$$X(\lambda_n a_n) = \lambda_n(\sigma_n - \sigma_{n-1})$$

where $\sigma_n = X(s_n)$, $\sigma_{-1} = 0$.

Consequently, in view of [2] and [4], for $X \in \mathscr{H}_\lambda$, $\sum_{n=0}^{\infty} a_n$ is summable $|X, \gamma|_\beta$ if and only if $\sum_{n=1}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |X(\lambda_n a_n)|^\beta < \infty$.

Our primary object is to prove four theorems which generalize results involving ordinary Hausdorff matrices (i.e., $\lambda_n = n$) due to Borwein ([1], Theorems 6, 9, 11 and Proposition (VI) (i)).

Preliminary results.

Lemma 1. *If* (X, f) *and* $(\tilde{X}, \tilde{f})$ *are members of* $\mathscr{H}_\lambda$ *with*

$$f(z) = \int_0^1 t^z \, d\chi(t) \quad \text{and} \quad \tilde{f}(z) = \int_0^1 t^z \, |d\chi(t)|$$

where $\chi \in BV$, *and if* $\beta \geqq 1$, *then, for any sequence* $\{w_n\}$,

$$|X(w_n)|^\beta \leqq M^{\beta-1} \tilde{X}(|w_n|^\beta)$$

where $M = \int_0^1 |d\chi(t)|$.

Proof. Let $X = (\lambda_{n,k})$ and $\tilde{X} = (\tilde{\lambda}_{n,k})$. Then, by Hölder's inequality,

$$|X(w_n)|^\beta = \left| \sum_{k=0}^{n} \lambda_{n,k} w_k \right|^\beta \leqq \left(\sum_{k=0}^{n} \tilde{\lambda}_{n,k} \right)^{\beta-1} \sum_{k=0}^{n} \tilde{\lambda}_{n,k} |w_k|^\beta \leqq M^{\lambda-1} \tilde{X}(|w_n|^\beta)$$

in view of (8) and (9).

Lemma 2. *Let* $\alpha \geqq 0$. *If either* $\alpha \leqq 1$ *or* $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$, *then there is a number* $M > 0$ *such that, for* $n \geqq k \geqq 0$,

$$(11) \qquad \left(1 + \frac{1}{\lambda_k + 1}\right)^\alpha \cdots \left(1 + \frac{1}{\lambda_n}\right)^\alpha \leqq M \left(1 + \frac{\alpha}{\lambda_k + 1}\right) \cdots \left(1 + \frac{\alpha}{\lambda_n}\right).$$

Proof. If $\alpha = 0$ or $\alpha = 1$, (11) is true as an equality with $M = 1$. If $0 < \alpha < 1$, a simple calculus argument shows that

$$\left(1 + \frac{1}{\lambda_n}\right)^\alpha \leqq 1 + \frac{\alpha}{\lambda_n}$$

so that (11) holds with $M = 1$. Finally, if $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$, then (11) follows from the order relation

$$\left(1 + \frac{1}{\lambda_n}\right)^\alpha \left(1 + \frac{\alpha}{\lambda_n}\right)^{-1} = 1 + O(\lambda_n^{-2}),$$

and this completes the proof.

We now introduce the notation:

$$\lambda_{n,k}^*(t) = \begin{cases} \lambda_{0,0}(t) & \text{for } n = k = 0, \\ \dfrac{\lambda_k}{\lambda_n} \lambda_{n,k}(t) & \text{for } 0 \leqq k \leqq n, \quad n \geqq 1. \end{cases}$$

It is known ([2], Lemma 2) that,

$$\sum_{n=k}^{\infty} \lambda_{n,k}^*(t) \leqq 1 \quad \text{for } 0 \leqq t \leqq 1, \quad k \geqq 0. \tag{12}$$

Lemma 3. *Suppose that $\alpha \geqq 0$ and that either $\alpha \leqq 1$ or $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$. Then there is a number M such that, for $0 \leqq t \leqq 1$, $k \geqq 0$,*

$$\sum_{n=k}^{\infty} \lambda_{n,k}^*(t)\, t^{\alpha} \left(\frac{D_n}{D_k}\right)^{\alpha} \leqq M.$$

Proof. It follows from (2), (11), and (12) that, for $0 \leqq t \leqq 1$, $k \geqq 0$,

$$\sum_{n=k}^{\infty} \lambda_{n,k}^*(t)\, t^{\alpha} \left(\frac{D_n}{D_k}\right)^{\alpha} = \sum_{n=k}^{\infty} \lambda_{n,k}^*(t)\, t^{\alpha} \left(1 + \frac{1}{\lambda_{k+1}}\right)^{\alpha} \cdots \left(1 + \frac{1}{\lambda_n}\right)^{\alpha}$$

$$\leqq M \sum_{n=k}^{\infty} \frac{\lambda_k + \alpha}{\lambda_n + \alpha} \lambda_{n,k}(t)\, t^{\alpha} \left(1 + \frac{\alpha}{\lambda_{k+1}}\right) \cdots \left(1 + \frac{\alpha}{\lambda_n}\right) \leqq M.$$

Main results.

Theorem 1. *Let $\chi \in BV$, let $\beta \geqq 1$ and let $X = (\lambda; f)$ where*

$$f(z) = \int_0^1 t^z \, d\chi(t).$$

Suppose that

$$\int_0^1 t^{-\gamma} \, |d\chi(t)| < \infty. \tag{13}$$

If either $\gamma\beta \leqq 1$ or $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$, then

$$\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} \, |X(\lambda_n a_n)|^{\beta} \leqq M \sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} \, |\lambda_n a_n|^{\beta} \tag{i}$$

where M is a constant independent of the sequence $\{a_n\}$ and

$$r = \begin{cases} 0 & \text{if } \lambda_0 > 0, \\ 1 & \text{if } \lambda_0 = 0, \end{cases}$$

and

(ii) $$|Q, d_n, \gamma|_\beta \Rightarrow |XQ, d_n, \gamma|_\beta \quad \text{for any matrix } Q.$$

(Note that condition (13) is redundant when $\gamma \leqq 0$).

Proof of (i). Let

$$S = \sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |\lambda_n a_n|^\beta.$$

Suppose first that $\gamma < 0$. By Lemma 1 and (12)

$$\begin{aligned}\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |X(\lambda_n a_n)|^\beta &= M_1^{\beta-1} \int_0^1 |d\chi(t)| \sum_{k=r}^{\infty} |\lambda_k a_k|^\beta \sum_{n=k}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} \lambda_{n,k}(t)\\ &\leqq M_1^{\beta-1} \int_0^1 |d\chi(t)| \sum_{k=r}^{\infty} D_k^{\gamma\beta} \lambda_k^{-1} |\lambda_k a_k|^\beta \sum_{n=k}^{\infty} \lambda_k \lambda_n^{-1} \lambda_{n,k}(t)\\ &\leqq M_1^\beta S\end{aligned}$$

where $M_1 = \int_0^1 |d\chi(t)|$.

Suppose now that $\gamma \geqq 0$ and $0 \leqq t \leqq 1$. Let

$$f_n(t) = \sum_{k=0}^{n} \lambda_{n,k}(t) \lambda_k a_k.$$

By Hölder's inequality and (8)

(14) $$|f_n(t)|^\beta \leqq \sum_{k=0}^{n} \lambda_{n,k}(t) |\lambda_k a_k|^\beta \left(\sum_{k=0}^{n} \lambda_{n,k}(t)\right)^{\beta-1} \leqq \sum_{k=0}^{n} \lambda_{n,k}(t) |\lambda_k a_k|^\beta.$$

Hence

(15) $$\begin{aligned}t^{\gamma\beta} \sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |f_n(t)|^\beta &= t^{\gamma\beta} \sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} \sum_{k=r}^{n} \lambda_{n,k}(t) |\lambda_k a_k|^\beta\\ &= \sum_{k=r}^{\infty} D_k^{\gamma\beta} \lambda_k^{-1} |\lambda_k a_k|^\beta \sum_{n=k}^{\infty} \left(\frac{D_n}{D_k}\right)^{\gamma\beta} \lambda_k \lambda_n^{-1} \lambda_{n,k}(t)\, t^{\gamma\beta}\\ &\leqq M_2 S\end{aligned}$$

by Lemma 3, M_2 being a constant independent of $\{a_n\}$. It follows, by a form of Minkowski's inequality, that

$$\begin{aligned}\left(\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |X(\lambda_n a_n)|\right)^{1/\beta} &= \left(\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} \left|\int_0^1 f_n(t)\, d\chi(t)\right|^\beta\right)^{1/\beta}\\ &\leqq \int_0^1 \left(\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |f_n(t)|^\beta\right)^{1/\beta} |d\chi(t)|\\ &\leqq (M_2 S)^{1/\beta} \int_0^1 t^{-\gamma} |d\chi(t)|\end{aligned}$$

and this completes the proof of (i).

Proof of (ii). In view of (2) and (4), it follows from (i) that $|I, d_n, \gamma|_\beta \Rightarrow |X, d_n, \gamma|_\beta$ where I is the identity matrix. Result (ii) is an immediate consequence.

Theorem 2. *Let* $\alpha > \beta \geqq 1, \frac{1}{p} = 1 + \frac{1}{\alpha} - \frac{1}{\beta}, \gamma \geqq 0$, *and let* $X = (\lambda; f)$ *where*

$$f(z) = \int_0^1 t^z \phi(t)\, dt$$

with $\phi(t) \in L(0, 1)$ *and* $t^{1-\gamma-1/p}\phi(t) \in L^p(0, 1)$. *If either* $0 \leqq \gamma\beta \leqq 1$ *or* $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$, *then*

$$\text{(i)} \qquad \left(\sum_{n=r}^{\infty} D_n^{\gamma\alpha} \lambda_n^{-1} |X(\lambda_n a_n)|^\alpha\right)^{1/\alpha} \leqq M\left(\sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |\lambda_n a_n|^\beta\right)^{1/\beta}$$

where M *is a constant independent of the sequence* $\{a_n\}$ *and*

$$r = \begin{cases} 0 & \text{if } \lambda_0 > 0, \\ 1 & \text{if } \lambda_0 = 0, \end{cases}$$

and

$$\text{(ii)} \qquad |Q, d_n, \gamma|_\beta \Rightarrow |XQ, d_n, \gamma|_\alpha \quad \textit{for many matrix } Q.$$

Proof of (i). Let $0 \leqq t \leqq 1$ and let $S, f_n(t)$ be defined as in the proof of Theorem 1 (i). The symbols M, M_1, M_2 will be used to denote positive numbers independent of n, t and the sequence $\{a_n\}$.

It follows from (14), (6) and Lemma 2 that

$$\text{(16)} \qquad \begin{aligned} D_n^{\gamma\beta} \int_0^1 t^{\gamma\beta-1} |f_n(t)|^\beta dt &\leqq D_n^{\gamma\beta} \sum_{k=r}^{n} |\lambda_k a_k|^\beta \int_0^1 t^{\gamma\beta-1} \lambda_{n,k}(t)\, dt \\ &\leqq D_n^{\gamma\beta} \sum_{k=r}^{n} |\lambda_k a_k|^\beta \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + \gamma\beta) \cdots (\lambda_n + \gamma\beta)} \\ &\leqq M_1 \sum_{k=r}^{n} |\lambda_k a_k|^\beta D_k^{\gamma\beta} (\lambda_k + \gamma\beta)^{-1} \leqq M_1 S. \end{aligned}$$

Now let $c = 1 - \gamma - \frac{1}{p}$, $\psi(t) = t^c \phi(t)$, and $K = \int_0^1 |\psi(t)|^p dt$. By hypothesis K is finite, and an application of Hölder's inequality yields

$$\begin{aligned} |X(\lambda_n a_n)| &= \left|\int_0^1 \psi(t)\, t^{-c} f_n(t)\, dt\right| \\ &\leqq K^{1-1/\beta} \left(\int_0^1 t^{\gamma\beta-1} |f_n(t)|^\beta dt\right)^{1/\beta - 1/\alpha} \left(\int_0^1 |\psi(t)|^p t^{\gamma\beta} |f_n(t)|^\beta dt\right)^{1/\alpha}. \end{aligned}$$

Hence, for $n \geqq r$,

$$\begin{aligned} D_n^{\gamma\alpha} \lambda_n^{-1} |X(\lambda_n a_n)|^\alpha &\leqq K^{\alpha - \alpha/\beta} \left(D_n^{\gamma\beta} \int_0^1 t^{\gamma\beta-1} |f_n(t)|^\beta dt\right)^{\alpha/\beta - 1} \\ &\quad \cdot \int_0^1 |\psi(t)|^p t^{\gamma\beta} D_n^{\gamma\beta} \lambda_n^{-1} |f_n(t)|^\beta dt. \end{aligned}$$

In view of (15) and (16), it follows that

$$\sum_{n=r}^{\infty} D_n^{\gamma\alpha} \lambda_n^{-1} |X(\lambda_n a_n)|^\alpha \leqq K^{\alpha-\alpha/\beta}(M_1 S)^{\alpha/\beta-1} \int_0^1 |\psi(t)|^p t^{\gamma\beta} dt \sum_{n=r}^{\infty} D_n^{\gamma\beta} \lambda_n^{-1} |f_n(t)|^\beta$$
$$\leqq K^{\alpha-\alpha/\beta}(M_1 S)^{\alpha/\beta-1} K M_2 S = M S^{\alpha/\beta}$$

and this establishes (i).

Proof of (ii). It follows from (i) that $|I, d_n, \gamma|_\beta \Rightarrow |X, d_n, \gamma|_\alpha$, and (ii) is an immediate consequence.

Theorem 3. *Let $\beta \geqq 1$, $\alpha > -1$ and suppose that either $\gamma\beta \leqq 1$ or $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$.*

(i) *If $\gamma < \min(1, 1+\alpha)$, then $|C_\alpha, \gamma|_\beta \Rightarrow |H_\alpha, \gamma|_\beta$.*

(ii) *If $\gamma < 1$ or $\alpha = 2, 3, \ldots$ and $\gamma < 2$, then $|H_\alpha, \gamma|_\beta \Rightarrow |C_\alpha, \gamma|_\beta$.*

Proof. Let

$$w(z) = \frac{(z+1)^{-\alpha} \Gamma(z+\alpha+1)}{\Gamma(\alpha+1)\Gamma(z+1)}.$$

It is known (see [1], p. 131) that

$$w(z) = \int_0^1 t^z d\chi_1(t) \quad \text{and} \quad 1/w(z) = \int_0^1 t^z d\chi_2(t)$$

where $\chi_1, \chi_2 \in BV$,

$$\int_0^1 t^{-c} |d\chi_1(t)| < \infty \qquad \text{if } c < \min(1, 1+\alpha),$$

and

$$\int_0^1 t^{-c} |d\chi_2(t)| < \infty \text{ if } c < 1 \text{ or } \alpha = 2, 3, \ldots \text{ and } c < 2.$$

Let $X = (\lambda; w)$ and $Y = (\lambda; 1/w)$. Then $X C_\alpha = H_\alpha$ and $Y H_\alpha = C_\alpha$. Hence, by Theorem 1, if $\gamma < \min(1, 1+\alpha)$, then $|C_\alpha, \gamma|_\beta \Rightarrow |X C_\alpha, \gamma|_\beta$, and if $\gamma < 1$ or $\alpha = 2, 3, \ldots$ and $\gamma < 2$, then $|H_\alpha, \gamma|_\beta \Rightarrow |Y H_\alpha, \gamma|_\beta$. This completes the proof.

Theorem 4. *Let $\alpha \geqq \beta \geqq 1$, $\varrho > \frac{1}{\alpha} - \frac{1}{\beta}$, $\delta + 1 > \gamma \geqq 0$. If either $0 \leqq \gamma\beta \leqq 1$ or $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$ then*

$$|C_\delta Q, \gamma|_\beta \Rightarrow |C_{\delta+\varrho} Q, \gamma|_\alpha \quad \textit{for any matrix } Q.$$

Proof. In view of (10) we have

$$C_{\delta+\varrho} = C_{\delta+\varrho} C_\delta^{-1} C_\delta = X C_\delta$$

where $X = (\lambda; f)$ with

$$f(z) = \frac{\Gamma(\delta+\varrho+1)\Gamma(z+1)}{\Gamma(z+\delta+\varrho+1)} \cdot \frac{\Gamma(z+\delta+1)}{\Gamma(\delta+1)\Gamma(z+1)} = \int_0^1 t^z \phi(t)\, dt$$

and

$$\phi(t) = \frac{\Gamma(\delta + \varrho + 1)}{\Gamma(\varrho)\,\Gamma(\delta + 1)}\, t^{\delta}(1 - t)^{\varrho - 1}.$$

Suppose first that $\alpha = \beta$. Then, since $\delta - \gamma > -1$, we see that $t^{-\gamma}\phi(t) \in L(0, 1)$, and so by Theorem 1 (ii), $|C_\delta, \gamma|_\alpha \Rightarrow |C_{\delta+\varrho}, \gamma|_\alpha$. The required result is an immediate consequence.

Suppose now that $\alpha > \beta$ and let $\frac{1}{p} = 1 + \frac{1}{\alpha} - \frac{1}{\beta}$. Then $p(\varrho - 1) > -1$ and $p\left(\delta + 1 - \gamma - \frac{1}{p}\right) > -1$, so that $\phi(t) \in L(0, 1)$ and $t^{1-\gamma-1/p}\phi(t) \in L^p(0, 1)$. Hence, by Theorem 2 (ii), $|C_\delta, \gamma|_\beta \Rightarrow |C_{\delta+\varrho}, \gamma|_\alpha$ and again the required result follows.

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