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Nörlund matrices as bounded operators on l_p

By

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Introduction. Given a sequence $a = \{a_n\}$, the Nörlund matrix $N_a = \{a_{nk}\}$ is defined by

$$a_{nk} = \begin{cases} a_{n-k}/A_n & \text{for} \quad 0 \leqq k \leqq n \\ 0 & \text{for} \quad k > n, \end{cases}$$

where $A_n = \sum_{k=0}^{n} a_k$. The N_a-transform $y = \{y_n\}$ of the sequence $x = \{x_n\}$ is given by

$$y_n = \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Nörlund matrices have been extensively studied in summability theory. It is familiar that N_a is regular, and hence may be regarded as a bounded operator on c the space of convergent sequences, if and only if $\sum_{k=0}^{n} |a_k| = O(|A_n|)$ and $a_n/A_n \to 0$. Thus, if N_a is regular, either

(1) $$\sum_{k=0}^{\infty} |a_k| < \infty,$$

or

(2) $$|A_n| \to \infty.$$

For $1 \leqq p < \infty$, every bounded operator on l_p has a matrix representation. There is, however, a paucity of information about whether specific matrices are bounded operators on l_p when $1 < p < \infty$. This paper addresses the question for Nörlund matrices.

If (1) is satisfied, nothing essential is altered by taking the N_a-transform $y = \{y_n\}$ of the sequence $x = \{x_n\}$ to be defined by

(3) $$y_n = \sum_{k=0}^{n} a_{n-k} x_k \quad \text{for } n = 0, 1, 2, \ldots;$$

N_a is then the matrix of the analytic Toeplitz operator T_φ, with $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$, acting on the Hardy space H^2. See for example [7, p. 135 ff.]. It is shown below that when (1)

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holds, N_a, as defined by (3), is a bounded operator on l_p for $1 \leqq p \leqq \infty$; the spectra of N_a are determined and estimates of the operator norms are obtained.

When (2) is satisfied, the only results of a general nature concerning Nörlund matrices as bounded operators on l_p involve the Cesàro matrix C_α. It follows from an inequality of Hardy's (in a paper about Hausdorff matrices) that, for $p > 1$ and $\alpha > 0$, C_α is a bounded operator on l_p with norm $\Gamma(1+\alpha)\Gamma(1-1/p)/\Gamma(1+\alpha-1/p)$. See [8] and [9, p. 273 ff.]. Cesàro matrices are the only ones that are both Hausdorff and Nörlund [1]. The results obtained in this paper concerning N_a as a bounded operator on l_p when (2) holds, are detailed in Theorem 2 below; some discussion about the scope of the theorem is also given in the commentary following the statement of Theorem 3.

Borwein and Jakimovski have established results about generalized Hausdorff matrices as bounded operators l_p. See [2] and [3]. Other results about special matrices as bounded operators on l_p appear in [4], [5] and [10].

Results. For $1 \leqq p \leqq \infty$, let $B(l_p)$ denote the Banach algebra of bounded linear operators on l_p. For $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$, let $\|\varphi\|_\infty = \sup_{|z| \leqq 1} |\varphi(z)|$. Let $D = \{z \in C : |z| \leqq 1\}$. The spectrum of N_a in $B(l_p)$ is denoted by $\sigma_p(N_a)$, and $\|N_a\|_p$ is the norm of N_a in $B(l_p)$.

Theorem 1. *Let $1 \leqq p \leqq \infty$ and $\|a\|_1 = \sum_{k=0}^{\infty} |a_k| < \infty$. Then $N_a \in B(l_p)$, $\sigma_p(N_a) = \varphi(D)$ and $\|\varphi\|_\infty \leqq \|N_a\|_p \leqq \|a\|_1$.*

Theorem 2. *Let $1 < p < \infty$, $a_0 > 0$ and $a_n \geqq 0$ for $n = 1, 2, \ldots$. Suppose that $(n+1)a_n \leqq HA_n$, where H is a positive number. Then $N_a \in B(l_p)$ and*

$$\|N_a\|_p \leqq 2^{1/pq}(qH+1) \leqq 2^{1/4}(qH+1)$$

where $1/p + 1/q = 1$.

Theorem 3. *Let $1 < p < \infty$, $0 < \alpha < 1$ and $a_n = e^{n^\alpha}$. Then $N_a \notin B(l_p)$.*

Concerning the estimates of the norms of N_a in Theorem 1, it is known [7, Problem 196, Corollary 1] that $\|N_a\|_2 = \|\varphi\|_\infty$, and, by standard results in summability theory, that $\|N_a\|_1 = \|N_a\|_\infty = \|a\|_1$. If $a_k \geqq 0$ for $k = 0, 1, 2, \ldots$, then $\|N_a\|_p = \|a\|_1 = \|\varphi\|_\infty$ for $1 \leqq p \leqq \infty$. The Riesz convexity theorem [6, p. 525] implies that $\|N_a\|_{1/t}$, as a function of t, is convex and hence continuous on the interval [0, 1]. Consequently, $\|N_a\|_p$ is non-increasing for $1 \leqq p \leqq 2$ and non-decreasing for $2 \leqq p \leqq \infty$. That it is possible to have $\|\varphi\|_\infty < \|a\|_1$ is shown by the example $\varphi(z) = z^2 - z - 1$, for which $\|\varphi\|_\infty = \sqrt{5}$ and $\|a\|_1 = 3$.

The conditions of Theorem 2 are satisfied by the Cesàro matrices C_α for $\alpha > 0$. Since $\|C_1\|_p = q$, the norm estimate is not sharp. The conditions of Theorem 2 are also satisfied, with $H = 1$, if $a_n \geqq a_{n+1}$ for $n = 0, 1, 2, \ldots$, so the Nörlund matrix N_a with $a_n = 1/(n+1)$ belongs to $B(l_p)$ for $p > 1$. Since for a matrix $\{a_{nk}\}$ to belong to $B(l_p)$ it is necessary that $\sum_{n=0}^{\infty} |a_{nk}|^p < \infty$, the weighted mean matrix M_a with $a_n = 1/(n+1)$ does not belong to $B(l_p)$ for $p > 1$. ($M_a = \{a_{nk}\}$ where $a_{nk} = a_k/A_n$ for $0 \leqq k \leqq n$ and $a_{n,k} = 0$ for $k > n$.) On the other hand, since $M_a \in B(l_p)$ for $p > 1$ whenever the sequence a is non-decreasing (see [5] and [2]), Theorem 3 yields a case where $N_a \notin B(l_p)$ but $M_a \in B(l_p)$. It is interesting to note for $a_n = 1/(n+1)$ and $b_n = e^{\sqrt{n}}$, that $N_a \subset C_\alpha \subset N_b$ in the sense that the C_α-transform of a sequence converges whenever the N_a-transform converges and the N_b-transform converges whenever the C_α-transform does. See [9, p. 109f.].

Since $\alpha A_n \sim n^{1-\alpha} e^{n^\alpha}$ when $a_n = e^{n^\alpha}$, Theorem 3 shows that the condition $(n+1)a_n \leqq HA_n$ in Theorem 2 cannot be replaced by the condition $(n+1)^{1-\gamma} a_n \leqq HA_n$ for any $\gamma \in (0, 1)$. Moreover, in order that $N_a \in B(l_p)$ for $1 \leqq p \leqq \infty$, it is necessary that $\sum_{n=0}^{\infty} |a_n/A_n|^p < \infty$; if in addition $\{|a_n/A_n|\}$ is monotonic, this entails $(n+1)^{1/p} a_n = o(A_n)$.

Proofs of Theorems.

Proof of Theorem 1. For $1 \leqq p < \infty$, Hölder's inequality yields

$$|y_n|^p \leqq \left\{\sum_{k=0}^{n} |a_{n-k}||x_k|\right\}^p \leqq \left\{\sum_{k=0}^{n} |a_{n-k}||x_k|^p\right\} \|a\|_1^{p-1}.$$

Hence

$$\sum_{n=0}^{\infty} |y_n|^p \leqq \|a\|_1^{p-1} \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_{n-k}||x_k|^p = \|a\|_1^p \sum_{k=0}^{\infty} |x_k|^p.$$

When $p = \infty$ it is clear that

$$\sup_{n \geqq 0} |y_n| \leqq \|a\|_1 \sup_{n \geqq 0} |x_n|.$$

Thus $N_a \in B(l_p)$ and $\|N_a\|_p \leqq \|a\|_1$ for $1 \leqq p \leqq \infty$.

Let S denote the matrix of the unilateral shift. So $S = N_\tau$ where $\tau = \{0, 1, 0, 0, \ldots\}$. It is standard that $\sigma_p(S) = D$. Let

$$\varphi_r(z) = \varphi(rz) = \sum_{k=0}^{\infty} a_k r^k z^k.$$

Suppose now $0 < r < 1$. Then φ_r is holomorphic in a neighbourhood of $\sigma_p(S)$, so that, by the spectral mapping theorem,

$$\sigma_p(\varphi_r(S)) = \varphi_r(D).$$

Moreover, if $a_{(r)} = \{a_k r^k\}$, then $\varphi_r(S) = N_{a_{(r)}}$ and

$$\|a_{(r)}\|_1 \geqq \|\varphi_r(S)\|_p \geqq \sup\{|\lambda| : \lambda \in \varphi_r(D)\} = \|\varphi_r\|_\infty. \tag{4}$$

Now

$$\big| \|N_a\|_p - \|\varphi_r(S)\|_p \big| \leqq \|N_a - N_{a_{(r)}}\|_p = \|N_{a-a_{(r)}}\|_p \leqq \|a - a_{(r)}\|_1$$
$$= \sum_{k=0}^{\infty} (1 - r^k)|a_k| \to 0 \quad \text{as} \quad r \to 1^-$$

by Abel's theorem. Thus, letting $r \to 1^-$ in (4), it follows that

$$\|a\|_1 \geqq \|N_a\|_p \geqq \|\varphi\|_\infty.$$

It remains to show that $\sigma_p(N_a) = \varphi(D)$. Let $0 \notin \varphi(D)$. The Wiener-Levy theorem [11, p. 401, Ex. 8.] implies that N_a is invertible, so $0 \notin \sigma_p(N_a)$. Hence, if $\lambda \notin \varphi(D)$, then $0 \notin \sigma_p(N_{a-\lambda})$, so that $\lambda \notin \sigma_p(N_a)$. Thus

$$(5) \qquad \sigma_p(N_a) \subset \varphi(D).$$

On the other hand, if $\lambda \notin \sigma_p(N_a)$, then $N_a - \lambda$ is invertible in $B(l_p)$. Since the set of invertible elements of any Banach algebra is open, $N_{a(r)} - \lambda$ is invertible for r near 1. Hence, for some $r_0 \in (0, 1)$,

$$\lambda \notin \bigcup_{r_0 < r < 1} \sigma_p(N_{a(r)}) = \bigcup_{r_0 < r < 1} \varphi_r(D) = \varphi(D^0).$$

Thus $\sigma_p(N_a) \supseteq \varphi(D^0)$. Since $\varphi(D) = \overline{\varphi(D^0)}$ and $\sigma_p(N_a)$ is compact, the set inclusion

$$(6) \qquad \sigma_p(N_a) \supset \varphi(D)$$

holds. The desired conclusion follows from (5) and (6).

The proof of Theorem 2 is based on a result established in [3], which is a generalization of the Schur test (see for example [7, p. 22]). This result is stated as Lemma 1.

Lemma 1. *If $A = \{a_{nk}\}$ is a matrix with $a_{nk} > 0$ for $0 \leqq k \leqq n$, $a_{nk} = 0$ for $k > n$, if $b_n > 0$ for $n = 0, 1, 2, \ldots$,*

$$\sup_{n \geqq 0} \sum_{k=0}^{n} a_{nk}(b_k/b_n)^{1/p} = M_1 < \infty$$

and

$$\sup_{k \geqq 0} \sum_{n=k}^{\infty} a_{nk}(b_n/b_k)^{1/q} = M_2 < \infty,$$

where $1/p + 1/q = 1$, then $A \in B(l_p)$ and $\| A \|_p \leqq M_1^{1/q} M_2^{1/p}$.

Proof of Theorem 2. Let $b_n = 1/(n+1)$ and M_1, M_2 be as in Lemma 1. Then with $\delta = 1/p$,

$$\sum_{k=0}^{n} a_{nk}(b_k/b_n)^{\delta} = \frac{(n+1)^{\delta}}{A_n} \sum_{k=0}^{n} \frac{a_{n-k}}{(k+1)^{\delta}}.$$

For some integer m_n with $n/2 \leqq m_n \leqq n$,

$$\begin{aligned} \frac{(n+1)^{\delta}}{A_n} \sum_{0 \leqq k \leqq n/2} \frac{a_{n-k}}{(k+1)^{\delta}} &\leqq \frac{(n+1)^{\delta} a_{m_n}}{A_n} \sum_{0 \leqq k \leqq n/2} \frac{1}{(k+1)^{\delta}} \\ &\leqq \frac{(n+2)\, a_{m_n}}{2^{1-\delta} A_n (1-\delta)} \leqq \frac{2(m_n+1)\, a_{m_n}}{2^{1-\delta} A_{m_n}(1-\delta)} \leqq \frac{2^{\delta} H}{1-\delta}; \end{aligned}$$

and

$$\frac{(n+1)^{\delta}}{A_n} \sum_{n/2 < k \leqq n} \frac{a_{n-k}}{(k+1)^{\delta}} \leqq \frac{(n+1)^{\delta} 2^{\delta}}{(n+2)^{\delta}} \leqq 2^{\delta}.$$

Hence,

$$M_1 \leqq 2^{\delta}(H/(1-\delta) + 1) = 2^{1/p}(qH + 1).$$

Also

$$\sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n} (b_n/b_k)^{1/q} = \sum_{n=k}^{\infty} \frac{a_{n-k}}{A_n} \left(\frac{k+1}{n+1}\right)^{\mu}$$

where $\mu = 1/q$. But,

$$\begin{aligned}(k+1)^{\mu} \sum_{n=2k+1}^{\infty} \frac{a_{n-k}}{A_n (n+1)^{\mu}} &\leqq (k+1)^{\mu} \sum_{n=k+1}^{\infty} \frac{a_n}{A_n (n+1)^{\mu}} \\ &\leqq (k+1)^{\mu} H \sum_{n=k+1}^{\infty} \frac{1}{(n+1)^{\mu+1}} \leqq (k+1)^{\mu} H \int_{k+1}^{\infty} \frac{dx}{x^{\mu+1}} \\ &= qH;\end{aligned}$$

and

$$(k+1)^{\mu} \sum_{n=k}^{2k} \frac{a_{n-k}}{A_n (n+1)^{\mu}} \leqq 1.$$

Hence, $M_2 \leqq qH + 1$. The conclusion now follows from Lemma 1 and the fact that $pq \geqq 4$.

The proof of Theorem 3 uses the easily established fact that, for $0 < \alpha < 1$, $\alpha \sum_{k=0}^{n} e^{k^{\alpha}} \sim n^{1-\alpha} e^{n^{\alpha}}$. The following lemma is also needed.

Lemma 2. *If* $0 < \alpha < 1$ *and* $0 \leqq t \leqq n$, *then* $(n-t)^{\alpha} - n^{\alpha} \geqq -t n^{\alpha-1}$.

Proof. Let $f(t) = (n-t)^{\alpha} - n^{\alpha} + t n^{\alpha-1}$. Then $f(0) = f(n) = 0$ and, for $0 \leqq t \leqq n$, $f''(t) = \alpha(\alpha-1)(n-t)^{\alpha-2} \leqq 0$, so that $f(t) \geqq 0$.

Proof of Theorem 3. With $a_n = e^{n^{\alpha}}$, $b_n = 1/(n+1)$ and $\delta = 1/p$,

$$\begin{aligned}\sum_{k=0}^{n} a_{nk} (b_k/b_n)^{\delta} &= \frac{(n+1)^{\delta}}{A_n} e^{n^{\alpha}} \sum_{k=0}^{n} \frac{e^{(n-k)^{\alpha} - n^{\alpha}}}{(k+1)^{\delta}} \\ &\geqq \frac{(n+1)^{\delta} e^{n^{\alpha}}}{A_n} \sum_{k=0}^{n} e^{-kn^{\alpha-1}} (k+1)^{-\delta} = \frac{(n+1)^{\delta} e^{n^{\alpha}}}{A_n} e^{n^{\alpha-1}} \sum_{k=1}^{n+1} e^{-kn^{\alpha-1}} k^{-\delta} \\ &\geqq \frac{(n+1)^{\delta} e^{n^{\alpha}}}{A_n} e^{n^{\alpha-1}} \int_{1}^{n+2} e^{-tn^{\alpha-1}} t^{-\delta} dt \\ &= \frac{(n+1)^{\delta} e^{n^{\alpha}}}{A_n} e^{n^{\alpha-1}} n^{(1-\delta)(1-\alpha)} \int_{n^{\alpha-1}}^{n^{\alpha}+2n^{\alpha-1}} e^{-v} v^{-\delta} dv \\ &\sim \alpha n^{\alpha\delta} \Gamma(1-\delta) \to \infty.\end{aligned}$$

Theorem 4 in [3] now shows that $N_a \notin B(l_p)$ for $1 < p < \infty$.

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