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Analysis

On Summability Factors for the Strong Cesàro Method for Integrals

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Abstract. The paper is concerned with summability factors for the strong Cesàro method $[C,\lambda]_p$ of summability of integrals. For $p \geqslant 1$, $\lambda > 1 - \frac{1}{p}$, necessary conditions are obtained for a function ϕ to be such that $\int_1^\infty x(u)\phi(u)du$ is bounded (C) whenever $\int_1^\infty x(u)du$ is summable $[C,\lambda]_p$, and it is proved that these conditions are sufficient for $\int_1^\infty x(u)\phi(u)du$ to be summable $[C,\lambda]_p$ whenever $\int_1^\infty x(u)du$ is summable $[C,\lambda]_p$.

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1. Introduction

Let x be a measurable real-valued function defined on $(1,\infty)$ and Lebesgue integrable on finite intervals $(1,t)$ for all $t>1$. For $\lambda>0$, the λ*-th integral* of x is defined as

$$x_\lambda(t) = \frac{1}{\Gamma(\lambda)}\int_1^t (t-u)^{\lambda-1}x(u)du\,,$$

whenever the integral exists (in the Lebesgue sense). It is known that, for $\lambda \geqslant 1$, the integral $x_\lambda(t)$ exists for all $t>1$; that, for $0<\lambda<1$, $x_\lambda(t)$ exists for almost all $t>1$; and that, for $\lambda>0$, $x_\lambda \in L(1,w)$ for all $w>1$. Furthermore, for $\lambda>0$ and $\mu>0$, $(x_\lambda)_\mu(w) = x_{\lambda+\mu}(w)$ whenever the integrals exist (Zaanen [21, pp. 103–106]).

For $\lambda>-1$, the λ*-th Cesàro mean* of $\int_1^t x(u)du$ is defined as

$$m_\lambda x(t) = \Gamma(\lambda+1)t^{-\lambda}x_{\lambda+1}(t) = \int_1^t \left(1-\frac{u}{t}\right)^\lambda x(u)du.$$

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If s is a real number or $\pm\infty$, and if $m_\lambda x(t) \to s$ as $t \to \infty$, we write $\int_1^\infty x(u)du = s(C,\lambda)$. If s is finite, we say that $\int_1^\infty x(u)du$ is summable to s by the *(ordinary) Cesàro method* of order λ, or summable (C,λ) to s. The symbol (C,λ) also denotes the linear space of functions x such that $\int_1^\infty x(u)du$ is summable (C,λ), to some s depending upon x.

If $m_\mu x(t) = O(1)$ as $t \to \infty$, for some $\mu > -1$, we say that $\int_1^\infty x(u)du$ is bounded (C). The symbol $B(C)$ denotes the linear space of functions x such that $\int_1^\infty x(u)du$ is bounded (C).

If $\lambda > 0$, $p \geqslant 1$, s is real, and $\frac{1}{w}\int_1^w |m_{\lambda-1}x(t) - s|^p\, dt = o(1)$ as $w \to \infty$, we say that $\int_1^\infty x(u)du$ is summable to s by the *strong Cesàro method* of order λ and index p, or summable $[C,\lambda]_p$ to s, and write $\int_1^\infty x(u)du = s[C,\lambda]_p$. The symbol $[C,\lambda]_p$ also denotes the linear space of functions x such that $\int_1^\infty x(u)du$ is summable $[C,\lambda]_p$, to some s depending on x.

If P and Q are summability spaces such as (C,λ), $B(C)$, and $[C,\lambda]_p$, and ϕ is a function defined on $(1,\infty)$, then ϕ is said to be a *summability factor* from P to Q if the product function $x\phi$ is in Q whenever x is in P. The symbol $\{P; Q\}$ denotes the linear space of summability factors from P to Q.

In this paper we are concerned with determining necessary and sufficient conditions for a function ϕ to be a summability factor for the strong Cesàro method $[C,\lambda]_p$. That is, we want to characterize $\phi \in \{[C,\lambda]_p; Q\}$, where Q is one of the spaces $B(C)$, (C,λ), $[C,\lambda]_1$, or $[C,\lambda]_p$. A necessary condition for $\phi \in \{[C,\lambda]_p; Q\}$ is also necessary for $\phi \in \{[C,\lambda]_p; Q'\}$, where $Q' \subseteq Q$, and a sufficient condition for $\phi \in \{[C,\lambda]_p; Q\}$ is also sufficient for $\phi \in \{[C,\lambda]_p; Q''\}$, where $Q \subseteq Q''$.

For $\mu > \lambda > 0$ and $p > p' \geqslant 1$, Lemmas 2.2 and 2.4 (below) show that

$$[C,\lambda]_p \subseteq [C,\lambda]_{p'} \subseteq [C,\lambda]_1 \subseteq (C,\lambda) \subseteq (C,\mu) \subseteq B(C).$$

It thus suffices to determine conditions which are necessary for $\phi \in \{[C,\lambda]_p; B(C)\}$ and sufficient for $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$.

The conditions to be obtained involve quantities $M_n(h,\lambda,p)$ defined as follows. Suppose h is a measurable real-valued function defined on $[1,\infty)$, $\lambda > 0$, $p \geqslant 1$, and $\frac{1}{p} + \frac{1}{q} = 1$ if $p > 1$. Then we define

$$M_n(h,\lambda,p) = \begin{cases} \operatorname*{ess\,sup}_{2^n < v < 2^{n+1}} |v^\lambda h(v)| & \text{if } p = 1, \\ \left(\frac{1}{2^n}\int_{2^n}^{2^{n+1}} |v^\lambda h(v)|^q\, dv\right)^{1/q} & \text{if } p > 1, \end{cases}$$

for all integers $n \geqslant 0$.

Our objective is to prove the following two theorems.

Theorem 1. *If $p \geqslant 1$, $\lambda > 1 - \frac{1}{p}$, and $\phi \in \{[C,\lambda]_p; B(C)\}$, then there exist real numbers $c \geqslant 1$ and b, and a measurable real-valued function h defined on $[1,\infty)$ and vanishing on $[1,c)$, such that*

$$\phi \in L^\infty(1,c), \tag{1.1}$$

$$\sum_{n=0}^\infty M_n(h,\lambda,p) < \infty, \text{ and} \tag{1.2}$$

$$\phi(u) = b + \int_u^\infty (v-u)^{\lambda-1} h(v)dv \text{ for almost all } u > c. \tag{1.3}$$

Moreover, these conditions imply that

$$\phi \in L^\infty(1,\infty), \text{ and} \tag{1.4}$$

there is a function χ defined on $[1,\infty)$ such that $\phi(u) = \chi(u)$ for almost all $u > 1$, and $\chi(u) \to b$ as $u \to \infty$. (1.5)

Theorem 2. *Suppose $p \geqslant 1$, $\lambda > 1 - \frac{1}{p}$, $c \geqslant 1$, b is real, and ϕ and h satisfy* (1.1), (1.2), *and* (1.3). *Then $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$.*

Theorem 1 is proved in Section 4 and Theorem 2 is proved in Section 6.

When $\lambda > 1 - \frac{1}{p}$, Theorems 1 and 2 show that conditions (1.1), (1.2), and (1.3) are necessary and sufficient for $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$, and that $\{[C,\lambda]_p; [C,\lambda]_p\} = \{[C,\lambda]_p; B(C)\}$. The determination of necessary and sufficient conditions for $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$ when $p > 1$, $0 < \lambda \leqslant 1 - \frac{1}{p}$ is still an open problem.

As examples of $[C,\lambda]_p$ summability factors, we mention the following.

(1) Given $\lambda > 0$ and $\alpha > \lambda$, define

$$\phi(u) = \int_u^\infty (v-u)^{\lambda-1} v^{-\alpha} dv = B(\lambda, \alpha-\lambda) u^{\lambda-\alpha} \quad \text{for} \quad u \geq 1,$$

where $B(\lambda, \alpha-\lambda)$ denotes the beta function. Theorem 2 shows that $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$ for all $p \geq 1$ such that $\lambda > 1 - \frac{1}{p}$.

(2) Given $p \geq 1$, $\lambda > 1 - \frac{1}{p}$, and a sequence h_n such that $\sum_{n=0}^\infty |h_n| 2^{n\lambda} < \infty$, define $h(v) = h_n$ for v satisfying $2^n \leq v < 2^{n+1}$ and define $\phi(u) = \int_u^\infty (v-u)^{\lambda-1} h(v) dv$ for $u \geq 1$. Then $\phi \in \{[C,\lambda]_p; [C,\lambda]_p\}$.

For strong Cesàro summability of series, results similar to those of Theorems 1 and 2 have been obtained with restrictions on the values of λ and p. For $\lambda > 0$, Kuttner and Maddox [15] obtained necessary and sufficient conditions, analogous to (1.2) and (1.3), for $\phi_n \in \{[C,\lambda]_1; (C,\lambda)\}$. Also for $\lambda > 0$, Kuttner and Thorpe [16], and Jackson [14] independently, showed that these conditions are necessary and sufficient for $\phi_n \in \{[C,\lambda]_1; [C,\lambda]_1\}$ and hence that $\{[C,\lambda]_1; [C,\lambda]_1\} = \{[C,\lambda]_1; (C,\lambda)\}$. For $p > 1$ and $\lambda \geq 1$, Jackson [14] obtained necessary and sufficient conditions for $\phi_n \in \{[C,\lambda]_p; (C,\lambda)\}$ and, for $p \geq 1$ and $\lambda \geq 1$ an integer, showed that these conditions are sufficient for $\phi_n \in \{[C,\lambda]_p; [C,\lambda]_p\}$. In view of Theorems 1 and 2, it appears feasible to extend Jackson's results to the general case $p \geq 1$, $\lambda > 1 - \frac{1}{p}$. This possibility is not investigated in the present paper.

For results concerning summability factors for the ordinary and absolute Cesàro methods for series, see Bosanquet [6] and the references given there, and Bosanquet and Chow [7]. For results concerning summability factors for the ordinary and absolute Cesàro methods for integrals, see Hardy [11], Cossar [8, 9], Sargent [19], and Borwein [2, 4].

2. Preliminary Results

We require the following lemmas.

Lemma 2.1. *Suppose $\epsilon > 0$ and f is a measurable function defined on $(1,\infty)$ and integrable on finite intervals. Then*

(i) $$w^{-\epsilon} \int_1^w t^{\epsilon-1} |f(t)| dt \leq H \sup_{w>1} \frac{1}{w} \int_1^w |f(t)| dt, \text{ where } H = 1 + \left|1 - \frac{1}{\epsilon}\right|.$$

(ii) *If* $\frac{1}{w} \int_1^w |f(t)| dt = o(1)$ *as* $w \to \infty$, *then* $w^{-\epsilon} \int_1^w t^{\epsilon-1} |f(t)| dt = o(1)$ *as* $w \to \infty$.

Proof. Let $I(w) = w^{-\epsilon} \int_1^w t^{\epsilon-1} |f(t)| dt$, $F(t) = \int_1^t |f(u)| du$. Then

$$I(w) = \frac{1}{w} F(w) + (1-\epsilon) w^{-\epsilon} \int_1^w t^{\epsilon-2} F(t) dt.$$

(i) Let $M = \sup_{t>1} \frac{1}{t} F(t)$. Then

$$I(w) \leq M + |1-\epsilon| M w^{-\epsilon} \int_1^w t^{\epsilon-1} dt = M\left\{1 + \frac{|1-\epsilon|}{\epsilon}(1 - w^{-\epsilon})\right\} \leq HM.$$

(ii) Suppose $F(t) = o(t)$. Then

$$I(w) = o(1) + w^{-\epsilon} \int_1^w t^{\epsilon-1} o(1) dt = o(1).$$

Note. Here and in the sequel, when we say that a function is $o(t)$, $O(1)$, etc. we are referring to the behaviour of the function as the appropriate variable tends to ∞. The variable is explicitly identified only when there is a possibility of confusion.

For a series analogue of the following lemma, see Hyslop [13, Theorem 3].

Lemma 2.2. *Suppose $\lambda > 0$, $p \geq 1$. Then*

$$\int_1^\infty x(u) du = s[C,\lambda]_p \text{ if and only if } \int_1^\infty x(u) du = s(C,\lambda) \text{ and}$$

$$\frac{1}{w} \int_1^w \left| t^{-\lambda} \int_1^t (t-u)^{\lambda-1} u x(u) du \right|^p dt = o(1) \text{ as } w \to \infty. \tag{2.1}$$

Proof. We have the identity

$$m_{\lambda-1} x(t) - m_\lambda x(t) = t^{-\lambda} \int_1^t (t-u)^{\lambda-1} u x(u) du,$$

since $(t-u)^\lambda = (t-u)^{\lambda-1} t - (t-u)^{\lambda-1} u$.

If (2.1) holds and $\int_1^\infty x(u) du = s(C,\lambda)$, then by Minkowski's inequality,

$$\left(\frac{1}{w} \int_1^w |m_{\lambda-1} x(t) - s|^p dt\right)^{1/p} \leq \left(\frac{1}{w} \int_1^w |m_\lambda x(t) - s|^p dt\right)^{1/p}$$
$$+ \left(\frac{1}{w} \int_1^w |t^{-\lambda} \int_1^t (t-u)^{\lambda-1} u x(u) du|^p dt\right)^{1/p}$$
$$= o(1) + o(1) = o(1).$$

Hence $\int_1^\infty x(u)\,du = s[C,\lambda]_p$.

To prove the converse, suppose $\int_1^\infty x(u)\,du = s[C,\lambda]_p$. If $p > 1$, Hölder's inequality implies

$$\frac{1}{w}\int_1^w |f(t)|\,dt \leqslant \left(\frac{1}{w}\int_1^w |f(t)|^p\,dt\right)^{1/p}$$

for any $f \in L(1,w)$, and taking $f(t) = m_{\lambda-1}x(t) - s$ gives

$$\frac{1}{w}\int_1^w |m_{\lambda-1}x(t) - s|\,dt = o(1).$$

From the identity

$$m_\lambda x(w) = \frac{1}{\lambda}w^{-\lambda}\int_1^w t^{\lambda-1}m_{\lambda-1}x(t)\,dt,$$

we obtain

$$m_\lambda x(w) - s = \frac{1}{\lambda}w^{-\lambda}\int_1^w t^{\lambda-1}(m_{\lambda-1}x(t) - s)\,dt - \frac{s}{\lambda}w^{-\lambda}\int_0^1 t^{\lambda-1}\,dt.$$

The second term is $o(1)$, and Lemma 2.1 (ii), with $f(t) = m_{\lambda-1}x(t) - s$, shows that the first term is also $o(1)$. Hence $\int_1^\infty x(u)\,du = s\,(C,\lambda)$. Finally, (2.1) follows from

$$\left(\frac{1}{w}\int_1^w |t^{-\lambda}\int_1^t (t-u)^{\lambda-1}ux(u)\,du|^p\,dt\right)^{1/p} \leqslant \left(\frac{1}{w}\int_1^w |m_\lambda x(t) - s|^p\,dt\right)^{1/p}$$
$$+\left(\frac{1}{w}\int_1^w |m_{\lambda-1}x(t) - s|^p\,dt\right)^{1/p}.$$

Lemma 2.3. (Borwein [4]). *Suppose $\lambda > 0$. Then $\int_1^\infty x(u)\,du = s(C,\lambda)$, for some s, if and only if*

$$\int_1^w t^{-\lambda-1}\,dt\int_1^t (t-u)^{\lambda-1}ux(u)\,du \text{ is convergent as } w\to\infty. \tag{2.2}$$

Conditions (2.1) and (2.2) thus constitute a convenient test to determine whether $\int_1^\infty x(u)du$ is summable $[C,\lambda]_p$.

Lemma 2.4. *Suppose $\mu > \lambda > 0$, $p > p' \geqslant 1$.*

(i) *If* $\int_1^\infty x(u)\,du = s[C,\lambda]_p$, *then* $\int_1^\infty x(u)\,du = s[C,\lambda]_{p'}$.

(ii) *If* $\int_1^\infty x(u)\,du = s(C,\lambda)$, *then* $\int_1^\infty x(u)\,du = s(C,\mu)$.

(iii) *If* $\int_1^\infty x(u)\,du = s(C,\lambda)$, *then* $\int_1^\infty x(u)\,du$ *is bounded* (C).

Proof. Part (i) follows from an application of Hölder's inequality with exponent $r = \dfrac{p}{p'}$. (Cf. Flett [10, Theorem 1].)
Part (ii) is proved in Titchmarsh [20, section 1.15].
Part (iii) is obvious.

Lemma 2.5. *Suppose $p \geqslant 1$, $\lambda > 1 - \dfrac{1}{p}$, and $\int_1^\infty x(u)\,du = s(C,\lambda-1)$. Then $\int_1^\infty x(u)\,du = s[C,\lambda]_p$.*

Proof. Suppose without loss of generality that $s = 0$. By Hölder's inequality, we have, for $w > 1$.

$$\int_1^w |m_{\lambda-1}x(t)|^p\,dt \leqslant \left(\int_1^w |x(u)|\,du\right)^{p-1}\int_1^w t^{(1-\lambda)p}\,dt\int_1^t (t-u)^{(\lambda-1)p}|x(u)|\,du$$
$$= \left(\int_1^w |x(u)|\,du\right)^{p-1}\int_1^w |x(u)|\,du\int_u^w t^{(1-\lambda)p}(t-u)^{(\lambda-1)p}\,dt$$
$$< \infty,$$

since $(\lambda-1)p > -1$. It follows from this, and $m_{\lambda-1}x(t) = o(1)$, that $\frac{1}{w}\int_1^w |m_{\lambda-1}x(t)|^p\,dt = o(1)$, which completes the proof of the lemma.

Lemma 2.6. *Suppose $\mu > -1$ and $w > 1$.*

(i) *If x is an integrable function defined on $(1,\infty)$ and vanishing outside $(1,w)$, then $\int_1^\infty x(u)\,du$ is summable (C,μ) to the value of the Lebesgue integral $\int_1^\infty x(u)\,du$.*

(ii) *If x is a non-negative measurable function defined on $(1,\infty)$ and vanishing outside $(1,w)$, and if $\int_1^\infty x(u)\,du = \infty$, then $\int_1^\infty x(u)\,du = \infty\,(C,\mu)$.*

The proof of this lemma is immediate.

Lemma 2.7. *Suppose f is a measurable function defined on $(1,\infty)$ and integrable on finite intervals. If $\int_1^\infty \frac{1}{t}|f(t)|dt<\infty$, then $\frac{1}{w}\int_1^w f(t)dt = o(1)$.*

Proof. Let $g(t)=\frac{1}{t}f(t)$ and $I(w)=\frac{1}{w}\int_1^w tg(t)dt$, and integrate by parts to obtain

$$I(w) = g_1(w) - \frac{1}{w}\int_1^w g_1(t)dt.$$

Since $g\in L(1,\infty)$, $g_1(w)$ tends to $\int_1^\infty g(t)dt$ as $w\to\infty$, and $I(w)$ tends to $\int_1^\infty g(t)dt - \int_1^\infty g(t)dt = 0$.

Lemma 2.8. *Suppose $\gamma>0$, $\delta>0$, $v>0$. Then*

$$\int_v^\infty t^{-\delta-\gamma}(t-v)^{\delta-1}dt = B(\gamma,\delta)v^{-\gamma},$$

where $B(\gamma,\delta)$ denotes the beta function.

The proof is immediate by change of variable $u=\frac{v}{t}$.

Lemma 2.9. *Suppose $0<\epsilon<1$ and $0\leqslant a<b<t$. Then*

(i) $$(t-a)^\epsilon-(t-b)^\epsilon<(b-a)(t-a)^{\epsilon-1},$$

(ii) $$(t-b)^{\epsilon-1}-(t-a)^{\epsilon-1}<\frac{b-a}{t-a}(t-b)^{\epsilon-1}\leqslant\frac{b}{t}(t-b)^{\epsilon-1}, \text{ and}$$

(iii) $$\int_a^b (t-u)^{\epsilon-1}du<\frac{1}{\epsilon}(b-a)(t-a)^{\epsilon-1}.$$

Proof. Part (i) follows from

$$(t-a)^\epsilon-(t-b)^\epsilon<(t-a)(t-a)^{\epsilon-1}-(t-b)(t-a)^{\epsilon-1}.$$

Part (ii) follows from (i) and

$$(t-b)^{\epsilon-1}-(t-a)^{\epsilon-1} = \frac{(t-a)^{1-\epsilon}-(t-b)^{1-\epsilon}}{(t-a)^{1-\epsilon}(t-b)^{1-\epsilon}}.$$

Part (iii) also follows from (i).

Lemma 2.10. *Suppose $\epsilon>0$, $0<\delta<1$, $v>u>0$. Then*

$$\int_u^v t^{-\epsilon-1}(v-t)^{\delta-1}dt < 2^{1-\delta}\left(\frac{1}{\epsilon}+2^\epsilon\frac{1}{\delta}\right)v^{\delta-1}u^{-\epsilon}.$$

Proof. Let $I=\int_u^v t^{-\epsilon-1}(v-t)^{\delta-1}dt$. If $u\geqslant\frac{v}{2}$, then

$$I<2^{\epsilon+1}v^{-\epsilon-1}\int_{v/2}^v (v-t)^{\delta-1}dt = \frac{1}{\delta}2^{\epsilon+1-\delta}v^{\delta-\epsilon-1}<\frac{1}{\delta}2^{\epsilon+1-\delta}v^{\delta-1}u^{-\epsilon}.$$

If $u<\frac{v}{2}$, then

$$\int_u^{v/2} t^{-\epsilon-1}(v-t)^{\delta-1}dt\leqslant 2^{1-\delta}v^{\delta-1}\int_u^{v/2}t^{-\epsilon-1}dt<\frac{1}{\epsilon}2^{1-\delta}v^{\delta-1}u^{-\epsilon}, \text{ so}$$

$$I<2^{1-\delta}\left(\frac{1}{\epsilon}+2^\epsilon\frac{1}{\delta}\right)v^{\delta-1}u^{-\epsilon}.$$

Lemma 2.11. *Suppose $\epsilon>0$ and $\frac{1}{w}\int_1^w |f(u)|du = o(1)$. Then*

$$\frac{1}{w}\int_1^w t^{-\epsilon}dt\int_1^t (t-u)^{\epsilon-1}|f(u)|du = o(1).$$

Proof. $\frac{1}{w}\int_1^w t^{-\epsilon}dt\int_1^t(t-u)^{\epsilon-1}|f(u)|du = \frac{1}{w}\int_1^w|f(u)|du\int_u^w t^{-\epsilon}(t-u)^{\epsilon-1}dt.$

Choosing γ in $0<\gamma<1$ and multiplying by $\frac{w^{1-\gamma}}{t^{1-\gamma}}$, this is

$$\leqslant w^{-\gamma}\int_1^w|f(u)|du\int_u^w t^{-\epsilon-(1-\gamma)}(t-u)^{\epsilon-1}dt$$

$$\leqslant B(\epsilon,1-\gamma)w^{-\gamma}\int_1^w u^{\gamma-1}|f(u)|du, \text{ by Lemma 2.8,}$$

$$= o(1) \text{ by Lemma 2.1 (ii).}$$

The following lemma, due to M. Riesz, is of fundamental importance in calculations involving fractional integrals.

Lemma 2.12. (Riesz [18]). *Suppose $0<\lambda<1$, $1<t<w$, and $f\in L(1,t)$. Then*

$$\Gamma(1-\lambda)\int_1^t(w-u)^{\lambda-1}f(u)du=\lambda\int_1^t f_\lambda(v)dv\int_t^w(w-u)^{\lambda-1}(u-v)^{-\lambda-1}du.$$

Lemma 2.13. *Suppose* $0<\lambda<1$, $\alpha\geqslant 0$, $\beta\geqslant 0$, $\alpha+\beta=1$.

(i) *If* $0<v<w<y$, *then*

$$\int_v^w (y-t)^{\lambda-2}\,dt < (1-\lambda)^{-\beta}(w-v)(y-w)^{\lambda\beta-1}(y-v)^{\lambda\alpha-1}.$$

(ii) *If* $1<u<y$ *and* $f\in L(1,u)$, *then*

$$\left|\int_1^u (y-t)^{\lambda-2}\,dt\int_1^t (u-v)^{\lambda-1}f(v)\,dv\right|$$

$$\leqslant (1-\lambda)^{-\beta}\Gamma(\lambda+1)(y-u)^{\lambda\beta-1}\int_1^u (y-v)^{\lambda\alpha-1}\,|f_\lambda(v)|\,dv.$$

The case $\alpha=\beta=\frac{1}{2}$ of these results is proved in Borwein [3, Lemma 5 and 6], and the proof of the general case is essentially the same.

Suppose $\lambda>0$ and $f\in L(1,t)$. For v satisfying $1<v<t$, let

$$Q(v,t)=\int_1^v (t-u)^{\lambda-1}(v-u)^{\lambda-1}f(u)\,du. \tag{2.3}$$

This integral exists for almost all v satisfying $1<v<t$.

Lemma 2.14. *Suppose* $f\in L(1,t)$, *and let* $Q(v,t)$ *be given by* (2.3) *for any* $\lambda>0$.

(i) *If* $0<\lambda<1$, $\alpha\geqslant 0$, $\beta\geqslant 0$, $\alpha+\beta=1$, *then for almost all* v *satisfying* $1<v<t$,

$$|Q(v,t)-\Gamma(\lambda)(t-v)^{\lambda-1}f_\lambda(v)|$$

$$\leqslant (1-\lambda)^{\alpha}\lambda\Gamma(\lambda)(t-v)^{\lambda\beta-1}\int_1^v (t-u)^{\lambda\alpha-1}\,|f_\lambda(u)|\,du.$$

(ii) *If* $1<\lambda<2$, *then for all* v *satisfying* $1<v<t$,

$$|Q(v,t)-\Gamma(\lambda)(t-v)^{\lambda-1}f_\lambda(v)|\leqslant 2\Gamma(\lambda)\int_1^v (t-u)^{\lambda-2}\,|f_\lambda(u)|\,du.$$

Proof. (Cf. Borwein [3, page 316] and [4, page 205].)

(i) Integrating by parts,

$$Q(v,t)=(t-v)^{\lambda-1}\int_1^v (v-u)^{\lambda-1}f(u)\,du$$

$$+(\lambda-1)\int_1^v (t-u)^{\lambda-2}\,du\int_1^u (v-s)^{\lambda-1}f(s)\,ds.$$

Part (i) follows from Lemma 2.13 (ii).

(ii) Integrating by parts with integrand $f(u)$, we get

$$Q(v,t)=(\lambda-1)\int_1^v (t-u)^{\lambda-2}(v-u)^{\lambda-2}(t+v-2u)f_1(u)\,du.$$

Integrating by parts again with integrand $(v-u)^{\lambda-2}f_1(u)$, we get

$$Q(v,t)=\Gamma(\lambda)(t-v)^{\lambda-1}f_\lambda(v)$$

$$+(\lambda-1)\int_1^v (t-u)^{\lambda-3}\{\lambda(t-u)+(\lambda-2)(v-u)\}\,du\int_1^u (v-s)^{\lambda-2}f_1(s)\,ds.$$

Applying Lemma 2.12 with λ replaced by $\lambda-1$ to the integral $\int_1^u (v-s)^{\lambda-2}f_1(s)\,ds$, we obtain

$$|Q(v,t)-\Gamma(\lambda)(t-v)^{\lambda-1}f_\lambda(v)|$$

$$=\frac{(\lambda-1)^2}{\Gamma(2-\lambda)}\int_1^v (t-u)^{\lambda-3}\{\lambda(t-u)+(\lambda-2)(v-u)\}\,du$$

$$\cdot\int_1^u f_\lambda(y)\,dy\int_u^v (v-s)^{\lambda-2}(s-y)^{-\lambda}\,ds$$

$$\leqslant 2\frac{(\lambda-1)^2}{\Gamma(2-\lambda)}\int_1^v |f_\lambda(y)|\,dy\int_y^v (v-s)^{\lambda-2}(s-y)^{-\lambda}\,ds\int_y^s (t-u)^{\lambda-2}\,du,$$

since $|\lambda(t-u)+(\lambda-2)(v-u)|<2(t-u)$.

Applying Lemma 2.9 (iii) to the integral $\int_y^s (t-u)^{\lambda-2}du$, we get

$$|Q(v,t)-\Gamma(\lambda)(t-v)^{\lambda-1}f_\lambda(v)|$$

$$\leqslant 2\frac{(\lambda-1)}{\Gamma(2-\lambda)}\int_1^v (t-y)^{\lambda-2}\,|f_\lambda(y)|\,dy\int_y^v (v-s)^{\lambda-2}(s-y)^{-\lambda+1}\,ds$$

$$=2\Gamma(\lambda)\int_1^v (t-y)^{\lambda-2}\,|f_\lambda(y)|\,dy.$$

Lemma 2.15. (Cf. Hardy [12, §5.15]). *Suppose* $\lambda>0$ *and* x *is a function defined on* $(1,\infty)$ *and integrable on finite intervals. If* s *is finite, then*

$$\int_1^\infty x(u)\,du = s(C,\lambda)$$

if and only if $\frac{1}{t}\int_1^t m_{\lambda-1}\,x(u)\,du \to s$ *as* $t\to\infty$.

Proof. Let $\sigma(t) = m_\lambda x(t) = \int_1^t (1-\frac{u}{t})^\lambda x(u)du$, and let $\tau(t) = \frac{1}{t}\int_1^t m_{\lambda-1}\,x(u)du$.

The result follows from the following two identities, which may be established using integration by parts,

$$\sigma(t) = \lambda\tau(t) - (\lambda-1)\,\lambda t^{-\lambda}\int_1^t u^{\lambda-1}\tau(u)du, \text{ and}$$

$$\tau(t) = \frac{1}{\lambda}\,\sigma(t) + \frac{(\lambda-1)}{\lambda}\cdot\frac{1}{t}\int_1^t \sigma(u)du\,.$$

3. The Spaces W_p and $X_{\lambda,p}$

Let W_p $(p \geqslant 1)$ be the normed linear space of real-valued functions y defined on $[1,\infty)$ such that

$$\frac{1}{w}\int_1^w |y(t)-s|^p\,dt = o(1) \text{ as } w\to\infty, \text{ for some } s.$$

The norm on W_p is given by $\|y\| = \sup_{w>1}\left(\frac{1}{w}\int_1^w |y(t)|^p dt\right)^{1/p}$. In fact, W_p is a Banach space (cf. Maddox [17, problem 12 on page 101]) but the completeness of W_p is not needed in this paper.

Lemma 3.1. (Borwein [5]). *Suppose $p\geqslant 1$. If F is a continuous linear functional on W_p, then there exist a real number b and a measurable real-valued function α defined on $[1,\infty)$, such that*

$$\sum_{n=0}^{\infty} M_n(\alpha,1,p) < \infty, \text{ and} \tag{3.1}$$

$$F(y) = bs + \int_1^\infty \alpha(t)y(t)dt \text{ for all } y\in W_p, \tag{3.2}$$

where s satisfies $\frac{1}{w}\int_1^w |y(t)-s|^p\,dt = o(1)$. Conversely, if α satisfies (3.1) then the integral in (3.2) is absolutely convergent for all $y\in W_p$, F defined by (3.2) is a continuous linear functional on W_p, and

$$\|F\| \leqslant |b| + 2^{1/p}\sum_{n=0}^{\infty} M_n(\alpha,1,p).$$

Let $X_{\lambda,p}$ $(\lambda>0,\ p\geqslant 1)$ be the normed linear space of functions x such that $\int_1^\infty x(u)du = s[C,\lambda]_p$, for some s, with norm defined by

$$\|x\| = \|x\|_{\lambda,p} = \sup_{w>1}\left(\frac{1}{w}\int_1^w\left|\int_1^t\left(1-\frac{u}{t}\right)^{\lambda-1}x(u)du\right|^p dt\right)^{1/p}.$$

The space $X_{\lambda,p}$ is the same as the space $[C,\lambda]_p$ defined earlier. Define the map $T_{\lambda-1}: X_{\lambda,p}\to W_p$ by

$$T_{\lambda-1}(x)(t) = m_{\lambda-1}x(t) = \int_1^t\left(1-\frac{u}{t}\right)^{\lambda-1}x(u)du.$$

Then $T_{\lambda-1}$ is an isometry of $X_{\lambda,p}$ onto a subspace of W_p.

Let $F^c_{\lambda,p}$ $(\lambda>0,\ p\geqslant 1,\ c\geqslant 1)$ be the linear space of Lebesgue integrable functions which vanish outside of a finite interval contained in $[c,\infty)$. For $p\geqslant 1$, $\lambda > 1-\frac{1}{p}$, $F^c_{\lambda,p}$ is a subspace of $X_{\lambda,p}$, by Lemmas 2.6 (i) and 2.5.

Lemma 3.2. (Cf. Borwein [5, relation (8)]). *Suppose $\lambda>0$, $p\geqslant 1$, and h and f are measurable real-valued functions defined on $[1,\infty)$. Suppose*

$$\sum_{n=0}^{\infty} M_n(h,\lambda,p) < \infty, \text{ and}$$

$$A(w) = \left(\frac{1}{w}\int_1^w |u^{-\lambda}f(u)|^p du\right)^{1/p} = O(1).$$

Then $\int_1^\infty \frac{1}{v}\,|h(v)f(v)|\,dv < \infty$.

Proof. Let $I = \int_1^\infty \frac{1}{v}\,|h(v)f(v)|\,dv = \sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}\frac{1}{v}\,|h(v)f(v)|\,dv$.

If $p>1$, $\frac{1}{p}+\frac{1}{q} = 1$, then by Hölder's inequality,

$$I \leqslant \sum_{n=0}^{\infty}\left(\int_{2^n}^{2^{n+1}}\frac{1}{v}\,|v^\lambda h(v)|^q\,dv\right)^{1/q}\left(\int_{2^n}^{2^{n+1}}\frac{1}{v}\,|v^{-\lambda}f(v)|^p\,dv\right)^{1/p}$$

$$\leqslant 2^{1/p}\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\int_{2^n}^{2^{n+1}}|v^\lambda h(v)|^q\,dv\right)^{1/q}\left(\frac{1}{2^{n+1}}\int_{2^n}^{2^{n+1}}|v^{-\lambda}f(v)|^p\,dv\right)^{1/p}$$

$$\leqslant 2^{1/p}\left(\sum_{n=0}^{\infty} M_n(h,\lambda,p)\right)\sup_{w>1}A(w) < \infty.$$

If $p=1$, then

$$I \leqslant \sum_{n=0}^{\infty}\Big(\operatorname*{ess\,sup}_{2^n<v<2^{n+1}} |v^{\lambda}h(v)|\Big)\int^{2^{n+1}} \frac{1}{v}\,|v^{-\lambda}f(v)|\,dv$$

$$\leqslant 2\sum_{n=0}^{\infty} M_n(h,\lambda,1)\Big(\frac{1}{2^{n+1}}\int_{2^n}^{2^{n+1}2^n} |v^{-\lambda}f(v)|\,dv\Big)$$

$$\leqslant 2\Big(\sum_{n=0}^{\infty} M_n(h,\lambda,1)\Big)\sup_{w>1} A(w) < \infty.$$

Lemma 3.3. *Suppose* $\lambda>0$, $p\geqslant 1$, $\sum_{n=0}^{\infty} M_n(h,\lambda,p)<\infty$, $u\geqslant 1$, *and* $N=[\log_2 u]$. *Then*

(i) $$\int_u^{\infty} v^{\lambda-1}|h(v)|\,dv \leqslant \sum_{n=N}^{\infty} M_n(h,\lambda,p).$$

(ii) *If* $\lambda>1-\dfrac{1}{p}$, *then*

$$\int_u^{\infty} (v-u)^{\lambda-1}|h(v)|\,dv \leqslant H\sum_{n=N}^{\infty} M_n(h,\lambda,p), \tag{3.3}$$

where H is a constant depending only on λ *and p.*

(iii) *If* $p>1$ *and* $\dfrac{1}{p}+\dfrac{1}{q}=1$, *then*

$$\sup_{u>1}\Big(\frac{1}{u}\int_1^u |v^{\lambda}h(v)|^q\,dv\Big)^{1/q} \leqslant \sum_{n=0}^{\infty} M_n(h,\lambda,p).$$

Proof. The only case where the proof is not obvious is (ii), with $p>1$, $1-\dfrac{1}{p}<\lambda<1$. Let q satisfy $\dfrac{1}{p}+\dfrac{1}{q}=1$. Then, by Hölder's inequality and Lemma 2.8,

$$\int_u^{\infty}(v-u)^{\lambda-1}|h(v)|\,dv \leqslant \int_u^{2u}(v-u)^{\lambda-1}|h(v)|\,dv + \int_{2u}^{\infty}(v-u)^{\lambda-1}|h(v)|\,dv$$

$$\leqslant\Big(\int_u^{2u} v^{-\lambda p}(v-u)^{\lambda p-p}\,dv\Big)^{1/p}\Big(\int_u^{2u}|v^{\lambda}h(v)|^q\,dv\Big)^{1/q} + 2^{1-\lambda}\sum_{n=N+1}^{\infty} M_n(h,\lambda,p)$$

$$\leqslant B\Big(\frac{p}{q},\lambda p-\frac{p}{q}\Big)^{1/p} u^{-1/q}\Big\{(2^N)^{1/q}M_N+(2^{N+1})^{1/q}M_{N+1}\Big\} + 2^{1-\lambda}\sum_{n=N+1} M_n(h,\lambda\, p)$$

$$\leqslant H\sum_{n=N}^{\infty} M_n(h,\lambda,p).$$

Lemma 3.4. *Suppose* $p\geqslant 1$, $\lambda>1-\dfrac{1}{p}$, $c\geqslant 1$. *If f is a continuous linear functional on* $F^c_{\lambda,p}$ *then there exist a real number b and a measurable real-valued function h defined on* $[1,\infty)$ *and vanishing on* $[1,c)$, *such that*

$$\sum_{n=0}^{\infty} M_n(h,\lambda,p)<\infty, \text{ and} \tag{3.4}$$

$$f(x) = bs + \int_c^{\infty} x(u)\,du\int_u^{\infty}(v-u)^{\lambda-1}h(v)\,dv \tag{3.5}$$

for all $x\in F^c_{\lambda,p}$, *where s satisfies* $\int_1^{\infty} x(u)\,du = s\,[C,\lambda]_p$.

Conversely, if h satisfies (3.4), *then f defined by* (3.5) *is a continuous linear functional on* $F^c_{\lambda,p}$, *and*

$$\|f\| \leqslant |b|+2^{1/p}\sum_{n=0}^{\infty} M_n(h,\lambda,p).$$

Proof. Suppose h satisfies (3.4). If we define $\alpha(v)=v^{\lambda-1}h(v)$, then $M_n(\alpha,1,p)=M_n(h,\lambda,p)$ and α satisfies (3.1). Applying Lemma 3.1, F defined by (3.2) is a continuous linear functional on W_p.
Lemma 3.3 (ii) shows that h satisfies (3.3). If $x\in F^c_{\lambda,p}$, then x is zero outside a finite interval $(1,t)$, and (3.3) and $x\in L(1,t)$ show that the Fubini inversion

$$\int_c^{\infty} x(u)\,du\int_u^{\infty}(v-u)^{\lambda-1}h(v)\,dv \tag{3.6}$$

$$= \int_c^{\infty} h(v)\,dv\int_c^{v}(v-u)^{\lambda-1}x(u)\,du$$

is valid. Hence, if f is defined by (3.5), we have $f=F\cdot T_{\lambda-1}$ (with $T_{\lambda-1}$ restricted to $F^c_{\lambda,p}$), so f is a continuous linear functional on $F^c_{\lambda,p}$. Since $T_{\lambda-1}$ is an isometry,

$$\|f\| \leqslant \|F\| \leqslant |b| + 2^{1/p}\sum_{n=0}^{\infty} M_n(\alpha,1,p)$$

$$= |b|+2^{1/p}\sum_{n=0}^{\infty} M_n(h,\lambda,p).$$

Conversely, suppose f is a continuous linear functional on $F^c_{\lambda,p}$. Then $f\cdot T^{-1}_{\lambda-1}$ is a continuous linear functional on a subspace of W_p, and by the Hahn-Banach theorem can be extended to a continuous linear functional F defined on all of W_p. Lemma 3.1 shows that F must have the form (3.2), where α satisfies (3.1). Define $h(v)=\begin{cases} v^{1-\lambda}\alpha(v) & \text{if } v>c\\ 0 & \text{otherwise.}\end{cases}$

Then $M_n(h, \lambda, p) \leqslant M_n(\alpha, 1, p)$, and (3.4) holds. For any x in $F^c_{\lambda,p}$,

$$f(x) = f \circ T^{-1}_{\lambda-1}(T_{\lambda-1}x) = F(T_{\lambda-1}x)$$

$$= bs + \int_1^\infty \alpha(v)\,dv \int_1^v \left(1-\frac{u}{v}\right)^{\lambda-1} x(u)\,du$$

$$= bs + \int_c^\infty h(v)\,dv \int_1^v (v-u)^{\lambda-1} x(u)\,du\,,$$

and (3.5) follows from (3.6).

Lemma 3.5. *Suppose* $\lambda > 0$, $p \geqslant 1$, *and* $x \in X_{\lambda,p}$. *Then*

$$\frac{1}{\lambda H}\|x\|_{\lambda+1,p} \leqslant \|x\|_{\lambda,1} \leqslant \|x\|_{\lambda,p},\ \textit{where}\ H = 1 + \left|1-\frac{1}{\lambda}\right|.$$

The first inequality follows from the identity $m_\lambda x(t) = \lambda t^{-\lambda}\int_1^t u^{\lambda-1} m_{\lambda-1}x(u)\,du$, by Lemma 2.1 (i), and the second inequality follows from Hölder's inequality.

4. Proof of Theorem 1

Lemma 4.1. *Suppose* $p \geqslant 1$, $\lambda > 1 - \frac{1}{p}$, *and* $\phi \in \{[C, \lambda]_p;\ B(C)\}$. *Then* ϕ *is essentially bounded on* $(1, w)$ *for any* $w < \infty$.

Proof. Suppose, on the contrary, that ϕ is not essentially bounded on $(1, w)$, for some w. For each positive integer n, let E_n be the set of u satisfying $1 < u < w$ such that $n \leqslant |\phi(u)| < n+1$, and let e_n be the measure of E_n. By the hypothesis, infinitely many of the e_n are nonzero. Let I be the set of n for which e_n is nonzero, and let $S = \sum_{n\in I} \frac{1}{n}$. Define a function x as follows:

$$x(u) = \begin{cases} \dfrac{1}{n^2 e_n}\operatorname{sgn}\phi(u) & \text{if } u \in E_n,\quad n\in I,\quad \text{and } S=\infty,\\[2mm] \dfrac{1}{n e_n}\operatorname{sgn}\phi(u) & \text{if } u \in E_n,\quad n\in I,\quad \text{and } S<\infty,\\[2mm] 0 & \text{otherwise}. \end{cases}$$

Then $\int_1^w |x(u)|\,du < \infty$, so $\int_1^\infty x(u)\,du$ is summable $[C, \lambda]_p$ by Lemmas 2.6 (i) and 2.5.

On the other hand $\int_1^\infty x(u)\phi(u)\,du = \infty$, so $\int_1^\infty x(u)\phi(u)\,du$ is not bounded (C) by Lemma 2.6 (ii), a contradiction which establishes the lemma.

Lemma 4.2. (Cf. Borwein [4, Lemma 2]). *Suppose* $p \geqslant 1$, $\lambda > 1 - \frac{1}{p}$, *and* $\phi \in \{[C, \lambda]_p;\ B(C)\}$. *Then there exists* $c \geqslant 1$ *such that* f *defined by*

$$f(x) = \int_1^\infty x(u)\phi(u)\,du, \qquad \textit{for } x \in F^c_{\lambda,p}, \tag{4.1}$$

is a continuous linear functional on $F^c_{\lambda,p}$.

Proof. ϕ is essentially bounded on finite intervals, by Lemma 4.1, so the integral in (4.1) exists for all x in $F^c_{\lambda,p}$ for any $c \geqslant 1$. The functional f is clearly linear. Suppose there is no $c \geqslant 1$ for which f is continuous on $F^c_{\lambda,p}$. Then we can find real numbers $c_0, c_1, \ldots$ and functions $x_1, x_2, \ldots$ such that $c_0 = 1$ and, for all $n \geqslant 1$,

$$x_n \in F^{c_{n-1}}_{\lambda,p}, \qquad \|x_n\|_{\lambda,p} < 2^{-n}, \qquad f(x_n) > 1, \qquad \text{and} \tag{4.2}$$

$$c_n = 2c_{n-1} + \sum_{r=1}^n \int_1^\infty t|x_r(t)\phi(t)|\,dt. \tag{4.3}$$

If $c_0, \ldots, c_{n-1}$ and $x_1, \ldots, x_{n-1}$ have been chosen, then since f is not continuous on $F^{c_{n-1}}_{\lambda,p}$ we can find x_n satisfying (4.2), and can then define c_n by (4.3). Now define $x(t) = x_1(t) + x_2(t) + \ldots$. This is a finite sum since $c_n \to \infty$. We claim that $\int_1^\infty x(u)\,du$ is summable $[C, \lambda]_p$, whereas $\int_1^\infty x(u)\phi(u)\,du$ is not bounded (C).

We will show that $\int_1^\infty x(u)\,du$ is summable $[C, \lambda]_p$ by showing that the conditions of Lemma 2.2 are satisfied.

The (C, λ) summability of $\int_1^\infty x(u)\,du$ follows from Lemma 2.15, since for any integer s we have

$$\limsup_{w>v\to\infty}\left|\frac{1}{v}\int_1^v m_{\lambda-1}x(u)\,du - \frac{1}{w}\int_1^w m_{\lambda-1}x(u)\,du\right|$$

$$\leqslant \sum_{r=1}^{s-1}\limsup_{w>v\to\infty}\left|\frac{1}{v}\int_1^v m_{\lambda-1}x_r(u)\,du - \frac{1}{w}\int_1^w m_{\lambda-1}x_r(u)\,du\right|$$

$$+ \limsup_{w>v\to\infty}\sum_{r=s}^\infty\left|\frac{1}{v}\int_1^v m_{\lambda-1}x_r(u)\,du - \frac{1}{w}\int_1^w m_{\lambda-1}x_r(u)\,du\right|$$

$$\leqslant 2\sum_{r=s}^\infty \sup_{w>1}\frac{1}{w}\int_1^w |m_{\lambda-1}x_r(u)|\,du$$

$$\leq 2 \sum_{r=s}^{\infty} \|x_r\|_{\lambda,1} \leq 2 \sum_{r=s}^{\infty} \|x_r\|_{\lambda,p} \leq 2^{-s+2}.$$

Also, applying Minkowski's inequality in the form

$$\left(\int_1^w \left|\sum_{r=1}^{\infty} f_r(t)\right|^p dt\right)^{1/p} \leq \sum_{r=1}^{\infty} \left(\int_1^w |f_r(t)|^p dt\right)^{1/p},$$

the $[C, \lambda]_p$ summability of $\int_1^\infty x(u)\,du$ follows from

$$\limsup_{w\to\infty} \left(\frac{1}{w}\int_1^w |t^{-\lambda} \int_1^t (t-u)^{\lambda-1} ux(u)\,du\,|^p\,dt\right)^{1/p}$$

$$\leq \sum_{r=s}^{\infty} \sup_{w>1} \left(\frac{1}{w}\int_1^w |t^{-\lambda} \int_1^t (t-u)^{\lambda-1} ux_r(u)\,du\,|^p\,dt\right)^{1/p}$$

$$= \sum_{r=s}^{\infty} \sup_{w>1} \left(\frac{1}{w}\int_1^w |m_{\lambda-1}x_r(t) - m_\lambda x_r(t)|^p\,dt\right)^{1/p}$$

$$\leq \sum_{r=s}^{\infty} \|x\|_{\lambda,p} + \sum_{r=s}^{\infty} \|x\|_{\lambda+1,p} \leq (1+\lambda H)\,2^{-s+2}, \text{ by Lemma 3.5.}$$

To see that $\int_1^\infty x(u)\phi(u)\,du$ is not bounded (C) we may apply the analogue of Lemma 2.3 with summability replaced by boundedness, since, for any $\mu > 1$ and any integer $n \geq 1$, we have

$$\int_1^{c_n} t^{-\mu-1}\,dt \int_1^t (t-u)^{\mu-1} ux(u)\phi(u)\,du$$

$$= \sum_{r=1}^{n} \int_1^{c_n} t^{-\mu-1}\,dt \int_1^t (t-u)^{\mu-1} ux_r(u)\phi(u)\,du$$

$$= \sum_{r=1}^{n} \left\{\int_1^\infty - \int_{c_n}^\infty\right\} t^{-\mu-1}\,dt \int_1^t (t-u)^{\mu-1} ux_r(u)\phi(u)\,du$$

$$\geq \sum_{r=1}^{n} \int_1^\infty ux_r(u)\phi(u)\,du \int_u^\infty (t-u)^{\mu-1}\,t^{-\mu-1}\,dt$$

$$- \sum_{r=1}^{n} \int_{c_n}^\infty t^{-2}\,dt \int_1^\infty u\,|x_r(u)\phi(u)|\,du$$

$$= \frac{1}{\mu} \sum_{r=1}^{n} f(x_r) - \frac{1}{c_n} \sum_{r=1}^{n} \int_1^\infty u\,|x_r(u)\phi(u)|\,du$$

$$\geq \frac{n}{\mu} - 1.$$

This contradicts the assumption that $\phi \in \{[C, \lambda]_p;\, B(C)\}$, and consequently there must exist $c \geq 1$ for which f is a continuous functional on $F^c_{\lambda,p}$.

Proof of Theorem 1. Suppose $\phi \in \{[C, \lambda]_p;\, B(C)\}$. By Lemma 4.2, there exists $c \geq 1$ such that f given by (4.1) is a continuous linear functional on $F^c_{\lambda,p}$. By Lemma 4.1, $\phi \in L^\infty(1, c)$, and by Lemma 3.4 there exists a function h satisfying (3.4) and such that f has the representation (3.5) for all $x \in F^c_{\lambda,p}$. Taking $x(t)$ as the characteristic function of (c, w), and equating (3.5) and (4.1),

$$\int_c^w \phi(u)\,du = b(w-c) + \int_c^w du \int_u^\infty (v-u)^{\lambda-1} h(v)\,dv,$$

for all $w > c$. Hence

$$\phi(u) = b + \int_u^\infty (v-u)^{\lambda-1} h(v)\,dv \quad \text{for almost all } u > c.$$

Finally, (3.4) and Lemma 3.3 (ii) imply that $\phi \in L^\infty(1, \infty)$ and, except on a set of measure zero, $\phi(u) \to b$ as $u \to \infty$.

5. Lemmas Required for the Proof of Theorem 2

Lemma 5.1. *Suppose* $\lambda > 0$, $p \geq 1$, $\sum_{n=0}^{\infty} M_n(h, \lambda, p) < \infty$, *and* $A(w) = \left(\frac{1}{w}\int_1^w |u^{-\lambda} f(u)|^p\,du\right)^{1/p} = O(1)$. *Then*

$$\frac{1}{w}\int_1^w |h(v)f(v)|\,dv = o(1).$$

Proof. This follows from Lemma 3.2, by an application of Lemma 2.7.

Lemma 5.2. *Suppose* $\epsilon > 0$, $p \geq 1$, $t > a > 0$, *and* f *is measurable on* (a, t). *Then*

$$\int_a^t (t-u)^{\epsilon-1} |f(u)|\,du \leq \epsilon^{1/p-1}\, t^{\epsilon(1-1/p)} \left\{\int_a^t (t-u)^{\epsilon-1} |f(u)|^p\,du\right\}^{1/p}.$$

Proof. If $p = 1$ the result is immediate. If $p > 1$, let q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and apply Hölder's inequality to get

$$\int_a^t (t-u)^{\epsilon-1} |f(u)| \, du \leqslant \left\{ \int_a^t (t-u)^{\epsilon-1} \, du \right\}^{1/q} \left\{ \int_a^t (t-u)^{\epsilon-1} |f(u)|^p \, du \right\}^{1/p}.$$

The result follows from $\int_a^t (t-u)^{\epsilon-1} du \leqslant \frac{1}{\epsilon} t^{\epsilon}$.

Lemma 5.3. *Suppose* $\epsilon > 0$, $\delta \geqslant 0$, n *is a nonnegative integer, and* f *is measurable on* $(1, 2^{n+1})$. *Then*

$$\int_{2^n}^{2^{n+1}} v^{-\delta} \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$$

$$\leqslant \frac{2^{\epsilon+1}}{\epsilon} (2^n)^{\epsilon-\delta+1} \left\{ \frac{1}{2^{n+1}} \int_1^{2^{n+1}} |f(u)| \, du \right\}.$$

Proof.

$$\int_{2^n}^{2^{n+1}} v^{-\delta} \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$$

$$= \int_1^{2^{n+1}} |f(u)| \, du \int_{\max(2^n, u)}^{2^{n+1}} v^{-\delta} (v-u)^{\epsilon-1} \, dv, \text{ by Fubini's theorem,}$$

$$\leqslant (2^n)^{-\delta} \int_1^{2^{n+1}} |f(u)| \, du \int_{\max(2^n, u)}^{2^{n+1}} (v-u)^{\epsilon-1} \, dv$$

$$\leqslant \frac{2^{\epsilon+1}}{\epsilon} (2^n)^{\epsilon-\delta+1} \left\{ \frac{1}{2^{n+1}} \int_1^{2^{n+1}} |f(u)| \, du \right\}.$$

Lemma 5.4. *Suppose* $\epsilon > 0$, $\delta \geqslant 0$, $\lambda > 0$, $p \geqslant 1$, *and* h *and* f *are measurable on* $(1, \infty)$. *Then*

$$\int_1^{\infty} v^{\lambda-\delta} |h(v)| \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$$

$$\leqslant \frac{1}{\epsilon} 2^{\epsilon+1/p} \sum_{n=0}^{\infty} \left\{ M_n(h, \lambda, p) (2^n)^{\epsilon-\delta+1} \left(\frac{1}{2^{n+1}} \int_1^{2^{n+1}} |f(u)|^p \, du \right)^{1/p} \right\}.$$

Proof. Let $I = \int_1^{\infty} v^{\lambda-\delta} |h(v)| \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$. Suppose $p = 1$. Then $|v^{\lambda} h(v)| \leqslant M_n(h, \lambda, 1)$ for almost all v satisfying $2^n < v < 2^{n+1}$, so by Lemma 5.3,

$$I \leqslant \sum_{n=0}^{\infty} M_n(h, \lambda, 1) \int_{2^n}^{2^{n+1}} v^{-\delta} \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$$

$$\leqslant \frac{2^{\epsilon+1}}{\epsilon} \sum_{n=0}^{\infty} M_n(h, \lambda, 1) (2^n)^{\epsilon-\delta+1} \left(\frac{1}{2^{n+1}} \int_1^{2^{n+1}} |f(u)| \, du \right).$$

Suppose $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, by Lemmas 5.2 and 5.3, and Hölder's inequality,

$$I = \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} v^{\lambda-\delta} |h(v)| \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)| \, du$$

$$\leqslant \epsilon^{-1/q} \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} v^{\lambda-\delta+\epsilon/q} |h(v)| \left\{ \int_1^v (v-u)^{\epsilon-1} |f(u)|^p \, du \right\}^{1/p} dv$$

$$\leqslant \epsilon^{-1/q} \sum_{n=0}^{\infty} 2^n M_n(h, \lambda, p) \left\{ \frac{1}{2^n} \int_{2^n}^{2^{n+1}} v^{-\delta p + \epsilon p/q} \, dv \int_1^v (v-u)^{\epsilon-1} |f(u)|^p \, du \right\}^{1/p}$$

$$\leqslant \frac{1}{\epsilon} 2^{\epsilon+1/p} \sum_{n=0}^{\infty} M_n(h, \lambda, p) (2^n)^{\epsilon-\delta+1} \left\{ \frac{1}{2^{n+1}} \int_1^{2^{n+1}} |f(u)|^p \, du \right\}^{1/p}.$$

Lemma 5.5. *Suppose* $\lambda > 1$, $p \geqslant 1$, $\sum_{n=0}^{\infty} M_n(f, \lambda, p) < \infty$, *and* $F(u) = \int_u^{\infty} f(v) \, dv$. *Then* $\sum_{n=0}^{\infty} M_n(F, \lambda - 1, p) < \infty$.

Proof. If $p = 1$, then

$$M_n(F, \lambda-1, p) = \operatorname*{ess\,sup}_{2^n < u < 2^{n+1}} |u^{\lambda-1} F(u)|$$

$$= \operatorname*{ess\,sup}_{2^n < u < 2^{n+1}} \left| u^{\lambda-1} \int_u^{\infty} f(v) \, dv \right|$$

$$\leqslant (2^{n+1})^{\lambda-1} \int_{2^n}^{\infty} |f(v)| \, dv.$$

If $p > 1$, then

$$M_n(F, \lambda-1, p) = \left(\frac{1}{2^n}\int_{2^n}^{2^{n+1}} |u^{\lambda-1}F(u)|^q\,du\right)^{1/q}$$

$$\leqslant \left(\frac{1}{2^n}\int_{2^n}^{2^{n+1}} \{(2^{n+1})^{\lambda-1}\int_{2^n}^{\infty} |f(v)|\,dv\}^q\,du\right)^{1/q}$$

$$= (2^{n+1})^{\lambda-1}\int_{2^n}^{\infty} |f(v)|\,dv.$$

Hence, in either case,

$$\sum_{n=0}^{\infty} M_n(F, \lambda-1, p) \leqslant \sum_{n=0}^{\infty} 2^{(n+1)(\lambda-1)} \sum_{m=n}^{\infty}\int_{2^m}^{2^{m+1}} |f(v)|\,dv$$

$$= \sum_{m=0}^{\infty}\int_{2^m}^{2^{m+1}} |f(v)|\,dv \sum_{n=0}^{m} (2^{\lambda-1})^{n+1}$$

$$\leqslant H\sum_{m=0}^{\infty} (2^m)^{\lambda-1}\int_{2^m}^{2^{m+1}} |f(v)|\,dv, \text{ with } H = \frac{2^{2\lambda-2}}{2^{\lambda-1}-1},$$

$$\leqslant H\sum_{m=0}^{\infty}\int_{2^m}^{2^{m+1}} |v^{\lambda-1}f(v)|\,dv$$

$$\leqslant H\sum_{m=0}^{\infty} M_n(f, \lambda, p), \text{ by Lemma 3.3 (i).}$$

Lemma 5.6. *Suppose $\delta > 0, f \in L(1, w)$ for all $w > 1$, and $F(t) = \dfrac{f_1(t)}{t}$. Then*

(i) $\dfrac{1}{t} f_{\delta+1}(t) = F_\delta(t) - \dfrac{\delta}{t} F_{\delta+1}(t)$, *and*

(ii) $\displaystyle\int_1^w t^{-\delta-1} f_{\delta+1}(t)\,dt = w^{-\delta} F_{\delta+1}(w)$.

Proof. (i) $F_\delta(t) - \dfrac{\delta}{t} F_{\delta+1}(t)$

$$= \frac{1}{\Gamma(\delta)}\int_1^t (t-u)^{\delta-1}\frac{f_1(u)}{u}\,du - \frac{\delta}{t\Gamma(\delta+1)}\int_1^t (t-u)^{\delta}\frac{f_1(u)}{u}\,du$$

$$= \frac{1}{\Gamma(\delta)}\int_1^t (t-u)^{\delta-1}\left(1 - \frac{t-u}{t}\right) u^{-1} f_1(u)\,du$$

$$= \frac{1}{t}\cdot\frac{1}{\Gamma(\delta)}\int_1^t (t-u)^{\delta-1} f_1(u)\,du$$

$$= \frac{1}{t} f_{\delta+1}(t)\,dt.$$

(ii) This is Lemma 1 of Borwein [1], with $\rho = 0$ and $\alpha = \delta + 1$.

Lemma 5.7. *Suppose $\delta > 0, p \geqslant 1, f \in L(1, w)$ for all $w < \infty$, and $F(t) = \dfrac{f_1(t)}{t}$. Then $\frac{1}{w}\int_1^w |t^{-\delta} F_\delta(t)|^p\,dt = o(1)$ if and only if $\frac{1}{w}\int_1^w |t^{-\delta-1} f_{\delta+1}(t)|^p\,dt = o(1)$.*

Proof. Using the identity of Lemma 5.6 (ii), and Hölder's inequality if $p > 1$, it is easily shown that either $\frac{1}{w}\int_1^w |t^{-\delta} F_\delta(t)|^p\,dt = o(1)$ or $\frac{1}{w}\int_1^w |t^{-\delta-1} f_{\delta+1}(t)|^p\,dt = o(1)$ implies $\frac{1}{w}\int_1^w |t^{-\delta-1} F_{\delta+1}(t)|^p\,dt = o(1)$.

The equivalence follows from Lemma 5.6 (i).

Lemma 5.8. *Suppose $\lambda > 1, p \geqslant 1$, and $\int_1^\infty x(u)\,du$ is summable $[C, \lambda]_p$. Let $y(u) = u^{-2}\int_1^u v x(v)\,dv$. Then $\int_1^\infty y(u)\,du$ is summable $[C, \lambda-1]_p$.*

Proof. $\int_1^\infty x(u)\,du$ is summable (C, λ) by Lemma 2.2, so by Theorem 1 of Borwein [1], $\int_1^\infty y(u)\,du$ is summable $(C, \lambda-1)$. Also, letting $z(u) = uy(u)$,

$$\frac{1}{w}\int_1^w \left|t^{-\lambda+1}\int_1^t (t-u)^{\lambda-2} uy(u)\,du\right|^p dt$$

$$= \Gamma(\lambda-1)^p\frac{1}{w}\int_1^w |t^{-\lambda+1} z_{\lambda-1}(t)|^p\,dt = o(1),$$

by Lemma 5.7 with $\delta = \lambda - 1$, $f(v) = vx(v)$, $F(u) = z(u)$. Then $\int_1^\infty y(u)\,du$ is summable $[C, \lambda-1]_p$ by Lemma 2.2.

Lemma 5.9. *Suppose* $\lambda > 1$, $p \geq 1$, $t > 1$, $f \in L(1,t)$, k *is a measurable function defined on* $(1,\infty)$, *and* $\sum_{n=0}^{\infty} M_n(k,\lambda,p) < \infty$. *Let* $\theta(u) = \int_u^\infty (v-u)^{\lambda-1} k(v)\,dv$ *and* $\psi(u) = \int_u^\infty (v-u)^{\lambda-2} v k(v)\,dv$. *Then*

$$\int_1^t (t-u)^{\lambda-1} f(u)\theta(u)\,du = (\lambda-1)t\int_1^t (t-u)^{\lambda-2}\frac{f_1(u)}{u}\theta(u)\,du + (\lambda-1)\int_1^t (t-u)^{\lambda-1}\frac{f_1(u)}{u}\psi(u)\,du - 2(\lambda-1)\int_1^t (t-u)^{\lambda-1}\frac{f_1(u)}{u}\theta(u)\,du.$$

Proof. Since $\lambda > 1$, $\int_u^\infty (v-u)^{\lambda-1}|k(v)|\,dv \leq \int_u^\infty v^{\lambda-1}|k(v)|\,dv < \infty$, and $\theta(u)$ is absolutely convergent on $(1,\infty)$. Also,

$$\int_u^\infty dt \int_t^\infty (v-t)^{\lambda-2}|k(v)|\,dv = \int_u^\infty |k(v)|\,dv \int_u^v (v-t)^{\lambda-2}\,dt = \frac{1}{\lambda-1}\int_u^\infty (v-u)^{\lambda-1}|k(v)|\,dv, \text{ so}$$

$\theta'(u) = -(\lambda-1)\int_u^\infty (v-u)^{\lambda-2}k(v)\,dv$ for almost all $u > 1$. We have

$$\psi(u) - \theta(u) = u\int_u^\infty (v-u)^{\lambda-2}k(v)\,dv = -\frac{1}{\lambda-1}u\theta'(u)$$

for almost all $u > 1$, so

$$\int_1^t \frac{f_1(u)}{u}\psi(u)\,du - \int_1^t \frac{f_1(u)}{u}\theta(u)\,du = -\frac{1}{\lambda-1}\int_1^t f_1(u)\theta'(u)\,du = \frac{1}{\lambda-1}\int_1^t f(u)\theta(u)\,du - \frac{1}{\lambda-1}f_1(t)\theta(t). \tag{5.1}$$

Taking the $(\lambda-1)$-st integral of (5.1) and multiplying by $\Gamma(\lambda)$,

$$\int_1^t (t-u)^{\lambda-1}\frac{f_1(u)}{u}\psi(u)\,du - \int_1^t (t-u)^{\lambda-1}\frac{f_1(u)}{u}\theta(u)\,du$$

$$= \frac{1}{\lambda-1}\int_1^t (t-u)^{\lambda-1} f(u)\theta(u)\,du - \int_1^t (t-u)^{\lambda-2} f_1(u)\theta(u)\,du$$

$$= \frac{1}{\lambda-1}\int_1^t (t-u)^{\lambda-1} f(u)\theta(u)\,du - t\int_1^t (t-u)^{\lambda-2}\frac{f_1(u)}{u}\theta(u)\,du + \int_1^t (t-u)^{\lambda-1}\frac{f_1(u)}{u}\theta(u)\,du.$$

The lemma follows.

6. Proof of Theorem 2

Proof. Suppose that ϕ, h, c, b satisfy (1.1), (1.2), and (1.3). Relations (1.4) and (1.5) follow from Theorem 1.

If $\phi(u) = b$ for almost all $u > c$ – that is, h is essentially zero – then $\phi \in \{[C,\lambda]_p ; [C,\lambda]_p\}$ follows from Lemmas 2.6 (i) and 2.5. Thus it suffices to consider only the case $c = 1$, $b = 0$. We write $M_n = M_n(h,\lambda,p)$. For convenience, define $h(v) = 0$ for $v < 1$. If $p > 1$, we take q to satisfy $\frac{1}{p} + \frac{1}{q} = 1$. H, H_1, etc. denote positive constants depending on λ and p only, not necessarily the same on each occurrence.

Suppose $\int_1^\infty x(u)$ is summable $[C,\lambda]_p$. Let $g(u) = ux(u)$.

Lemmas 2.3 and 2.2 give

$$\int_1^w t^{-\lambda-1} g_\lambda(t)\,dt \text{ is convergent as } w \to \infty, \text{ and} \tag{6.1}$$

$$\frac{1}{w}\int_1^w |t^{-\lambda} g_\lambda(t)|^p\,dt = o(1). \tag{6.2}$$

Write $A(w) = \frac{1}{w}\int_1^w |t^{-\lambda}g_\lambda(t)|^p\,dt$ and $B(t) = \int_1^t (t-u)^{\lambda-1}g(u)\phi(u)\,du$.

The latter integral exists for almost all $t > 1$, since (3.3) implies that $\phi \in L^\infty(1,t)$. Also, (6.2) implies that $A(w)$ is bounded for all $w > 1$.

We have to show that

$$I(w) = \int_1^w t^{-\lambda-1}B(t)\,dt \text{ is convergent as } w \to \infty, \text{ and} \tag{6.3}$$

$$J(w) = \frac{1}{w}\int_1^w |t^{-\lambda} B(t)|^p \, dt = o(1). \tag{6.4}$$

The proof is divided into four cases:

(a) $p \geqslant 1$ and $1 - \frac{1}{p} < \lambda < 1$.

(b) $p \geqslant 1$ and $\lambda = 1$.

(c) $p \geqslant 1$ and $1 < \lambda < 2$.

(d) $p \geqslant 1$ and $\lambda \geqslant 2$.

Case (a): Suppose $p \geqslant 1$ and $1 - \frac{1}{p} < \lambda < 1$. Since $g \in L(1,t)$ and (3.3) holds, we may apply Fubini's theorem to the double integral

$$B(t) = \int_1^t (t-u)^{\lambda-1} g(u) du \int_u^\infty (v-u)^{\lambda-1} h(v) dv$$

for almost all $t > 1$ to obtain

$$B(t) = \int_1^\infty h(v) dv \int_1^{\min(v,t)} (t-u)^{\lambda-1} (v-u)^{\lambda-1} g(u) du.$$

Write $B(t) = B_1(t) + B_2(t) + B_3(t)$ where

$$B_1(t) = \int_1^t h(v) dv \int_1^v (t-u)^{\lambda-1} (v-u)^{\lambda-1} g(u) du,$$

$$B_2(t) = \int_t^\infty h(v) dv \int_1^t (t-u)^{\lambda-1} \{(v-u)^{\lambda-1} - v^{\lambda-1}\} g(u) du, \text{ and}$$

$$B_3(t) = \int_t^\infty v^{\lambda-1} h(v) dv \int_1^t (t-u)^{\lambda-1} g(u) du$$

$$= \Gamma(\lambda) g_\lambda(t) \int_t^\infty v^{\lambda-1} h(v) dv.$$

Then (6.3) can be written as $I_1(w) + I_2(w) + I_3(w)$, where $I_k(w) = \int_1^w t^{-\lambda-1} B_k(t) dt$ for $k = 1, 2, 3$. We will show that $I_1(\infty)$ and $I_2(\infty)$ are absolutely convergent and that $I_3(w)$ is convergent as $w \to \infty$.

Applying Lemma 2.14 (i) with $\alpha = \beta = \frac{1}{2}$.

$\int_1^\infty t^{-\lambda-1} |B_1(t)| dt \leqslant \Gamma(\lambda) \{I_4 + I_5\}$ where

$$I_4 = \int_1^\infty t^{-\lambda-1} dt \int_1^t (t-v)^{\lambda-1} |h(v) g_\lambda(v)| dv \text{ and}$$

$$I_5 = \int_1^\infty t^{-\lambda-1} dt \int_1^t (t-v)^{\lambda/2-1} |h(v)| dv \int_1^v (t-u)^{\lambda/2-1} |g_\lambda(u)| du.$$

$$I_4 = \int_1^\infty |h(v) g_\lambda(v)| dv \int_v^\infty t^{-\lambda-1} (t-v)^{\lambda-1} dt$$

$$= B(\lambda, 1) \int_1^\infty \frac{1}{v} |h(v) g_\lambda(v)| dv < \infty,$$

by Lemma 2.8 and by Lemma 3.2 with $f(v) = g_\lambda(v)$.

$$I_5 = \int_1^\infty |h(v)| dv \int_1^v |g_\lambda(u)| du \int_v^\infty t^{-\lambda-1} (t-v)^{\lambda/2-1} (t-u)^{\lambda/2-1} dt$$

$$\leqslant B\left(\frac{\lambda}{2}, 1 + \frac{\lambda}{2}\right) \int_1^\infty v^{\lambda/2-1} |h(v)| dv \int_1^v (v-u)^{\lambda/2-1} |u^{-\lambda} g_\lambda(u)| du$$

$$\leqslant H \sum_{n=0}^\infty M_n \left\{ \frac{1}{2^{n+1}} \int_1^{2^{n+1}} |u^{-\lambda} g_\lambda(u)|^p du \right\}^{1/p} < \infty,$$

by Lemmas 2.8 and 5.4, with $\epsilon = \frac{\lambda}{2}$, $\delta = \frac{\lambda}{2} + 1$, and

$H = B\left(\frac{\lambda}{2}, 1 + \frac{\lambda}{2}\right) \frac{1}{\lambda} 2^{\lambda/2+1+1/p}$. Thus $I_1(\infty)$ is absolutely convergent.

Applying Lemma 2.14 (i) with v and t interchanged and with $\alpha = \beta = \frac{1}{2}$, we have

$$\int_1^\infty t^{-\lambda-1} |B_2(t)| dt \leqslant \Gamma(\lambda) \{I_6 + I_7\}, \text{ where}$$

$$I_6 = \int_1^\infty t^{-\lambda-1} |g_\lambda(t)| dt \int_t^\infty \{(v-t)^{\lambda-1} - v^{\lambda-1}\} |h(v)| dv, \text{ and}$$

$$I_7 = \int_1^\infty t^{-\lambda-1}\,dt \int_t^\infty (v-t)^{\lambda/2-1}\,|h(v)|\,dv \int_1^t (v-u)^{\lambda/2-1}\,|g_\lambda(u)|\,du.$$

Then

$$I_6 \leqslant \int_1^\infty t^{-\lambda}\,|g_\lambda(t)|\,dt \int_t^\infty \frac{1}{v}(v-t)^{\lambda-1}\,|h(v)|\,dv$$

$$= \int_1^\infty \frac{1}{v}\,|h(v)|\,dv \int_1^v (v-t)^{\lambda-1}\,|t^{-\lambda} g_\lambda(t)|\,dt$$

$$\leqslant \frac{1}{\lambda} 2^{\lambda+1/p} \sum_{n=0}^\infty M_n \left\{\frac{1}{2^{n+1}} \int_1^{2^{n+1}} |t^{-\lambda} g_\lambda(t)|^p\,dt\right\}^{1/p}$$

$$= \frac{1}{\lambda} 2^{\lambda+1/p} \sum_{n=0}^\infty M_n A(2^{n+1}) < \infty,$$

by Lemmas 2.9(ii) and 5.4, with $\epsilon = \lambda$, $\delta = \lambda + 1$.

$$I_7 = \int_1^\infty |h(v)|\,dv \int_1^v (v-u)^{\lambda/2-1}\,|g_\lambda(u)|\,du \int_u^v t^{-\lambda-1}(v-t)^{\lambda/2-1}\,dt$$

$$\leqslant H_1 \int_1^\infty v^{\lambda/2-1}\,|h(v)|\,dv \int_1^v (v-u)^{\lambda/2-1}\,|u^{-\lambda} g_\lambda(u)|\,du$$

$$\leqslant H_2 \sum_{n=0}^\infty M_n A_n (2^{n+1}) < \infty,$$

by Lemma 2.10 with $\epsilon = \lambda$, $\delta = \frac{\lambda}{2}$, $H_1 = 2^{1-\lambda/2}(2^{\lambda+1}+1)\frac{1}{\lambda}$, and by Lemma 5.4 with $\epsilon = \frac{\lambda}{2}$, $\delta = \frac{\lambda}{2} + 1$, and

$$H_2 = \frac{1}{\lambda} 2^{\lambda/2+1+1/p}, \qquad H_1 = 2^{2+1/p}(2^{\lambda+1}+1)\frac{1}{\lambda^2}.$$

This shows that $I_2(\infty)$ is absolutely convergent.

Since $\int_1^\infty v^{\lambda-1}\,|h(v)|\,dv \leqslant \sum_{n=0}^\infty M_n < \infty$, by Lemma 3.3 (i), the function $\psi(t) = \int_t^\infty v^{\lambda-1} h(v)\,dv$ is of bounded variation on $[1, \infty)$, and convergence of $I_3(w)$ as $w \to \infty$ follows from (6.1). This completes the proof of (6.3) for the case $p \geqslant 1$, $1 - \frac{1}{p} < \lambda < 1$.

To prove (6.4), in case $p = 1$, $0 < \lambda < 1$, define

$$K_k(w) = \frac{1}{w} \int_1^w |t^{-\lambda} B_k(t)|\,dt \quad \text{for} \quad k = 1, 2, 3.$$

We have $J(w) \leqslant K_1(w) + K_2(w) + K_3(w)$. $K_1(w) = o(1)$ and $K_2(w) = o(1)$ follow from the absolute convergence of $I_1(\infty)$ and $I_2(\infty)$ respectively.

$$K_3(w) \leqslant \Gamma(\lambda) \frac{1}{w} \int_1^w t^{-\lambda}\,|g_\lambda(t)|\,dt \int_t^\infty v^{\lambda-1}\,|h(v)|\,dv$$

$$\leqslant \Gamma(\lambda)\left(\sum_{n=0}^\infty M_n\right) \frac{1}{w} \int_1^w t^{-\lambda}\,|g_\lambda(t)|\,dt = o(1),$$

by Lemma 3.3 (i).

This completes the proof of the theorem, when $p = 1$ and $0 < \lambda < 1$.

Suppose $p > 1$, $1 - \frac{1}{p} < \lambda < 1$. Define

$$J_1(w) = \frac{1}{w} \int_1^w \left| t^{-\lambda} \int_1^t h(v)\,dv \int_1^v (t-u)^{\lambda-1}(v-u)^{\lambda-1} g(u)\,du \right|^p dt, \text{ and}$$

$$J_2(w) = \frac{1}{w} \int_1^w \left| t^{-\lambda} \int_t^\infty h(v)\,dv \int_1^t (t-u)^{\lambda-1}(v-u)^{\lambda-1} g(u)\,du \right|^p dt.$$

Then $J(w)^{1/p} \leqslant J_1(w)^{1/p} + J_2(w)^{1/p}$.

Choose β in $\frac{1}{\lambda q} < \beta < 1$, so that $\lambda\beta > 1 - \frac{1}{p}$, and let $\alpha = 1 - \beta$.

$$J_1(w)^{1/p} \leqslant \Gamma(\lambda)\left\{J_3(w)^{1/p} + J_4(w)^{1/p}\right\}, \text{ where}$$

$$J_3(w) = \frac{1}{w} \int_1^w \left\{ t^{-\lambda} \int_1^t (t-v)^{\lambda-1}\,|h(v) g_\lambda(v)|\,dv \right\}^p dt, \text{ and}$$

$$J_4(w) = \frac{1}{w} \int_1^w \left\{ t^{-\lambda} \int_1^t (t-v)^{\lambda\beta-1}\,|h(v)|\,dv \int_1^v (t-u)^{\lambda\alpha-1}\,|g_\lambda(u)|\,du \right\}^p dt,$$

by Lemma 2.14 (i). Then

$$J_3(w) \leqslant \frac{1}{w} \int_1^w \left\{ \frac{1}{t} \int_1^t |v^\lambda h(v)|^q\,dv \right\}^{p/q} t^{-\lambda p + p/q}\,dt \int_1^t (t-v)^{\lambda p - p}\,|v^{-\lambda} g_\lambda(v)|^p\,dv$$

$$= o(1),$$

by Hölder's inequality and Lemmas 3.3 (iii) and 2.11.

The conditions on β imply that $\lambda\alpha<1$, and hence $(t-u)^{\lambda\alpha-1} \leqslant (v-u)^{\lambda\alpha-1}$ for $1<v<t$. We have

$$J_4(w) \leqslant \frac{1}{w}\int_1^w \left\{\frac{1}{t}\int_1^t |v^\lambda h(v)|^q\,dv\right\}^{p/q} t^{-\lambda p+p/q}\,dt$$

$$\cdot \int_1^t (t-v)^{\lambda\beta p-p}\left\{\int_1^v (t-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|\,du\right\}^p dv$$

$$\leqslant (\lambda\alpha)^{1-p}\left(\sum_{n=0}^\infty M_n\right)^p \frac{1}{w}\int_1^w t^{-\lambda p+p/q}\,dt \int_1^t v^{\lambda\alpha p/q}(t-v)^{\lambda\beta p-p}\,dv$$

$$\cdot \int_1^v (t-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du\,,$$

by Hölder's inequality, Lemma 3.3 (iii), and Lemma 5.4 with t replaced by v.

$$J_4(w) \leqslant (\lambda\alpha)^{1-p}\left(\sum_{n=0}^\infty M_n\right)^p \frac{1}{w}\int_1^w v^{\lambda\alpha p/q}\,dv \int_1^v (v-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du$$

$$\cdot \int_v^w t^{-\lambda p+p/q}(t-v)^{\lambda\beta-p-p}\,dt$$

$$\leqslant H\left(\sum_{n=0}^\infty M_n\right)^p \frac{1}{w}\int_1^w v^{-\lambda\alpha}\,dv \int_1^v (v-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du, \text{ by Lemma 2.8,}$$

where $H=(\lambda\alpha)^{1-p}B(\lambda\beta p-p/q,\ \lambda\alpha p)$, and this is o(1) by Lemma 2.11. This shows that $J_1(w)=o(1)$.

$$J_2(w)^{1/p} \leqslant \Gamma(\lambda)\left\{J_5(w)^{1/p}+J_6(w)^{1/p}\right\}, \text{ where}$$

$$J_5(w) = \frac{1}{w}\int_1^w |t^{-\lambda}g_\lambda(t)\phi(t)|^p\,dt \text{ and}$$

$$J_6(w)=\frac{1}{w}\int_1^w\left\{t^{-\lambda}\int_t^\infty (v-t)^{\lambda\beta-1}\,|h(v)|\,dv\int_1^t (v-u)^{\lambda\alpha-1}\,|g_\lambda(u)|\,du\right\}^p dt,$$

by Lemma 2.14 (i). $J_5(w)=o(1)$ follows from (1.4). Also,

$$t^{-\lambda}\int_t^\infty (v-t)^{\lambda\beta-1}\,|h(v)|\,dv\int_1^t (v-u)^{\lambda\alpha-1}\,|g_\lambda(u)|\,du$$

$$\leqslant \left(\int_t^\infty |v^{\lambda-1/q}h(v)|^q\,dv\right)^{1/q}$$

$$\cdot\left(\int_t^\infty v^{-\lambda p+p/q}(v-t)^{(\lambda\beta-1)p}\left\{\int_1^t (v-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|\,du\right\}^p dv\right)^{1/p}.$$

By Lemma 5.2, with $\epsilon=\lambda\alpha$, and $f(u)=u^{-\lambda}g_\lambda(u)$ for $1\leqslant u\leqslant t$, $f(u)=0$ for $t<u<v$, this is

$$\leqslant\left(\frac{1}{\lambda\alpha}\right)^{1/q}\left(\sum_{n=0}^\infty M_n\right)\left(\int_t^\infty v^{-\lambda p+p/q+\lambda\alpha p/q}(v-t)^{(\lambda\alpha-1)p}\,dv\right.$$

$$\left.\cdot\int_1^t (v-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du\right)^{1/p}$$

$$\leqslant\left(\frac{1}{\lambda\alpha}\right)^{1/q}\left(\sum_{n=0}^\infty M_n\right)\left(\int_1^t (t-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du\right.$$

$$\left.\cdot\int_t^\infty v^{-\lambda p+p/q+\lambda\alpha p/q}(v-t)^{(\lambda\beta-1)p}\,dv\right)^{1/p}.$$

Since $-\lambda p+\dfrac{p}{q}+\lambda\alpha\dfrac{p}{q}+\lambda\beta p-p = -\lambda\alpha-1$, this is

$$\leqslant H\left(\sum_{n=0}^\infty M_n\right)\left(t^{-\lambda\alpha}\int_1^t (t-u)^{\lambda\alpha-1}\,|u^{-\lambda}g_\lambda(u)|^p\,du\right)^{1/p}, \text{ where}$$

$H=\left(\dfrac{1}{\lambda\alpha}\right)^{1/q}B(\lambda\beta p-\dfrac{p}{q},\ \lambda\alpha)^{1/p}$. Hence $J_6(w)=o(1)$ by Lemma 2.11 and this shows that $J_2(w)=o(1)$.

This proves (6.4) when $p>1$, $1-\dfrac{1}{p}<\lambda<1$, and completes the proof of the theorem for $p\geqslant 1$, $1-\dfrac{1}{p}<\lambda<1$.

Case (b): Suppose $\lambda=1$, $p\geqslant 1$,

$$I(w) = \int_1^w t^{-2}\,dt\int_1^t g(u)\,\phi(u)\,du$$

$$= \int_1^w t^{-2}\,dt \int_1^t g(u)\,du \int_u^\infty h(v)\,dv$$

$$= \int_1^w t^{-2}\,dt \int_1^t h(v) g_1(v)\,dv + \int_1^w t^{-2} g_1(t)\,\phi(t)\,dt$$

$$= I_1(w) + I_2(w), \text{ say.}$$

$$I_1(w) = \int_1^w h(v) g_1(v)\,dv \int_v^w t^{-2}\,dt$$

$$= \int_1^w \frac{1}{v} h(v) g_1(v)\,dv - \frac{1}{w}\int_1^w h(v) g_1(v)\,dv\,,$$

which are convergent by Lemmas 3.2 and 5.1. Also, ϕ is of bounded variation on $[1, \infty)$, so $I_2(w)$ is convergent. This proves (6.3) in case $\lambda = 1$.

Similarly, to prove (6.4),

$$J(w)^{1/p} \leqslant \left(\frac{1}{w}\int_1^w \left\{\frac{1}{t}\int_1^t |h(v) g_1(v)|\,dv\right\}^p dt\right)^{1/p} + \left(\frac{1}{w}\int_1^w \left\{\frac{1}{t}\,|g_1(t)\,\phi(t)|\right\}^p dt\right)^{1/p}$$

$$= o(1), \text{ by Lemma 5.1 and (1.4).}$$

Case (c): Now suppose $1 < \lambda < 2$. Define $B_k(t)$ and $I_k(w)$ as in (a), for $k = 1, 2, 3$.

To show that $I_1(\infty)$ is absolutely convergent, we have, by Lemma 2.14 (ii),

$\int_1^\infty t^{-\lambda-1}\,|B_1(t)|\,dt \leqslant 2\Gamma(\lambda)\{I_4 + I_5\}$, where

$$I_4 = \int_1^\infty t^{-\lambda-1}\,dt \int_1^t (t-v)^{\lambda-1}\,|h(v) g_\lambda(v)|\,dv, \text{ and}$$

$$I_5 = \int_1^\infty t^{-\lambda-1}\,dt \int_1^t |h(v)|\,dv \int_1^v (t-u)^{\lambda-2}\,|g_\lambda(u)|\,du\,.$$

$I_4 < \infty$ follows as in (a). By Fubini's theorem and Lemma 2.8,

$$I_5 = \int_1^\infty |h(v)|\,dv \int_1^v |g_\lambda(u)|\,du \int_v^\infty t^{-\lambda-1}(t-u)^{\lambda-2}\,dt$$

$$\leqslant B(\lambda-1, 2)\int_1^\infty v^{-2}\,|h(v)|\,dv \int_1^v |g_\lambda(u)|\,du$$

$$\leqslant B(\lambda-1, 2)\int_1^\infty v^{\lambda-2}\,|h(v)|\,dv \int_1^v |u^{-\lambda} g_\lambda(u)|\,du, \text{ and this is}$$

$$\leqslant 2^{1+1/p}\,B(\lambda-1, 2)\sum_{n=0}^\infty M_n A(2^{n+1}) < \infty,$$

by Lemma 5.4 with $\delta = 2$, $\epsilon = 1$.

For $I_2(\infty)$ we have, by Lemma 2.14 (ii),

$\int_1^\infty t^{-\lambda-1}\,|B_2(t)|\,dt \leqslant 2\Gamma(\lambda)\{I_6 + I_7\}$, where

$$I_6 = \int_1^\infty t^{-\lambda-1}\,|g_\lambda(t)|\,dt \int_t^\infty \left\{v^{\lambda-1} - (v-t)^{\lambda-1}\right\}|h(v)|\,dv, \text{ and}$$

$$I_7 = \int_1^\infty t^{-\lambda-1}\,dt \int_t^\infty |h(v)|\,dv \int_1^t (v-u)^{\lambda-2}\,|g_\lambda(u)|\,du\,.$$

$$I_6 \leqslant \int_1^\infty t^{-\lambda}\,|g_\lambda(t)|\,dt \int_t^\infty v^{\lambda-2}\,|h(v)|\,dv\,,$$

by Lemma 2.9(i), and this is $< \infty$, as for I_5.

$$I_7 = \int_1^\infty |h(v)|\,dv \int_1^v (v-u)^{\lambda-2}\,|g_\lambda(u)|\,du \int_u^v t^{-\lambda-1}\,dt$$

$$\leqslant \frac{1}{\lambda}\int_1^\infty |h(v)|\,dv \int_1^v (v-u)^{\lambda-2}\,|u^{-\lambda} g_\lambda(u)|\,du$$

$$\leqslant \frac{1}{\lambda(\lambda-1)}\,2^{\lambda-1+1/p}\sum_{n=0}^\infty M_n A(2^{n+1}) < \infty,$$

by Lemma 5.4 with $\delta = \lambda$, $\epsilon = \lambda - 1$. This shows that $I_2(\infty)$ is absolutely convergent. $I_3(w)$ is convergent as $w \to \infty$, as in (a).

Define $J_1(w)$ and $J_2(w)$ as in (a). Then, by Lemma 2.14 (ii),

$$J_1(w)^{1/p} \leqslant 2\Gamma(\lambda)\left\{J_3(w)^{1/p} + J_4(w)^{1/p}\right\}, \text{ where}$$

$$J_3(w) = \frac{1}{w}\int_1^w \left\{t^{-\lambda}\int_1^t (t-v)^{\lambda-1}\,|h(v) g_\lambda(v)|\,dv\right\}^p dt, \text{ and}$$

$$J_4(w) = \frac{1}{w}\int_1^w \left\{t^{-\lambda}\int_1^t |h(v)|\,dv \int_1^v (t-u)^{\lambda-2}\,|g_\lambda(u)|\,du\right\}^p dt.$$

Since $\lambda > 1$, $J_3(w) \leqslant \frac{1}{w}\int_1^w \left\{\frac{1}{t}\int_1^t |h(v) g_\lambda(v)|\,du\right\}^p dt = o(1)$ by Lemma 5.1.

Letting $N_t = [\log_2 t]$,

$$J_4(w) \leqslant \frac{1}{w}\int_1^w \left\{t^{-\lambda}\int_1^t v^\lambda |h(v)|\,dv \int_1^v (t-u)^{\lambda-2}\,|u^{-\lambda} g_\lambda(u)|\,du\right\}^p dt$$

$$\leqslant H\frac{1}{w}\int_1^w \left\{t^{-\lambda}\sum_{n=0}^{N_t+1} M_n (2^n)^\lambda A(2^{n+1})\right\}^p dt = o(1),$$

by Lemma 5.4 with $\epsilon = \lambda - 1$, $\delta = 0$. Also by Lemma 2.14 (ii),

$$J_2(w)^{1/p} \leqslant 2\Gamma(\lambda)\left\{J_5(w)^{1/p} + J_6(w)^{1/p}\right\}, \text{ where}$$

$$J_5(w) = \frac{1}{w}\int_1^w |t^{-\lambda} g_\lambda(t)\phi(t)|^p\,dt \text{ and}$$

$$J_6(w) = \frac{1}{w}\int_1^w \left\{t^{-\lambda}\int_t^\infty |h(v)|\,dv \int_1^t (v-u)^{\lambda-2}\,|g_\lambda(u)|\,du\right\}^p dt.$$

$J_5(w) = o(1)$ follows from (1.4), and $J_6(w) = o(1)$ follows as in (a) if we let $\beta = \frac{1}{\lambda}$, $\alpha = 1 - \frac{1}{\lambda}$. This proves the theorem for $1 < \lambda < 2$.

Case (d): The remainder of the proof is by induction on the hypothesis that the theorem holds for $\lambda - 1$. We may assume that $\lambda \geqslant 2$.

Define $j(v) = vh(v)$ and $\psi(u) = \int_u^\infty (v-u)^{\lambda-2} j(v)\,dv$.

$$M_n(j, \lambda - 1, p) = M_n(h, \lambda, p) \text{ and}$$

$$\int_u^\infty (v-u)^{\lambda-2}\,|j(v)|\,dv \leqslant \int_u^\infty v^{\lambda-1}\,|h(v)| \leqslant \sum_{n=0}^\infty M_n < \infty,$$

so j and ψ satisfy the conditions (1.1) – (1.3) for $\lambda - 1$. Also, $h_1(u) = \int_u^\infty h(v)\,dv$ and $\frac{1}{\lambda-1}\phi(u) = \int_u^\infty (v-u)^{\lambda-2} h_1(v)\,dv$ satisfy the conditions for $\lambda - 1$, by Lemma 5.5. If y is defined by $y(u) = u^{-2} g_1(u)$, then $\int_1^\infty y(u)\,du$ is summable $[C, \lambda - 1]_p$ by Lemma 5.8.

Now, by the inductive hypothesis, $\int_1^\infty y(u)\phi(u)\,du$ and $\int_1^\infty y(u)\psi(u)\,du$ are summable $[C, \lambda - 1]_p$, and hence summable $[C, \lambda]_p$. That is, (2.1) and (2.2) become

$$\frac{1}{w}\int_1^w \left|t^{-\lambda+1}\int_1^t (t-u)^{\lambda-2} u^{-1} g_1(u)\phi(u)\,du\right|^p dt = o(1),$$

and $\int_1^w t^{-\lambda}\,dt \int_1^t (t-u)^{\lambda-2} u^{-1} g_1(u)\phi(u)\,du$ is convergent as $w \to \infty$, and similarly with ϕ replaced by ψ or λ replaced by $\lambda + 1$. Applying Lemma 5.9,

$$\int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} g(u)\phi(u)\,du$$

$$= (\lambda-1)\int_1^w t^{-\lambda}\,dt \int_1^t (t-u)^{\lambda-2} u^{-1} g_1(u)\phi(u)\,du$$

$$+ (\lambda-1)\int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} u^{-1} g_1(u)\psi(u)\,du$$

$$- 2(\lambda-1)\int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} u^{-1} g_1(u)\phi(u)\,du$$

is convergent as $w \to \infty$, and

$$\left(\frac{1}{w}\int_1^w \left|t^{-\lambda}\int_1^t (t-u)^{\lambda-1} g(u)\phi(u)\,du\right|^p dt\right)^{1/p}$$

$$\leqslant (\lambda-1)\left(\frac{1}{w}\int_1^w \left|t^{-\lambda+1}\int_1^t (t-u)^{\lambda-2} u^{-1} g_1(u)\phi(u)\,du\right|^p dt\right)^{1/p}$$

$$+ (\lambda-1)\left(\frac{1}{w}\int_1^w \left|t^{-\lambda}\int_1^t (t-u)^{\lambda-1} u^{-1} g_1(u)\psi(u)\,du\right|^p dt\right)^{1/p}$$

$$+ 2(\lambda-1)\left(\frac{1}{w}\int_1^w \left|t^{-\lambda}\int_1^t (t-u)^{\lambda-1} u^{-1} g_1(u)\phi(u)\,du\right|^p dt\right)^{1/p}$$

$$= o(1).$$

This completes the proof of Theorem 2.

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