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TAUBERIAN THEOREMS BETWEEN THE LOGARITHMIC AND ABEL-TYPE SUMMABILITY METHODS

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The object of this paper is to show that if a series is summable by the logarithmic method L, then the series is also summable by the Abel method A_λ, provided a tauberian condition of the "slowly decreasing" type is satisfied.

1. Introduction. Suppose throughout that $\{s_n\}$ is a sequence of numbers, λ real is real, $\varepsilon_0^\lambda = 1$, $\varepsilon_n^\lambda = \binom{n+\lambda}{n}$ for $n = 1, 2, 3, \cdots$, and

$$v_n^\lambda = \frac{\varepsilon_n^\lambda \Gamma(\lambda+1)}{(n+1)^\lambda} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

We are concerned with the methods of summability A_λ introduced and studied by Borwein [1] and the logarithmic method L. They are defined as follows. Let

$$(1) \qquad \sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n, \quad \text{and}$$

$$(2) \qquad L(y) = \frac{1}{\log(1+y)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1} .$$

If $\sigma_\lambda(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that the sequence $\{s_n\}$ is A_λ-convergent to s and write $s_n \to s(A_\lambda)$. The method A_0 is the ordinary Abel method.

If $L(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that $\{s_n\}$ is L-convergent to s and write $s_n \to s(L)$.

Evidently, $s_n \to s(L)$ if and only if

$$\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

converges for $0 < x < 1$ and tends to s as $x \to 1^-$.

LEMMA 1. *A_λ is regular for $\lambda > -1$. [That is, $s_n \to s$ implies $s_n \to s(A_\lambda)$].*

LEMMA 2. *L is regular.*

LEMMA 3. *$A_{\lambda+\varepsilon} \subset A_\lambda$ for $\lambda > -1$, and $\varepsilon > 0$. [That is, $s_n \to s(A_{\lambda+\varepsilon})$ implies $s_n \to s(A_\lambda)$ and there exists a sequence $\{s_n\}$, depending on λ and ε, such that $\{s_n\}$ is A_λ-convergent but not $A_{\lambda+\varepsilon}$-convergent.]*

LEMMA 4. *$A_\lambda \subset L$ for $\lambda > -1$.*

Lemmas 1 and 3 were established by Borwein in [1]. Lemma 4 was proved by Borwein in [2] as a particular case of a more general inclusion theorem on methods of summability based on power series. Lemma 2 is a standard result found, for example, in [4].

2. The main theorem. Suppose that Φ is a nonnegative, continuous, strictly increasing function on $[a, \infty)$, for some a, such that $\Phi(t) \to \infty$ as $t \to \infty$.

The real-valued function f is said to be *slowly decreasing with respect to* Φ if $\liminf \{f(y) - f(x)\} \geqq 0$ whenever $y \geqq x \to \infty$ and $\Phi(y) - \Phi(x) \to 0$.

THEOREM 1. *For $\lambda > -1$, if $s_n \to s(L)$ and $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log \log t$, then $s_n \to s(A_\lambda)$.*

In connection with the methods A_λ, we proved the following lemma in [3].

LEMMA 5. *For $\lambda > -1$ and $\varepsilon > 0$, if $s_n \to s(A_\lambda)$ and $\sigma_{\lambda+\varepsilon}(t)$ is slowly decreasing with respect to $\log t$, then $s_n \to s(A_{\lambda+\varepsilon})$.*

3. Methods of summability based on power series. Suppose that $p_n \geqq 0$, $q_n \geqq 0$, $\sum_{v=n}^{\infty} p_v > 0$, and $\sum_{v=n}^{\infty} q_v > 0$ for $n = 0, 1, 2, \cdots$. Set

$$p(x) = \sum_{n=0}^{\infty} p_n x^n , \quad \text{and}$$

$$q(x) = \sum_{n=0}^{\infty} q_n x^n .$$

Let ρ_p and ρ_q denote their respective radii of convergence. We also write

$$p_s(x) = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n$$

$$q_s(x) = \frac{1}{q(x)} \sum_{n=0}^{\infty} q_n s_n x^n .$$

The power series method P is defined as follows. If $\rho_p > 0$, $\sum_{n=0}^{\infty} p_n s_n x^n$ converges for $0 < x < \rho_p$ and $\lim_{x \to \rho_p -} p_s(x) = s$, then we write $s_n \to s(P)$.

The method Q is defined similarly.

Borwein has proved [2] the following lemma.

LEMMA 6. (i) *If $0 < \rho_p < \infty$, then a necessary and sufficient condition for P to be regular is that $\sum_{n=0}^{\infty} p_n (\rho_p)^n = \infty$.*
(ii) *If $\rho_p = \infty$ then P is regular.*

Suppose that $\chi(t)$ is a function of bounded variation on $[0, 1]$, and $\chi^*(t)$ is its associated normalized function. That is,

$$\chi^*(t) = \begin{cases} 0 & t = 0 \\ \frac{1}{2}\{\chi(t+) + \chi(t-)\} - \chi(0) & 0 < t < 1 \\ \chi(1) - \chi(0) & t = 1 . \end{cases}$$

A sequence $\{\mu_n\}$ is called an *m-sequence* if, for some χ,

$$\mu_n = \int_0^1 t^n d\chi(t) \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

If, in addition,

$$\mu_n \geqq \delta \int_0^1 t^n |d\chi^*(t)| \quad \text{for} \quad 0 < \delta \leqq 1 \quad \text{and}$$

$n = N, N+1, \cdots$, then $\{\mu_n\}$ is called an *$\bar{m}$-sequence*.

LEMMA 7. *If $p_n = \mu_n q_n (n = N, N+1, \cdots)$, $\{\mu_n\}$ is an $\bar{m}$-sequence, $\rho_p = \rho_q > 0$, and P is regular, then $Q \subseteq P$. (That is, $s_n \to s(Q)$ implies $s_n \to s(P)$.)*

This result is due to Borwein (see [2], Theorem A′).
We require the following two lemmas.

LEMMA 8. *An m-sequence which converges to a positive limit is an $\bar{m}$-sequence.*

LEMMA 9. *The sequences $\{v_n^\lambda\}$ and $\{1/v_n^\lambda\}$ are $\bar{m}$-sequences for $\lambda > -1$.*

The proof of Lemma 8 is straightforward and Lemma 9 was established in [4], Theorem 211.

The next result is used in the proof of Theorem 1.

THEOREM 2. *Let Q be a regular power series method and suppose that $\{\mu_n\}$ is an $\bar{m}$-sequence such that $\mu_n \to a > 0$. Then $\mu_n s_n \to as(Q)$*

whenever $s_n \to s(Q)$.

Proof. Suppose that $s_n \to s(Q)$. Set $p_n = \mu_n q_n$ for $n = 0, 1, 2, \cdots$. Since $\mu_n \geqq 0$ and $\mu_n \to a$ it is easy to verify that $\rho_p = \rho_q$. If $\rho_p = \infty$, then P is regular by Lemma 6(ii). Otherwise, since $p_n \sim aq_n$, P is regular by Lemma 6(i).

Therefore, by Lemma 7, $s_n \to s(P)$. That is,

$$(3) \qquad \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n \longrightarrow s \quad \text{as} \quad x \longrightarrow \rho_P^- .$$

In addition, since Q is regular,

$$(4) \qquad \frac{p(x)}{q(x)} = \frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n q_n x^n \longrightarrow a \quad \text{as} \quad x \longrightarrow \rho_q^- .$$

Application of Q to $\{\mu_n s_n\}$ *yields*

$$\frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n s_n q_n x^n$$

$$= \frac{p(x)}{q(x)} \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n$$

$$\longrightarrow as \quad \text{as} \quad x \longrightarrow \rho_q^- = \rho_p^- \text{ by (3) and (4)} .$$

This completes the proof.

COROLLARY TO THEOREM 2. $s_n \to s(L)$ *if and only if* $v_n^\lambda s_n \to s(L)$.

This is immediate in view of Lemmas 8 and 9, and the fact that $v_n^\lambda \to 1$ as $n \to \infty$.

4. An integral transformation.

The integral transformation $J_\lambda(w)$ of the function $f(t)$, for $\lambda > -1$ and $w > 0$, is defined as follows.

$$(5) \qquad J_\lambda(w) = \frac{1}{\log(1+w)} \int_0^w (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)}\right)^\lambda f(t)dt .$$

THEOREM 3. *If* $\lambda > -1$ *and* $f(t) = \sigma_\lambda(t)$ *is convergent for all* $t > 0$, *then* $J_\lambda(w) \to s$ *as* $w \to \infty$ *if and only if* $s_n \to s(L)$.

Proof. Setting $u = (t(1+w))/(w(1+t))$ in $J_\lambda(w)$ gives

$$J_\lambda(w)$$

$$= \frac{1}{\log(1+w)} \int_0^w (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)}\right)^\lambda (1+t)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n dt$$

$$= \frac{1}{\log(1+w)} \int_0^1 \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} u^n \left(\log \frac{1}{u}\right)^\lambda du$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} \int_0^1 u^n \left(\log \frac{1}{u}\right)^\lambda du$$

$$= \frac{\Gamma(\lambda+1)}{\log(1+w)} \sum_{n=0}^{\infty} \frac{\varepsilon_n^\lambda}{(n+1)^{\lambda+1}} s_n \left(\frac{w}{1+w}\right)^{n+1}$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n^\lambda s_n}{n+1} \left(\frac{w}{1+w}\right)^{n+1} .$$

The convergence, for $t > 0$, of the series defining $\sigma_\lambda(t)$ implies its absolute convergence. This justifies the integration term by term and, in view of the corollary to Theorem 2, the proof is complete.

5. Additional lemmas.

LEMMA 10. *For* $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ *is absolutely convergent for* $|x| < 1$ *if and only if* $\sum_{n=0}^{\infty} (s_n/(n+1))x^n$ *is absolutely convergent for* $|x| < 1$.

We omit the simple proof.

LEMMA 11. *For* $0 < t < w$,

$$\log \frac{w(1+t)}{t(1+w)} > \frac{w-t}{w(1+t)} .$$

Proof. For $x > 1$,

$$\log x = \log x - \log 1 = \frac{x-1}{\theta} > \frac{x-1}{x}$$

where $1 < \theta < x$. The result follows by observing that, for $0 < t < w$, $x = (w(1+t))/(t(1+w)) > 1$.

LEMMA 12. *For fixed* $\gamma > 1$ *and* $\lambda > -1$,

$$I(x) = \int_0^x (1+t)^{\lambda-1} \left(\left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda - \left(\log \frac{x(1+t)}{t(1+x)}\right)^\lambda \right) dt$$

$$= O(1) .$$

Proof. Suppose $\lambda \geqq 1$. Then, for $x \geqq 1$,

$$|I(x)| = I(x)$$

$$\leqq \lambda \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^x (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda-1} dt$$

$$\leqq \lambda \log \frac{x^{\gamma}(1+x)}{x(1+x^{\gamma})}\Big(\int_0^1 + \int_1^x\Big)(1+t)^{\lambda-1}\Big(\log \frac{1+t}{t}\Big)^{\lambda-1} dt$$
$$= I_1(x) + I_2(x) .$$

Now,

$$\int_0^1 (1+t)^{\lambda-1}\Big(\log \frac{1+t}{t}\Big)^{\lambda-1} dt < \infty .$$

Hence,

$$I_1(x) = O(1) .$$

Also,

$$I_2(x) = O(1) \log \frac{x^{\gamma}(1+x)}{x(1+x^{\gamma})} \int_1^x dt$$
$$= O(1) x \log \frac{1+x}{x} = O(1) .$$

Suppose $0 < \lambda < 1$. By Lemma 11 we have,

$$|I(x)| = I(x)$$
$$\leqq \lambda \log \frac{x^{\gamma}(1+x)}{x(1+x^{\gamma})} \int_0^x (1+t)^{\lambda-1}\Big(\log \frac{x(1+t)}{t(1+x)}\Big)^{\lambda-1} dt$$
$$< \lambda \frac{M}{x} \int_0^x (1+t)^{\lambda-1}\Big(\frac{x-t}{x(1+t)}\Big)^{\lambda-1} dt$$

since $x \log (x^{\gamma}(1+x))/(x(1+x^{\gamma})) \leqq M$.

Therefore

$$I(x) \leqq \lambda \frac{M}{x^{\lambda}} \int_0^x (x-t)^{\lambda-1} dt = M .$$

Suppose $-1 < \lambda < 0$. Then

$$|I(x)| = -I(x)$$
$$= \Big(\int_0^{x/2} + \int_{x/2}^x\Big)(1+t)^{\lambda-1}\Big(\Big(\log \frac{x(1+t)}{t(1+x)}\Big)^{\lambda} - \Big(\log \frac{x^{\gamma}(1+t)}{t(1+x^{\lambda})}\Big)^{\lambda}\Big) dt$$
$$= I_1(x) + I_2(x) .$$

Using Lemma 11 and the fact that

$$\Big| x \log \frac{x(1+x^{\gamma})}{(1+x)x^{\gamma}}\Big| \leqq M$$

we have

$$0 \leqq I_1(x) \leqq \lambda\Big(\log \frac{x(1+x^{\gamma})}{x^{\gamma}(1+x)}\Big) \int_0^{x/2} (1+t)^{\lambda-1}\Big(\log \frac{x(1+t)}{t(1+x)}\Big)^{\lambda-1} dt$$
$$\leqq -\frac{\lambda M}{x} \int_0^{x/2} (1+t)^{\lambda-1}\Big(\frac{x-t}{x(1+t)}\Big)^{\lambda-1} dt$$
$$= M((1/2)^{\lambda} - 1) .$$

For $I_2(x)$, since $1 + t > x/2$,

$$0 \leqq I_2(x) \leqq \int_{x/2}^x (1+t)^{\lambda-1}\Big(\log \frac{x(1+t)}{t(1+x)}\Big)^{\lambda} dt$$
$$\leqq \int_{x/2}^x (1+t)^{\lambda-1}\Big(\frac{x-t}{x(1+t)}\Big)^{\lambda} dt$$
$$= \frac{1}{x^{\lambda}} \int_{x/2}^x (x-t)^{\lambda} \frac{dt}{1+t}$$
$$\leqq \frac{2}{x^{\lambda+1}} \int_{x/2}^x (x-t)^{\lambda} dt$$
$$= \frac{1}{(\lambda+1)2^{\lambda}} .$$

Hence, $I(x) = O(1)$ in this case.

Finally, since the case $\lambda = 0$ is trivial, the lemma is established.

LEMMA 13. *For* $\gamma > 1$, *and* $\lambda > -1$,

$$\int_x^{x^{\lambda}} (1+t)^{\lambda-1}\Big(\log \frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda} dt$$
$$= (\gamma - 1)\log(1+x) + o(\log(1+x)) .$$

Proof. Set $\{s_n\} = \{1\}$. Then $\sigma_{\lambda}(t) = 1$ and, by Theorem 3, putting $f(t) = \sigma_{\lambda}(t)$ in (5) gives

$$J_{\lambda}(x) = 1 + o(1) \quad \text{as} \quad x \longrightarrow \infty .$$

Now by Lemma 12,

$$\int_x^{x^{\lambda}} (1+t)^{\lambda-1}\Big(\log \frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda} dt$$
$$= \Big(\int_0^{x^{\lambda}} - \int_0^x\Big)(1+t)^{\lambda-1}\Big(\log \frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda} dt$$
$$= \log(1+x^{\gamma}) + o(\log(1+x^{\gamma})) - \log(1+x) + o(\log(1+x)) + o(1)$$
$$= (\gamma - 1)\log(1+x) + o(\log(1+x)) .$$

This establishes the lemma.

6. A general tauberian result.

THEOREM 4. *Suppose that the following conditions hold:*

(6) $K(w,t)$ *is defined, real-valued, and nonnegative for* $w>0$, $t\geqq 0$; *moreover,* $\int_0^\infty K(w,t)dt$ *exists in the sense of Lebesgue for each* $w>0$,

(7) $$\int_0^\infty K(w,t)dt \longrightarrow 1 \quad as \quad w\longrightarrow\infty,$$

(8) f *is real-valued and continuous on* $(0,\infty)$,

(9) $F(w)=\int_0^\infty K(w,t)f(t)dt$ *exists in the Cauchy-Lebesgue sense for each* $w>0$,

(10) $\liminf\{f(y)-f(x)\}\geqq -\mu$ *for some fixed finite nonnegative* μ, *whenever* $y\geqq x\to\infty$ *and* $\Phi(y)-\Phi(x)\to 0$,

(11) $$\Phi(x)-\Phi(x-1)\longrightarrow 0 \quad as \quad x\longrightarrow\infty,$$

(12) $$\int_0^x K(w,t)dt\longrightarrow 0 \quad whenever \quad w>x\longrightarrow\infty \quad and \quad \Phi(w)-\Phi(x)\longrightarrow\infty,$$

(13) $$\int_x^\infty K(w,t)(\Phi(t)-\Phi(x))dt\longrightarrow 0 \quad whenever \quad x>w\longrightarrow\infty \quad and \quad \Phi(x)-\Phi(w)\longrightarrow\infty, \quad and$$

(14) $$F(w)=O(1) \quad for \quad w>0.$$

Then $f(t)=O(1)$ *for* $t>0$.

This result was established in [5]. A version of this theorem with (10) replaced by the stronger condition that f be slowly decreasing with respect to Φ can be found in [3]. The proofs are very similar.

7. A theorem on boundedness.

In this section we deduce a weakened form of Theorem 1 from the general tauberian result of § 6.

THEOREM 5. *If* $\lambda>-1$, $\infty>\mu\geqq 0$, $s_n\to s(L)$, *and* $\liminf\{\sigma_\lambda(y)-\sigma_\lambda(x)\}\geqq-\mu$ *whenever* $y\geqq x\to\infty$ *and* $\Phi(y)-\Phi(x)\to 0$, *then* $\sigma_\lambda(t)=O(1)$.

Proof. Set

$$K(w,t)=\begin{cases}\dfrac{1}{\log(1+w)}(1+t)^{\lambda-1}\left(\log\dfrac{w(1+t)}{t(1+w)}\right)^{\lambda} & 0<t<w\\ 0 & \text{otherwise},\end{cases}$$

$$\Phi(t)=\begin{cases}t/e^e & 0\leqq t<e^e\\ \log\log t & e^e\leqq t,\end{cases}$$

and

$$f(t)=\sigma_\lambda(t).$$

First, note that if $\{s_n\}=\{1\}$, then $s_n\to 1(L)$ and $\sigma_\lambda(t)=1$. Hence, by Theorem 3 with $f(t)=\sigma_\lambda(t)=1$ in (5), we have

$$\begin{aligned}\int_0^\infty K(w,t)dt &= \frac{1}{\log(1+w)}\int_0^w(1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda}dt\\ &= J_\lambda(w)\longrightarrow 1 \quad \text{as} \quad w\longrightarrow\infty.\end{aligned}$$

This establishes (6) and (7).

Conditions (8), (9), (10) and (14) hold by hypotheses, and (11) clearly holds.

Furthermore, condition (13) is immediate since $K(w,t)=0$ whenever $t\geqq w$. It remains to show (12). Suppose $-1<\lambda<0$. Then, by Lemma 11, we have

$$\begin{aligned}\int_0^x K(w,t)dt &= \frac{1}{\log(1+w)}\int_0^x(1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda}dt\\ &\leqq \frac{1}{\log(1+w)}\int_0^x(1+t)^{\lambda-1}\left(\frac{w-t}{w(1+t)}\right)^{\lambda}dt\\ &= \frac{1}{\log(1+w)}\int_0^x(1-t/w)^{\lambda}\frac{dt}{1+t}\\ &\leqq \frac{(1-x/w)^{\lambda}}{\log(1+w)}\int_0^x\frac{dt}{1+t}\\ &= (1-x/w)^{\lambda}\frac{\log(1+x)}{\log(1+w)}=o(1)\end{aligned}$$

as $w>x\to\infty$ and $\log\log w-\log\log x\to\infty$, since the latter implies $\log x/\log w\to 0$ and $x/w\to 0$.

Suppose $\lambda\geqq 0$ and $x>1$. Then

$$\log(1+w)\int_0^x K(w,t)dt = \int_0^x (1+t)^{\lambda-1}\Big(\log\frac{w(1+t)}{t(1+w)}\Big)^{\lambda} dt$$

$$\leq \Big(\int_0^1 + \int_1^x\Big)(1+t)^{\lambda-1}\Big(\log\frac{1+t}{t}\Big)^{\lambda} dt$$

$$= I_1 + I_2 .$$

Setting $u = 1/t$ in I_1 gives

$$I_1 = \int_1^\infty (1+1/u)^{\lambda-1}(\log(1+u))^{\lambda}\frac{du}{u^2}$$

$$= O(1) .$$

Furthermore,

$$I_2 = O(1)\int_1^x (1+t)^{-1}dt$$

$$= O(1)\log(1+x) - O(1) .$$

Therefore,

$$\int_0^x K(w,t)dt$$

$$= \frac{1}{\log(1+w)}\{I_1 + I_2\}$$

$$= o(1) + O(1)\frac{\log(1+x)}{\log(1+w)} = o(1)$$

as $w > x \to \infty$ and $\log\log w - \log\log x \to \infty$.

This completes the proof.

8. Proof of Theorem 1. Assign $\varepsilon > 0$. Since $\sigma_\lambda(t)$ is slowly decreasing with respect to $\Phi(t) = \log\log t$, there exist positive numbers X and δ such that $\sigma_\lambda(y) - \sigma_\lambda(x) > -\varepsilon$ whenever $y > x > X$ and $\log\log y - \log\log x < \delta$; or equivalently, writing $\delta = \log \gamma$

(15) $$\sigma_\lambda(x) - \varepsilon < \sigma_\lambda(y) \quad \text{whenever} \quad X < x < y < x^{\gamma} .$$

Suppose, without loss of generality, that $s = 0$. Then $J_\lambda(w) \to 0$ as $w \to \infty$.

Relation (15) implies, for $x > X$, that

$$I_1 = \int_x^{x^{\gamma}} (1+t)^{\lambda-1}\Big(\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda}(\sigma_\lambda(x) - \varepsilon)dt$$

$$\leq \int_x^{x^{\gamma}} (1+t)^{\lambda-1}\Big(\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda}\sigma_\lambda(t)dt$$

$$= I_2 .$$

Now, by Theorem 5 and Lemma 12,

$$I_2 = \Big(\int_0^{x^{\gamma}} - \int_0^x\Big)(1+t)^{\lambda-1}\Big(\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda}\sigma_\lambda(t)dt$$

$$= \log(1+x^{\gamma})J_\lambda(x^{\gamma}) - \log(1+x)J_\lambda(x) + O(1)$$

$$= o(\log(1+x^{\gamma})) + o(\log(1+x))$$

$$= o(\log(1+x)) .$$

By Lemma 13,

$$I_1 = (\sigma_\lambda(x) - \varepsilon)\int_x^{x^{\gamma}} (1+t)^{\lambda-1}\Big(\log\frac{x^{\gamma}(1+t)}{t(1+x^{\gamma})}\Big)^{\lambda} dt$$

$$= (\sigma_\lambda(x) - \varepsilon)((\gamma - 1)\log(1+x) + o(\log(1+x))) .$$

But $I_1 \leq I_2$ implies

$$\sigma_\lambda(x) - \varepsilon \leq \frac{o(1)}{(\gamma - 1) + o(1)} .$$

Therefore,

(16) $$\limsup_{x\to\infty} \sigma_\lambda(x) \leq \varepsilon .$$

In a similar fashion, we can show that

(17) $$-\varepsilon \leq \liminf_{x\to\infty} \sigma_\lambda(x) .$$

Combining (16) and (17) completes the proof of theorem.

9. A counterexample. In this section we give an example which shows that Theorem 1 would be false if $\log\log t$ were replaced by $\log t$. That is, a more delicate tauberian condition on $\sigma_\lambda(t)$ is required than what is obtained by using the standard definition of slowly decreasing.

LEMMA 14. *If $f(x)$ is absolutely continuous on $[0, T]$ for each $T > 0$ and $f'(x) > -M/x$ for all $x > 0$, then $f(x)$ is slowly decreasing with respect to $\log x$.*

Proof. Assign $\varepsilon > 0$. Then if $y > x > 0$

$$f(y) - f(x) = \int_x^y f'(t)dt$$

$$> -M\int_x^y \frac{1}{t}dt$$

$$= -M(\log y - \log x) > -\varepsilon$$

whenever $\log y - \log x < \varepsilon/M$. This completes the proof.

THEOREM 6. *There exists a sequence* $\{s_n\}$ *such that* $s_n \to s(L)$ *and, for every* $\lambda > -1$, $\sigma_\lambda(t)$ *is slowly decreasing with respect to* $\log t$, *but* $\{s_n\}$ *is not* A_λ*-convergent.*

Proof. Let $\{s_n\}$ be the real part of the sequence $\{\varepsilon_n^i\}$. For any $\lambda > -1$, $\sigma_\lambda(t)$ exists for $t > 0$, and we have

$$\varepsilon_n^i = \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \frac{\varepsilon_n^{\lambda+i}}{\varepsilon_n^\lambda} + o(1) .$$

Therefore, $\sigma_\lambda(t)$ is the real part of

$$(1 + t)^{-\lambda-1} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \varepsilon_n^{\lambda+i} \left(\frac{t}{1 + t}\right)^n + (1 + t)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda o(1) \left(\frac{t}{1 + t}\right)^n$$
$$= \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} (1 + t)^i + o(1) .$$

The first term above has a derivative which is $O(1/t)$ and, hence, the real part of the first term has a derivative which is $O(1/t)$. The second term is $o(1)$ since A_λ is regular. Hence, the real part of this term is slowly decreasing with respect to any Φ. Therefore, by Lemma 14, $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log t$.

Next, it is clear that $\{s_n\}$ is not A_λ-convergent.

However,

$$J_0(w) = \frac{1}{\log(1 + w)} \int_0^w (1 + t)^{-1} \sigma_0(t) dt$$
$$= \frac{1}{\log(1 + w)} \int_0^w \frac{\cos \log(1 + t)}{1 + t} dt$$
$$= \frac{\sin \log(1 + w)}{\log(1 + w)} \longrightarrow 0 \quad \text{as} \quad w \longrightarrow \infty .$$

Hence, by Theorem 3, $s_n \to O(L)$. This completes the proof.

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