ON THE ABSCISSAE OF SUMMABILITY OF A DIRICHLET SERIES

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1. Suppose throughout that (l_n) is an unboundedly increasing sequence of positive numbers with $l_0 > 1$ and that

$$D = \overline{\lim_{n\to\infty}} \frac{\log n}{\log l_n} < \infty.$$

Then D is the lower bound of real numbers σ for which $\Sigma l_n^{-\sigma}$ converges.

Let σ_k, $\bar{\sigma}_k$ be respectively the abscissae† of summability (R, l, k), $|R, l, k|$ of the Dirichlet series $\Sigma a_n l_n^{-s}$. It has been proved by Bosanquet‡ that, for $k = 0, 1, \ldots,$

$$\bar{\sigma}_k \leqslant \sigma_k + D; \tag{1}$$

and by Austin§ that, for $k \geqslant 0$, $0 < x \leqslant 1$,

$$\bar{\sigma}_{k+x} \leqslant \sigma_k + (1-x)D. \tag{2}$$

By investigating the continuity of σ_k as a function of k, Austin deduced from (2) that (1) must hold whenever $k \geqslant 0$ and $k \neq k_0$, where k_0 is the lower bound of numbers k such that $\sigma_k < \infty$.

The object of this paper is to prove directly that (1) is true for every $k \geqslant 0$.

2. Let

$$A(u) = \sum_{l_n \leqslant u} a_n, \quad B(u) = \sum_{l_n \leqslant u} b_n \quad (u \geqslant 1).$$

Write, for $\mu > -1$, $w \geqslant 1$,

$$\Gamma(\mu+1) A_\mu(w) = \sum_{l_n \leqslant w} (w - l_n)^\mu a_n = \int_1^w (w-u)^\mu \, dA(u),$$

where $w \neq l_n$ if $\mu < 0$, and define $B_\mu(w)$ similarly.

Then||, for $\mu > -1$, $\lambda > 0$, $\lambda + \mu > 0$, $w \geqslant 1$,

$$A_{\lambda+\mu}(w) = \frac{1}{\Gamma(\lambda)} \int_1^w (w-u)^{\lambda-1} A_\mu(u) \, du.$$

It will be sufficient for our purpose to prove the following

Theorem. *If* $k \geqslant 0$, $k+p > -1$ *and* $A_k(w) = O(w^{k+p})$, *then, for any* $\sigma > p + D$, $\Sigma a_n l_n^{-\sigma}$ *is summable* $|R, l, k|$.

We require some lemmas.

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† See G. H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series* (Cambridge Tract No. 18, 1915), 45–46; and N. Obrechkoff, *Math. Zeit.*, 30 (1928), 357–386.

‡ L. S. Bosanquet, *Journal London Math. Soc.*, 22 (1947), 190–195.

§ M. C. Austin, *Journal London Math. Soc.*, 27 (1952), 189–198.

|| See Austin, *loc. cit.*

3. LEMMA 1. *If $0<\delta\leqslant 1$, $\mu\geqslant 0$ and $1\leqslant x\leqslant w$, then*

$$\frac{1}{\Gamma(\delta)}\left|\int_1^x (w-u)^{\delta-1}A_\mu(u)\,du\right| \leqslant \max_{1\leqslant u\leqslant x}|A_{\mu+\delta}(u)|.$$

This is due to Riesz*.

LEMMA 2. *If $k\geqslant 0$, $k+p\geqslant 0$ and $A_k(w)=O(w^{k+p})$, then, for $\mu=0,1,\ldots,[k]$ and $l_m\leqslant w<l_{m+1}$,*

$$A_\mu(w)=O\left\{w^\mu l_m^{\,p}\left(\frac{l_{m+1}}{l_{m+1}-l_m}\right)^{k-\mu}\right\}.$$

Bosanquet has proved this result† for integral values of k.

Suppose then that k is not an integer. Put $s=[k]$, $\delta=k-s$, and let $l_m\leqslant w\leqslant l_{m+1}$. Then, by Lemma 1,

$$\begin{aligned}I(w)&=\int_{l_m}^{w}(w-u)^{\delta-1}A_s(u)\,du\\&=\int_1^w(w-u)^{\delta-1}A_s(u)\,du-\int_1^{l_m}(w-u)^{\delta-1}A_s(u)\,du=O(w^{k+p}).\end{aligned}$$

Also

$$\begin{aligned}I(w)&=\frac{1}{s!}\int_{l_m}^{w}(w-u)^{\delta-1}\,du\sum_{n=0}^{m}(u-l_n)^s a_n\\&=\frac{1}{s!}\sum_{\mu=0}^{s}\sum_{n=0}^{m}(l_m-l_n)^\mu a_n\binom{s}{\mu}\int_{l_m}^{w}(w-u)^{\delta-1}(u-l_m)^{s-\mu}\,du\\&=\sum_{\mu=0}^{s}\frac{\Gamma(\delta)}{\Gamma(k+1-\mu)}(w-l_m)^{k-\mu}A_\mu(l_m).\end{aligned}$$

Let $0<(s+1)h\leqslant l_{m+1}-l_m$ and put $w=l_m+(\nu+1)h$ in the above to get, for $\nu=0,1,\ldots,s$,

$$\sum_{\mu=0}^{s}c_{\nu,\mu}h^{k-\mu}A_\mu(l_m)=I\big(l_m+(\nu+1)h\big)=O\Big\{\big(l_m+(s+1)h\big)^{k+p}\Big\},$$

where $c_{\nu,\mu}=(\nu+1)^{k-\mu}\Gamma(\delta)/\Gamma(k+1-\mu)$.

Since the determinant $|c_{\nu,\mu}|$ is non-zero, we deduce that, for $\mu=0,1,\ldots,s$,

$$A_\mu(l_m)=O\Big\{\big(l_m+(s+1)h\big)^{k+p}h^{\mu-k}\Big\}.$$

If $l_{m+1}-l_m\leqslant l_m$ we take $(s+1)h=l_{m+1}-l_m$ to get

$$A_\mu(l_m)=O\{l_{m+1}^{k+p}(l_{m+1}-l_m)^{\mu-k}\}=O\left\{l_m^{p+\mu}\left(\frac{l_{m+1}}{l_{m+1}-l_m}\right)^{k-\mu}\right\};$$

and if $l_{m+1}-l_m>l_m$ we obtain the same result on taking $(s+1)h=l_m$.

* M. Riesz, *Acta Litt. ac Sci. Univ. Hungaricae (Szeged)*, 1 (1922–3), 114–126.

† Bosanquet, *loc. cit.*, Lemma 3.

With minor adjustments Bosanquet's concluding argument can now be used to complete the proof.

LEMMA 3. *If $k\geqslant 0$, $k+p>-1$, $A_k(w)=O(w^{k+p})$ and $b_n=a_nl_n^{\,q}$, where q is a positive integer, then*

$$B_k(w)=O(w^{k+p+q}).$$

For $q=1$ we have

$$B_k(w)=wA_k(w)-(k+1)A_{k+1}(w)=O(w^{k+p+1}),$$

and the result follows by induction.

LEMMA 4. *If $k>0$, $\sigma+k+q>0$, $b_n=a_nl_n^{\,q}$ and*

$$\int_1^\infty w^{-\sigma-k-q}|B_{k-1}(w)|\,dw<\infty,$$

then $\Sigma a_n l_n^{-\sigma}$ is summable $|R,l,k|$.

This follows from a result* given by Austin.

4. *Proof of the theorem. Suppose first that $k=0$.* Then

$$a_n=A(l_n)-A(l_n-0)=O(l_n^{\,p})+O(l_n^{\,p})=O(l_n^{\,p}).$$

Since $\Sigma l_n^{-\sigma+p}<\infty$ for $\sigma>p+D$, it follows that $\Sigma a_n l_n^{-\sigma}$ is absolutely convergent for such σ.

Suppose next that $k>0$, $\sigma>p+D$ and that $s+1$, q are positive integers such that $s<k\leqslant s+1$, $1-\sigma<q$. Let $\delta=k-s$ and $b_n=a_nl_n^{\,q}$.

Note that
$$\Sigma l_n^{-\sigma+p}<\infty, \tag{3}$$

and that, by Lemma 4, it is sufficient to prove that

$$\sum_{m=0}^{\infty}\int_{l_m}^{l_{m+1}}w^{-\sigma-k-q}|B_{k-1}(w)|\,dw<\infty.$$

Since $B(u)$ is constant for $l_m\leqslant u<l_{m+1}$, we have, for $l_m<w<l_{m+1}$,

$$\begin{aligned}\Gamma(k)B_{k-1}(w)&=\int_1^w(w-u)^{k-1}\,dB(u)\\&=(w-l_m)^{k-1}B(l_m)+(k-1)\int_1^{l_m}(w-u)^{k-2}B(u)\,du.\end{aligned}$$

Integrating s times by parts we get, for $l_m<w<l_{m+1}$,

$$B_{k-1}(w)=\sum_{\mu=0}c_\mu(w-l_m)^{k-1-\mu}B_\mu(l_m)+c_{s+1}\int_1^{l_m}(w-u)^{\delta-2}B_s(u)\,du, \tag{4}$$

where the c's are constants, and $c_{s+1}=0$ if k is an integer.

* Austin, *loc. cit.*, Lemma 2.

Now, by Lemmas 2 and 3, since $k+p > -1$ and $k+p+q > 0$,

$$B_\mu(w) = O\left\{w^\mu l_m^{p+q}\left(\frac{l_{m+1}}{l_{m+1}-l_m}\right)^{k-\mu}\right\}$$

for $\mu = 0, 1, \ldots, s$ and $l_m < w < l_{m+1}$.

Hence, for $\mu = 0, 1, \ldots, s$,

$$\int_{l_m}^{l_{m+1}} w^{-\sigma-k-q}(w-l_m)^{k-1-\mu}\,|B_\mu(l_m)|\,dw$$

$$= O\left\{l_m^{-\sigma-q+1}\, l_m^{p+q-1}\left(\frac{l_{m+1}}{l_{m+1}-l_m}\right)^{k-\mu}\int_{l_m}^{l_{m+1}}\left(1-\frac{l_m}{w}\right)^{k-1-\mu}\frac{l_m}{w^2}\,dw\right\}$$

$$= O(l_m^{-\sigma+p}). \tag{5}$$

If k is an integer, then, since $c_{s+1} = 0$, the result follows from (3), (4) and (5).

*Suppose finally that k is not an integer**. Then $0 < \delta < 1$ and

$$\sum_{m=0}^{\infty}\int_{l_m}^{l_{m+1}} w^{-\sigma-k-q}\,dw\int_1^{l_m}(w-u)^{\delta-2}\,|B_s(u)|\,du$$

$$= \sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\int_{l_m}^{l_{m+1}} w^{-\sigma-k-q}\,dw\int_{l_n}^{l_{n+1}}(w-u)^{\delta-2}\,|B_s(u)|\,du$$

$$= \sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty}\int_{l_m}^{l_{m+1}} w^{-\sigma-k-q}\,dw\int_{l_n}^{l_{n+1}}(w-u)^{\delta-2}\,|B_s(u)|\,du$$

$$= \sum_{n=0}^{\infty}\int_{l_{n+1}}^{\infty} w^{-\sigma-k-q}\,dw\int_{l_n}^{l_{n+1}}(w-u)^{\delta-2}\,|B_s(u)|\,du$$

$$= \sum_{n=0}^{\infty} I_n, \tag{6}$$

where

$$I_n = O\left\{l_{n+1}^{-\sigma-k-q}\int_{l_n}^{l_{n+1}}|B_s(u)|\,du\int_{l_{n+1}}^{\infty}(w-u)^{\delta-2}\,dw\right\}$$

$$= O\left\{l_{n+1}^{-\sigma-k-q}\int_{l_n}^{l_{n+1}}(l_{n+1}-u)^{\delta-1}\,|B_s(u)|\,du\right\}$$

$$= O\left\{l_{n+1}^{-\sigma-k-q}\, l_n^{p+q}\left(\frac{l_{n+1}}{l_{n+1}-l_n}\right)^{\delta}\int_{l_n}^{l_{n+1}}(l_{n+1}-u)^{\delta-1}\,u^s\,du\right\}$$

$$= O(l_{n+1}^{-\sigma-k-q}\, l_n^{p+q}\, l_{n+1}^{k}) = O(l_n^{-\sigma+p}). \tag{7}$$

The required result now follows from (3), (4), (5), (6) and (7), and the proof of the theorem is thus completed.

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