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GENERALIZATION OF THE HAUSDORFF MOMENT PROBLEM

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1. Introduction. Suppose throughout that $\{k_n\}$ is a sequence of positive integers, that

$$0 \leqq l_0 < l_1 < l_2 < \ldots < l_n,\ l_n \to \infty, \sum_{n=1}^{\infty} \frac{k_n}{l_n} = \infty,$$

that $k_0 = 1$ if $l_0 = 1$, and that $\{u_n^{(r)}\}$ $(r = 0, 1, \ldots, k_n - 1,\ n = 0, 1, \ldots)$ is a sequence of real numbers. We shall be concerned with the problem of establishing necessary and sufficient conditions for there to be a function α satisfying

$$(1) \qquad (-1)^r u_n^{(r)} = \int_0^1 t^{l_n} \log^r t\, d\alpha(t)$$

$$\text{for } r = 0, 1, \ldots, k_n - 1, \quad n = 0, 1, \ldots$$

and certain additional conditions. The case $l_0 = 0$, $k_n = 1$ for $n = 0, 1, \ldots$ of the problem is the version of the classical moment problem considered originally by Hausdorff [**5**], [**6**], [**7**]; the above formulation will emerge as a natural generalization thereof. An alternative formulation of the problem is to express it as the "infinite Hermite interpolation problem" of establishing necessary and sufficient conditions for a function F to be a Laplace transform of the form

$$F(z) = \int_0^{\infty} e^{-uz} d\gamma(u)$$

and to satisfy

$$F^{(r)}(l_n) = (-1)^r u_n^{(r)} \text{ for } r = 0, 1, \ldots, k_n - 1,\ n = 0, 1, \ldots.$$

Considerable simplification is obtained by adoption of the following notation. Construct a monotonic sequence $\{\lambda_s\}$ from $\{l_n\}$ by repeating each l_n k_n times. Then

$$0 \leqq \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n,\ \lambda_1 > 0, \quad \lambda_n \to \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

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For each s there is an integer $n(s)$ such that $\lambda_s = l_{n(s)}$. Let $m_s = k_{n(s)}$ and construct a sequence $\{\mu_s^{(r)}\}$ $(r = 0, 1, \ldots, m_s - 1,\ s = 0, 1, \ldots)$ from $\{u_n^{(r)}\}$ by setting $\mu_s^{(r)} = u_{n(s)}^{(r)}$. Then m_s is the multiplicity of λ_s, i.e., it is the number of indices j for which $\lambda_j = \lambda_s$; and $\mu_j^{(r)} = \mu_s^{(r)}$ whenever $\lambda_j = \lambda_s$. Formula (1) can be written in the equivalent form

$$(2) \qquad (-1)^r \mu_s^{(r)} = \int_0^1 t^{\lambda_s} \log^r t\, d\alpha(t)$$

$$\text{for } r = 0, 1, \ldots, m_s - 1, \quad s = 0, 1, \ldots.$$

For $0 \leqq k \leqq s \leqq n$, let $m_s(k, n)$ be the multiplicity of λ_s among $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$. By a standard result on Hermite interpolation (see [**3**, p. 29]) there is a unique polynomial $P_n(z)$ of degree at most n such that

$$(3) \qquad P_n^{(r)}(\lambda_s) = (-1)^r \mu_s^{(r)} \quad \text{for } r = 0, 1, \ldots, m_s(0, n) - 1,\ s = 0, 1, \ldots, n.$$

It is known (see [**11**, p. 45]) that

$$P_n(z) = \sum_{k=0}^{n} u[\lambda_k, \ldots, \lambda_n](\lambda_{k+1} - z) \ldots (\lambda_n - z)$$

where the divided difference $u[\lambda_k, \ldots, \lambda_n]$ is given by

$$u[\lambda_k, \ldots, \lambda_n] = -\frac{1}{2\pi i} \int_{C_{kn}} \frac{P_n(z)dz}{(\lambda_k - z) \ldots (\lambda_n - z)},$$

C_{kn} being a positively sensed Jordan contour enclosing $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$. For $0 \leqq k \leqq n$, $0 < t \leqq 1$, let

$$\lambda_{nk} = \lambda_{k+1} \ldots \lambda_n u[\lambda_k, \ldots, \lambda_n],$$

$$(4) \qquad \lambda_{nk}(t) = -\lambda_{k+1} \ldots \lambda_n \frac{1}{2\pi i} \int_{C_{kn}} \frac{t^z dz}{(\lambda_k - z) \ldots (\lambda_n - z)},$$

$$\lambda_{nk}(0) = \lambda_{nk}(0+),$$

with the convention that products such as $\lambda_{k+1} \ldots \lambda_n = 1$ when $k = n$.

If $f(z)$ is analytic inside and on C_{kn} then, by the theory of residues,

$$\int_{C_{kn}} \frac{f(z)dz}{(\lambda_k - z) \ldots (\lambda_n - z)}$$

is a linear combination, with coefficients depending only on $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$, of the values $f^{(r)}(\lambda_s)$, $r = 0, 1, \ldots, m_s(k, n) - 1$, $s = k, k + 1, \ldots, n$. It follows that $\lambda_{nk}(t)$ is a linear combination of the functions $t^{\lambda_s} \log^r t$, $r = 0, 1, \ldots, m_s(k, n) - 1$, $s = k, k + 1, \ldots, n$ and that λ_{nk} is the same linear combination with $(-1)^r \mu_s^{(r)}$ substituted for $t^{\lambda_s} \log^r t$. Consequently, if $\alpha \in \mathrm{BV}$, where BV is the space of normalized functions of bounded variation on [0, 1], i.e., $\alpha(0) = 0$, $2\alpha(t) = \alpha(t+) + \alpha(t-)$ for $0 < t < 1$, and if

$$(-1)^r \mu_s^{(r)} = \int_0^1 t^{\lambda_s} \log^r t \, d\alpha(t) \quad \text{for } 0 \leqq r < m_s(k, n), \quad k \leqq s \leqq n,$$

then

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t) d\alpha(t).$$

An explicit formula for $u[\lambda_k, \ldots, \lambda_n]$ can be obtained by evaluating

$$\frac{1}{2\pi i} \int_{C_{kn}} \frac{t^z dz}{(\lambda_k - z) \ldots (\lambda_n - z)}$$

and substituting $(-1)^r \mu_s^{(r)}$ for $t^{\lambda_s} \log^r t$ in the result.

Let

$$D_0 = (1 + \lambda_0) d_0 = 1, D_n = \left(1 + \frac{1}{\lambda_1}\right) \ldots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n) d_n \quad \text{for } n \geqq 1.$$

Then, for $n \geqq 0$,

$$D_n = \lambda_{n+1} d_{n+1} = \frac{\lambda_0}{1 + \lambda_0} + \sum_{k=0}^{n} d_k,$$

and, for $n > k \geqq 0$,

$$(5) \quad \sum_{j=k+1}^{n} \frac{1}{1 + \lambda_j} = \sum_{j=k+1}^{n} \frac{d_j}{D_j} \leqq \sum_{j=k+1}^{n} \int_{D_{j-1}}^{D_j} \frac{dx}{x} = \log \frac{D_n}{D_k} \leqq \sum_{j=k+1}^{n} \frac{d_j}{D_{j-1}} = \sum_{j=k+1}^{n} \frac{1}{\lambda_j}.$$

Further, it is known that if all the λ_n's are different, then

$$(6) \quad 0 \leqq \lambda_{ns}(t) \leqq \sum_{k=0}^{n} \lambda_{nk}(t) \leqq 1 \quad \text{for } 0 \leqq t \leqq 1, \quad 0 \leqq s \leqq n,$$

by [**10**, Lemma 1] and

$$(7) \quad \int_0^1 \lambda_{nk}(t) dt = \frac{d_k}{D_n} \quad \text{for } 0 \leqq k \leqq n$$

by [**6**, p. 294]. A simple continuity argument applied to (4) shows that (6) and (7) remain valid when different λ_n's are allowed to coalesce.

Let θ be an even continuous convex function such that $\theta(u)/u \to 0$ as $u \to 0$ and $\theta(u)/u \to \infty$ as $u \to \infty$. Associated with this function is the *Orlicz class* L_θ of all functions x Lebesgue integrable over [0, 1] for which

$$\int_0^1 \theta(x(t)) dt < \infty.$$

Let L_∞ be the space of measurable functions x on [0, 1] with finite norm

$$\|x\|_\infty = \text{ess. sup}_{0<t<1} |x(t)|.$$

Let

$$M_\theta(n) = \sum_{k=0}^{n} \frac{d_k}{D_n} \theta\left(\frac{D_n}{d_k} \lambda_{nk}\right),$$

$$M_1(n) = \sum_{k=0}^{n} |\lambda_{nk}|,$$

$$M_\infty(n) = \max_{0 \leqq k \leqq n} |\lambda_{nk}| \frac{D_n}{d_k},$$

and let

$$M_\theta = \sup_{n \geqq 0} M_\theta(n), \; M_1 = \sup_{n \geqq 0} M_1(n), \; M_\infty = \sup_{n \geqq 0} M_\infty(n).$$

The following two theorems are the main results established in the present paper.

THEOREM 1. *A necessary and sufficient condition for there to be a function*

(i) $\alpha \in$ BV *satisfying* (1) *is that* $M_1 < \infty$;

(ii) $\beta \in L_\infty$ *satisfying*

$$(8) \quad (-1)^r u_n^{(r)} = \int_0^1 t^{l_n} \log^r t \, \beta(t) dt \quad \text{for } r = 0, 1, \ldots, k_n - 1, n = 0, 1, \ldots$$

is that $M_\infty < \infty$;

(iii) $\beta \in L_\theta$ *satisfying* (8) *is that* $M_\theta < \infty$.

Furthermore

(iv) *if* (1) *is satisfied by a function* $\alpha \in$ BV, *then* $M_1 = \int_0^1 |d\alpha(t)| - \delta|\alpha(0+)|$ *where* $\delta = 0$ *when* $l_0 = 0$, $\delta = 1$ *when* $l_0 > 0$; *moreover* α *is unique when* $l_0 = 0$, *and when* $l_0 > 0$ *it differs by a constant, over the interval* $0 < t \leqq 1$, *from any other function in* BV *satisfying* (1);

(v) *if* (8) *is satisfied by a function* $\beta \in L_\infty$, *then* β *is essentially unique and* $M_\infty = \|\beta\|_\infty$;

(vi) *if* (8) *is satisfied by a function* $\beta \in L_\theta$, *then* β *is essentially unique and*

$$M_\theta = \int_0^1 \theta(\beta(t)) dt.$$

THEOREM 2. *For* $n = 0, 1, \ldots$,

$$M_1(n) \leqq M_1(n+1), \; M_\infty(n) \leqq M_\infty(n+1), \; M_\theta(n) \leqq M_\theta(n+1);$$

and

$$\lim_{n\to\infty} M_1(n) = M_1, \lim_{n\to\infty} M_\infty(n) = M_\infty, \lim_{n\to\infty} M_\theta(n) = M_\theta.$$

The case $l_0 = 0$, $k_n = 1$ for $n = 0, 1, \ldots$ of Theorem 1(i) was established by Hausdorff [**5**], [**6**] and Schoenberg [**13**] subsequently gave a different proof; the case $l_0 > 0$, $k_n = 1$ for $n = 0, 1, \ldots$ was proved by Leviatan [**9**]. (See also [**4**].)

The case $l_n = n$, $k_n = 1$ for $n = 0, 1, \ldots$ of Theorem 1(ii) is due to Hausdorff [**7**].

The case $l_n = n$, $k_n = 1$ for $n = 0, 1, \ldots, \theta(u) = |u|^p, 1 < p < \infty$, of Theorem 1(iii) is due to Hausdorff [**7**] and the case $k_n = 1$ for $n = 0, 1, \ldots$ to Leviatan [**9**], [**10**]. (See also [**1**] and [**2**].)

See [**2**] and the references there given for known special cases of Theorem 2.

2. Preliminary results.

LEMMA 1. *Let r, a be non-negative integers, let $0 < \lambda < \lambda_{a+1}$, and let*

$$\delta_{nk} = \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \cdots \left(1 - \frac{\lambda}{\lambda_n}\right)\left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^r.$$

Then (i) δ_{nk} *is uniformly bounded for* $n > k \geqq a$,

(ii) $\lim_{n\to\infty} \delta_{nk} = 0$ *for* $k \geqq a$,

(iii) $\delta_{nk} - \left(\frac{D_k}{D_n}\right)^\lambda \log^r \frac{D_n}{D_k} \to 0$ *uniformly when* $n > k \to \infty$.

Proof. Let $0 < \epsilon < \lambda$, $\alpha = \lambda - \epsilon$, $\beta = \lambda + \epsilon$, let

$$\gamma = \gamma_{nk} = \sum_{j=k+1}^{n} \frac{1}{\lambda_j},$$

and, for $n > a$, let

$$u_n = 1 - \frac{\lambda}{\lambda_n} = e^{-\alpha_n/\lambda_n}, v_n = \left(1 + \frac{1}{\lambda_n}\right)^{-\lambda} = e^{-\beta_n/\lambda_n}.$$

Then $\alpha_n \to \lambda$, $\beta_n \to \lambda$ and so we can choose a positive integer $N \geqq a$ so large that

$$|\alpha_n - \lambda| < \epsilon, |\beta_n - \lambda| < \epsilon \quad \text{for } n > N.$$

First, for $n > k \geqq N$, we have that

$$0 < \delta_{nk} = u_{k+1} \cdots u_n \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^r \leqq e^{-\alpha\gamma}\gamma^r \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r.$$

Since $\gamma_{nk} \to \infty$ as $n \to \infty$, it follows that (i) and (ii) hold for $k \geqq N$. The extension of these conclusions to the range $N > k \geqq a$ is simple.

Next, let

$$a_{nk} = |u_{k+1} \cdots u_n - v_{k+1} \cdots v_n| \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^r,$$

$$b_{nk} = v_{k+1} \cdots v_n \left\{\left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^r - \log^r \frac{D_n}{D_k}\right\}.$$

Then, for $n > k \geqq N$, we have that

$$(9) \quad 0 \leqq a_{nk} \leqq (e^{-\alpha\gamma} - e^{-\beta\gamma})\gamma^r \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r$$
$$\leqq \gamma(\beta - \alpha)e^{-\alpha\gamma}\gamma^r \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r$$
$$\leqq \frac{2\gamma\epsilon\gamma^r(r+1)!}{(\alpha\gamma)^{r+1}} \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r = \frac{2(r+1)!}{(\lambda - \epsilon)^{r+1}} \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r \epsilon,$$

and, by (5), that

$$(10) \quad 0 \leqq b_{nk} \leqq v_{k+1} \cdots v_n r \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^{r-1} \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda} - \log \frac{D_n}{D_k}\right)$$
$$\leqq v_{k+1} \cdots v_n r \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^{r-1} \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda} - \frac{1}{1 + \lambda_j}\right)$$
$$\leqq v_{k+1} \cdots v_n r \left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - \lambda}\right)^r \frac{\lambda + 1}{\lambda_{k+1}}$$
$$\leqq e^{-\alpha\gamma} r\gamma^r \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r \frac{\lambda + 1}{\lambda_{k+1}} \leqq \frac{rr!}{(\lambda - \epsilon)^r} \left(\frac{\lambda_{k+1}}{\lambda_{k+1} - \lambda}\right)^r \frac{\lambda + 1}{\lambda_{k+1}}.$$

It follows from (9) that $a_{nk} \to 0$ uniformly when $n > k \to \infty$, and from (10) that $b_{nk} \to 0$ uniformly when $n > k \to \infty$. Since

$$\left|\delta_{nk} - \left(\frac{D_k}{D_n}\right)^\lambda \log^r \frac{D_n}{D_k}\right| \leqq a_{nk} + b_{nk} \quad \text{for } n > k \geqq N,$$

conclusion (iii) follows.

LEMMA 2. *Let* $\psi(t) = (\lambda_{k+1} - t) \cdots (\lambda_n - t)$ *where* $0 \leqq k < n$ *and* $0 < t < \lambda_{k+1}$, *and let r be a positive integer. Then*

$$\left|\psi^{(r)}(t) - (-1)^r \psi(t)\left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - t}\right)^r\right| \leqq \frac{M\psi(t)}{\lambda_{k+1} - t}\left(\sum_{j=k+1}^{n} \frac{1}{\lambda_j - t}\right)^{r-1}$$

where M is a number independent of t, k and n.

Proof. The result is evidently true with $M = 0$ when $r = 1$. Suppose therefore that $r \geqq 2$ and let

$$\gamma_j = \frac{1}{\lambda_j - t}.$$

As easy inductive argument shows that

$$\frac{\psi^{(r)}(t)}{\psi(t)} - (-1)^r\left(\sum_{j=k+1}^{n}\gamma_j\right)^r$$

is equal to a linear combination with constant coefficients of terms of the form

$$\left(\sum_{j=k+1}^{n}\gamma_j^{a_1}\right)^{b_1}\left(\sum_{j=k+1}^{n}\gamma_j^{a_2}\right)^{b_2}\cdots\left(\sum_{j=k+1}^{n}\gamma_j^{a_m}\right)^{b_m}$$

where the a_i's and b_i's are positive integers, $a_1 > 1$ and

$$a_1b_1 + a_2b_2 + \dots + a_mb_m = r.$$

Each of the terms is no greater than

$$\gamma_{k+1}\left(\sum_{j=k+1}^{n}\gamma_j^{a_1-1}\right)\left(\sum_{j=k+1}^{n}\gamma_j^{a_1}\right)^{b_1-1}\left(\sum_{j=k+1}^{n}\gamma_j^{a_2}\right)^{b_2}\cdots\left(\sum_{j=k+1}^{n}\gamma_j^{a_m}\right)^{b_m}$$

$$\leqq \gamma_{k+1}\left(\sum_{j=k+1}^{n}\gamma_j\right)^{a_1-1+a_1(b_1-1)+a_2b_2+\dots+a_mb_m} = \gamma_{k+1}\left(\sum_{j=k+1}^{n}\gamma_j\right)^{r-1}.$$

The desired conclusion follows.

LEMMA 3. *Let* $\psi(t) = (\lambda_{s+1} - t)\dots(\lambda_n - t)$, $\Phi(t) = (\lambda_s - t)^a\psi(t)$ *where* a *is a positive integer,* $0 \leqq s < n$ *and* $\lambda_s < \lambda_{s+1}$. *Then* $\Phi^{(r)}(\lambda_s) = 0$ *when* $0 \leqq r < a$, *and when* $r \geqq a$,

$$|\Phi^{(r)}(\lambda_s)| \leqq M\psi(\lambda_s)\left(\sum_{j=s+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^{r-a}$$

where M *is a number independent of* s *and* n.

Proof. The first part is evident. For the second part we observe that, when $r \geqq a$,

$$|\Phi^{(r)}(\lambda_s)| = r(r-1)\dots(r-a+1)\psi^{(r-a)}(\lambda_s),$$

and, as in the proof of Lemma 2, that $\psi^{(r-a)}(\lambda_s)/\psi(\lambda_s)$ can be expressed as a linear combination with constant coefficients of terms each with absolute value no greater than

$$\left(\sum_{j=s+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^{r-a}.$$

The desired conclusion follows.

LEMMA 4. *If* $M_1 < \infty$, $\lambda_s < \lambda_{s+1}$ *and* $r = 0, 1, \dots, m_s - 1$, *then*

$$\mu_s^{(r)} = \lim_{n\to\infty}\sum_{k=s}^{n}\lambda_{nk}\left(1 - \frac{\lambda_s}{\lambda_{k+1}}\right)\dots\left(1 - \frac{\lambda_s}{\lambda_n}\right)\left(\sum_{j=k+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^r.$$

Proof. For $r = 0$ the above sum is equal to $\mu_s^{(0)}$ for every $n \geqq s$ by

(3). Suppose therefore that $1 \leqq r \leqq m_s - 1$. Then, by Lemmas 2 and 3 we have, for $n \geqq s$, that

$$(11)\quad \left|(-1)^rP_n^{(r)}(\lambda_s) - \sum_{k=s}^{n}\lambda_{nk}\left(1 - \frac{\lambda_s}{\lambda_{k+1}}\right)\dots\left(1 - \frac{\lambda_s}{\lambda_n}\right)\times\left(\sum_{j=k+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^r\right|$$

$$\leqq M\sum_{k=s}^{n}|\lambda_{nk}|\frac{w_{nk}}{\lambda_{k+1} - \lambda_s} + Mw_{ns}\sum_{k=s-m_s+1}^{s-1}|\lambda_{nk}|$$

where M is a positive number independent of s and n, and

$$w_{nk} = \left(1 - \frac{\lambda_s}{\lambda_{k+1}}\right)\dots\left(1 - \frac{\lambda_s}{\lambda_n}\right)\left(\sum_{j=k+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^{r-1}.$$

Since $\sum_{k=0}^{n}|\lambda_{nk}| \leqq M_1$ for $n \geqq 0$, and, by Lemma 1(i) and (ii), w_{nk} is uniformly bounded and $\lim_{n\to\infty} w_{nk} = 0$ for $k \geqq s$, the right-hand side of (11) tends to 0 as $n \to \infty$. In view of (3), this establishes the desired conclusion.

LEMMA 5. *If* $M_1 < \infty$ *and* $r = 0, 1, \dots, m_s - 1$, *then*

$$(-1)^r\mu_s^{(r)} = \lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{nk}\left(\frac{D_k}{D_n}\right)^{\lambda_s}\log^r\frac{D_k}{D_n}.$$

Proof. Suppose, without loss in generality, that $\lambda_s < \lambda_{s+1}$, and let

$$\delta_{nk} = \left(1 - \frac{\lambda_s}{\lambda_{k+1}}\right)\dots\left(1 - \frac{\lambda_s}{\lambda_n}\right)\left(\sum_{j=k+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^r.$$

Then, by Lemma 1(ii) and (iii),

$$\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{nk}\left\{\delta_{nk} - \left(\frac{D_k}{D_n}\right)^{\lambda_s}\log^r\frac{D_n}{D_k}\right\} = 0$$

since $\sum_{k=0}^{n}|\lambda_{nk}| \leqq M_1$ for $n \geqq 0$ and $D_n \to \infty$; and, by Lemma 1(ii) and Lemma 4,

$$\lim_{n\to\infty}\sum_{k=0}^{n}\lambda_{nk}\delta_{nk} = \mu_s^{(r)}.$$

The desired conclusion follows.

LEMMA 6. *If a function* $x \in$ BV *is such that*

$$\int_0^1 t^{\lambda_s}\log^r t\,dx(t) = 0 \quad \textit{for } r = 0, 1, \dots, m_s - 1, \quad s = 0, 1, \dots,$$

then $x(t) = x(0+)$ *for* $0 < t \leqq 1$. *If, in addition,* $\lambda_0 = 0$, *then* $x(0+) = 0$.

Proof. When $\lambda_0 = 0$ it follows from a known result (see [**11**, Theorem 8.2]) that

$$\int_0^1 t^n dx(t) = 0 \quad \text{for } n = 0, 1, \ldots .$$

The proof can now be completed in the same way as in the proof of Lemma 3 in [**2**].

3. Proofs of the main results.

Proofs of the necessity parts of Theorem 1(i), (ii) *and* (iii).

Part (i). Suppose the function $\alpha \in \mathrm{BV}$ satisfies (1). For $0 \leqq k \leqq n$, we have that

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t)d\alpha(t),$$

and thus, by (6),

$$\sum_{k=0}^{n} |\lambda_{nk}| \leqq \int_0^1 |d\alpha(t)| \sum_{k=0}^{n} \lambda_{nk}(t) \leqq \int_0^1 |d\alpha(t)|.$$

Hence

$$(12) \quad M_1 \leqq \int_0^1 |d\alpha(t)|.$$

Part (ii). Suppose the function $\beta \in L_\infty$ satisfies (8). For $0 \leqq k \leqq n$, we have that

$$(13) \quad \lambda_{nk} = \int_0^1 \lambda_{nk}(t)\beta(t)dt$$

and thus, by (6) and (7),

$$|\lambda_{nk}| \leqq ||\beta||_\infty \frac{d_k}{D_n}.$$

Hence

$$(14) \quad M_\infty \leqq ||\beta||_\infty.$$

Part (iii). Suppose the function $\beta \in L_\theta$ satisfies (8). It follows from (13) and (7), by Jensen's inequality (see [**15**, pp. 23–24]) that

$$\Theta\left(\frac{D_n}{d_k}\lambda_{nk}\right) \leqq \frac{D_n}{d_k}\int_0^1 \lambda_{nk}(t)\Theta(\beta(t))dt \quad \text{for } 0 \leqq k \leqq n.$$

Hence, by (6),

$$\sum_{k=0}^{n} \frac{d_k}{D_n}\Theta\left(\frac{D_n}{d_k}\lambda_{nk}\right) \leqq \int_0^1 \Theta(\beta(t))dt$$

and so

$$(15) \quad M_\Theta \leqq \int_0^1 \Theta(\beta(t))dt.$$

Proofs of the sufficiency parts of Theorem 1(i), (ii) *and* (iii).

We first observe that

$$\sum_{k=0}^{n} |\lambda_{nk}| \leqq M_\infty \sum_{k=0}^{n} \frac{d_k}{D_n} \leqq M_\infty,$$

and, by Young's inequality (see [**8**, p. 12]), that

$$\frac{D_n}{d_k}|\lambda_{nk}| \leqq N(1) + \Theta\left(\frac{D_n}{d_k}\lambda_{nk}\right)$$

where N is the convex function complementary to Θ (see [**8**, p. 11]). Hence

$$\sum_{k=0}^{n} |\lambda_{nk}| \leqq N(1)\sum_{k=0}^{n} \frac{d_k}{D_n} + \sum_{k=0}^{n} \frac{d_k}{D_n}\Theta\left(\frac{D_n}{d_k}\lambda_{nk}\right) \leqq N(1) + M_\Theta.$$

It follows that $M_1 \leqq M_\infty$, $M_1 \leqq N(1) + M_\theta$ and so $M_1 < \infty$ under each of the three hypotheses of the sufficiency parts of Theorem 1(i), (ii) and (iii). Suppose therefore that $M_1 < \infty$.

For $n = 0, 1, \ldots$, define the function α_n on $[0, 1]$ by setting

$$\alpha_n(t) = \begin{cases} 0 & \text{for } 0 \leqq t < 1/D_n, \\ \sum_{D_k \leqq tD_n} \lambda_{nk} & \text{for } 1/D_n \leqq t \leqq 1, \end{cases}$$

so that

$$\int_0^1 |d\alpha_n(t)| = \sum_{k=0}^{n} |\lambda_{nk}| \leqq M_1.$$

Consequently, by Helly's theorem (see [**14**, p. 29]), there is an increasing sequence of positive integers $\{n_i\}$ and a function α of bounded variation on $[0, 1]$ such that

$$(16) \quad \lim_{i\to\infty} \alpha_{n_i}(t) = \alpha(t) \quad \text{for } 0 \leqq t \leqq 1$$

and

$$(17) \quad \int_0^1 |d\alpha(t)| \leqq M_1.$$

Part (i). By Lemma 5, we have that

$$(-1)^r \mu_s^{(r)} = \lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk}\left(\frac{D_k}{D_n}\right)^{\lambda_s} \log^r \frac{D_k}{D_s} = \lim_{n\to\infty} \int_0^1 t^{\lambda_s} \log^r t \, d\alpha_n(t)$$

for $r = 0, 1, \ldots, m_s - 1$, $s = 0, 1, \ldots$. It follows, by the Helly-Bray theorem, (see [**14**, p. 31]) that α satisfies (2) and hence (1).

Part (ii). Suppose $M_\infty < \infty$. Let $0 \leqq x < y \leqq 1$. Then for n sufficiently large there are integers a, b (depending on n) such that $-1 \leqq a < b \leqq n$ and

$$\frac{D_a}{D_n} \leqq x < \frac{D_{a+1}}{D_n} \leqq \frac{D_b}{D_n} \leqq y < \frac{D_{b+1}}{D_n} \quad (D_{-1} = 0),$$

since

$$\max_{0 \leqq k \leqq n} \frac{d_k}{D_n} = \max_{0 \leqq k \leqq n} \frac{D_k}{D_n} \frac{1}{1+\lambda_k} \to 0 \quad \text{as } n \to \infty.$$

Now

$$\frac{|\alpha_n(y) - \alpha_n(x)|}{\sum_{k=a+1}^{b} \frac{d_k}{D_n}} = \frac{\left|\sum_{k=a+1}^{b} \lambda_{nk}\right|}{\sum_{k=a+1}^{b} \frac{d_k}{D_n}} \leqq M_\infty,$$

and

$$\lim_{n\to\infty} \sum_{k=a+1}^{b} \frac{d_k}{D_n} = y - x.$$

In view of (16), it follows that

$$\frac{|\alpha(y) - \alpha(x)|}{y - x} \leqq M_\infty.$$

Hence

$$\alpha(t) = c + \int_0^t \beta(u)du \quad \text{for } 0 \leqq t \leqq 1$$

where $\beta \in L_\infty$ and $\|\beta\|_\infty \leqq M_\infty$. Further, β satisfies (8) since α satisfies (1).

Part (iii). Suppose $M_\theta < \infty$. Let $0 = x_0 < x_1 < \ldots < x_m = 1$. Then, for n sufficiently large, there exist integers $a_0, a_1, \ldots, a_m$ (depending on n) such that $-1 = a_0 < a_1 < \ldots < a_m = n$ and

$$\frac{D_{a_j}}{D_n} \leqq x_j < \frac{D_{1+a_j}}{D_n} \quad \text{for } j = 1, 2, \ldots, m-1,$$

so that

$$\alpha_n(x_{j+1}) - \alpha_n(x_j) = \sum_{k=1+a_j}^{a_{j+1}} \lambda_{nk} \quad \text{for } j = 0, 1, \ldots, m-1.$$

Let

$$\sigma_{jn} = \left(\sum_{k=1+a_j}^{a_{j+1}} \frac{d_k}{D_n}\right) \theta\left(\frac{\alpha_n(x_{j+1}) - \alpha_n(x_j)}{\sum_{k=1+a_j}^{a_{j+1}} \frac{d_k}{D_n}}\right).$$

Then, by Jensen's inequality (see [**15**, pp. 23–24]),

$$\sigma_{jn} \leqq \sum_{k=1+a_j}^{a_{j+1}} \frac{d_k}{D_n} \theta\left(\frac{D_n}{d_k}\lambda_{nk}\right) \quad \text{for } j = 0, 1, \ldots, m-1,$$

and so

$$\sum_{j=0}^{m-1} \sigma_{jn} \leqq M_\theta.$$

Also

$$\lim_{n\to\infty} \sum_{k=1+a_j}^{a_{j+1}} \frac{d_k}{D_n} = x_{j+1} - x_j \quad \text{for } j = 0, 1, \ldots, m-1.$$

In view of (16), it follows that

$$\lim_{n\to\infty} \sum_{j=0}^{m-1} \sigma_{jn} = \sum_{j=0}^{m-1} (x_{j+1} - x_j)\Theta\left(\frac{\alpha(x_{j+1}) - \alpha(x_j)}{x_{j+1} - x_j}\right) \leqq M_\Theta,$$

and, by a theorem of Medvedev [**12**], this implies that

$$\alpha(t) = c + \int_0^t \beta(u)du \quad \text{for } 0 \leqq t \leqq 1$$

where $\beta \in L_\theta$ and $\int_0^1 \theta(\beta(t))dt \leqq M_\theta$. Further, β satisfies (8) since α satisfies (1).

Proofs of Theorem 1(iv), (v) *and* (vi).

Part (iv). Suppose that $l_0 = 0$. By Lemma 6 the function $\alpha \in BV$ satisfying (1) is unique. By (12), (17) and the proof of the sufficiency part of Theorem 1(i), we have that

$$M_1 \leqq \int_0^1 |d\alpha(t)| \leqq M_1.$$

Suppose that $l_0 > 0$, and let $\gamma(0) = 0$, $\gamma(t) = \alpha(t) - \alpha(0+)$ for $0 < t \leqq 1$. Then $\gamma \in BV$ and satisfies (1). Hence, by (12),

$$M_1 \leqq \int_0^1 |d\gamma(t)|.$$

Further, by (17) and the proof of the sufficiency part of Theorem 1(i), there is a function $\tilde{\alpha} \in BV$ satisfying (1) and

$$\int_0^1 |d\tilde{\alpha}(t)| \leqq M_1.$$

By Lemma 6, $\gamma(t) = \tilde{\alpha}(t) - \tilde{\alpha}(0+)$ for $0 < t \leqq 1$. Since $\gamma(0+) = \gamma(0)$, we have that

$$M_1 \leqq \int_0^1 |d\gamma(t)| \leqq \int_0^1 |d\tilde{\alpha}(t)| \leqq M_1.$$

Hence

$$M_1 = \int_0^1 |d\alpha(t)| - |\alpha(0+)|.$$

Part (v). By Lemma 6, the function $\beta \in L_\infty$ satisfying (8) is essentially unique. By (14) and the proof of the sufficiency part of Theorem 1(ii), we have that $M_\infty \leqq \|\beta\|_\infty \leqq M_\infty$.

Part (vi). This part can be established by the proof of Part (v) with certain obvious modifications.

Proof of Theorem 2. Let $0 \leqq k \leqq n$. Then

$$\left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\lambda_{n+1,k} + \frac{\lambda_{k+1}}{\lambda_{n+1}}\lambda_{n+1,k+1}$$

$$= -\lambda_{k+1}\dots\lambda_{n+1}\left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{1}{2\pi i}\int_{C_{k,n+1}}\frac{P_{n+1}(z)dz}{(\lambda_k - z)\dots(\lambda_{n+1} - z)}$$

$$-\lambda_{k+2}\dots\lambda_{n+1}\frac{\lambda_{k+1}}{\lambda_{n+1}}\frac{1}{2\pi i}\int_{C_{k,n+1}}\frac{P_{n+1}(z)dz}{(\lambda_{k+1} - z)\dots(\lambda_{n+1} - z)}$$

$$= -\lambda_{k+1}\dots\lambda_n\frac{1}{2\pi i}\int_{C_{k,n+1}}\frac{P_{n+1}(z)dz}{(\lambda_k - z)\dots(\lambda_n - z)} = \lambda_{nk};$$

and hence

$$(18)\quad \lambda_{nk}\frac{D_n}{d_k} = \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\lambda_{n+1,k}\frac{D_n}{d_k} + (1 + \lambda_k)\frac{\lambda_{n+1,k+1}}{\lambda_{n+1}}\frac{D_n}{d_{k+1}}.$$

It follows that

$$M_\infty(n) \leqq M_\infty(n+1)\left(1 + \frac{1}{\lambda_{n+1}}\right)\frac{D_n}{D_{n+1}} = M_\infty(n+1).$$

Since

$$\left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{D_n}{D_{n+1}} + (1 + \lambda_k)\frac{D_n}{\lambda_{n+1}D_{n+1}} = 1,$$

applying Jensen's inequality to (18) yields

$$\frac{d_k}{D_n}\Theta\left(\frac{D_n}{d_k}\lambda_{nk}\right)$$

$$\leqq \frac{d_k}{D_n}\left\{\left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{D_n}{D_{n+1}}\Theta\left(\lambda_{n+1,k+1}\frac{D_{n+1}}{d_k}\right)\right.$$

$$\left.+ (1 + \lambda_k)\frac{D_n}{\lambda_{n+1}D_{n+1}}\Theta\left(\lambda_{n+1,k+1}\frac{D_{n+1}}{d_{k+1}}\right)\right\}$$

$$= \left(1 - \frac{\lambda_k}{\lambda_{n+1}}\right)\frac{d_k}{D_{n+1}}\Theta\left(\lambda_{n+1,k}\frac{D_{n+1}}{d_k}\right) + \frac{\lambda_{k+1}}{\lambda_{n+1}}\frac{d_{k+1}}{D_{n+1}}\Theta\left(\lambda_{n+1,k+1}\frac{D_{n+1}}{d_{k+1}}\right).$$

Summing this inequality for $k = 0, 1, \dots, n$, we get that

$$M_\theta(n) \leqq M_\theta(n+1) - \frac{\lambda_0 d_0}{\lambda_{n+1}D_{n+1}}\Theta\left(\lambda_{n+1,0}\frac{D_{n+1}}{d_0}\right) \leqq M_\theta(n+1).$$

Since the above argument is valid when θ is any even continuous convex function, we can take $\theta(u) = |u|$ to obtain, in addition, that

$$M_1(n) \leqq M_1(n+1).$$

This completes the proof of Theorem 2.

Note. In all but Theorem 2 the condition that the sequences $\{l_n\}$ and $\{\lambda_n\}$ be monotonic is redundant and was imposed only to avoid non-essential and tedious complication in the proofs. Without the monotonicity condition, but with $\{l_n\}$ distinct, $\lambda_0 = l_0 \geqq 0$, $k_0 = 1$ if $l_0 = 0$, $l_n > 0$ for $n = 1, 2, \dots$, identities and inequalities such as (5), (6) (using (10) and (11) on p. 46 of [**11**] and the proof of Lemma 1 in [**10**]) and (7) can readily be shown to hold, and Lemmas 5 and 6 and Theorem 1 remain valid. Removal of the monotonicity condition involves changes in statements and proofs of lemmas as indicated below.

Statements.

LEMMA 1. Replace $0 < \lambda < \lambda_{a+1}$ by $0 < \lambda < \min_{k>a}\lambda_k$.

LEMMA 2. Replace $0 < t < \lambda_{k+1}$ by $0 < t \neq \lambda_j$ for $j > k$, and

$$\frac{\psi(t)}{\lambda_{k+1} - t}\left(\sum_{j=k+1}^{n}\frac{1}{\lambda_j - t}\right)^{r-1} \text{ by } \max_{i>k}\frac{|\psi(t)|}{|\lambda_i - t|}\left(\sum_{j=k+1}^{n}\frac{1}{|\lambda_j - t|}\right)^{r-1}.$$

LEMMA 3. Replace $\lambda_s < \lambda_{s+1}$ by $\lambda_s \neq \lambda_j$ for $n > j > s$, and

$$\psi(\lambda_s)\left(\sum_{j=s+1}^{n}\frac{1}{\lambda_j - \lambda_s}\right)^{r-a} \text{ by } |\psi(\lambda_s)|\left(\sum_{j=s+1}^{n}\frac{1}{|\lambda_j - \lambda_s|}\right)^{r-a}.$$

LEMMA 4. Replace $\lambda_s < \lambda_{s+1}$ by $\lambda_s \neq \lambda_j$ for $j > s$.

Proofs.

Lemma 1. Replace $\lambda_{k+1}/(\lambda_{k+1} - \lambda)$ by $\max_{j>k}\lambda_j/(\lambda_j - \lambda)$, and $1/\lambda_{k+1}$ by $\max_{j>k}1/\lambda_j$.

Lemma 2. In the inequalities replace γ_j by $|\gamma_j|$ and γ_{k+1} by $\max_{j>k}|\gamma_j|$.

Lemma 3. Replace $\lambda_j - \lambda_s$ by $|\lambda_j - \lambda_s|$.

Lemma 4. Replace $1/(\lambda_{k+1} - \lambda_s)$ by $\max_{j>k}1/|\lambda_j - \lambda_s|$, and take

$$w_{nk} = \left|\left(1 - \frac{\lambda_s}{\lambda_{k+1}}\right)\dots\left(1 - \frac{\lambda_s}{\lambda_n}\right)\right|\left(\sum_{j=k+1}^{n}\frac{1}{|\lambda_j - \lambda_s|}\right)^{r-1}.$$

Lemma 5. Replace $\lambda_s < \lambda_{s+1}$ by $\lambda_s \neq \lambda_j$ for $j > s$.

References

1. D. L. Berman, *Application of interpolatory polynomial operators to solve the moment problem*, Urkain Math. Z. *14* (1963), 184–190.
2. D. Borwein, *The Hausdorff moment problem*, Can. Math. Bull. *21* (1978), 257–265; Corrections *22* (1979), 128.
3. P. J. Davis, *Interpolation and approximation* (Blaisdell, New York, 1965).
4. K. Endl, *On systems of linear inequalities in infinitely many variables and generalized Hausdorff means*, Math. Z. *82* (1963), 1–7.
5. F. Hausdorff, *Summationsmethoden und Momentfolgen I*, Math. Z. *9* (1921), 74–109.
6. ——— *Summationsmethoden und Momentfolgen II*, Math. Z. *9* (1921), 280–299.
7. ——— *Momentprobleme für ein eindliches Interval*, Math. Z. *16* (1923), 220–248.
8. M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces* (P. Noordhoff, Groningen, 1961).
9. D. Leviatan, *A generalized moment problem*, Israel J. Math. *5* (1967), 97–103.
10. ——— *Some moment problems in a finite interval*, Can. J. Math. *20* (1968), 960–966.
11. G. G. Lorentz, *Bernstein polynomials* (University of Toronto Press, Toronto, 1953).
12. U. T. Medvedev, *Generalization of a theorem of F. Riesz*, Uspeshi, Mat. Nauk *8* (1953), 115–118.
13. I. J. Schoenberg, *On finite rowed systems of linear inequalities in infinitely many variables*, Trans. A.M.S. *34* (1932), 594–619.
14. D. V. Widder, *The Laplace transform* (Princeton, 1946).
15. A. Zygmund, *Trigonometric series I*, 2nd edition (Cambridge University Press, Cambridge, 1959).

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