Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 37 (1—3) (1981), pp. 29—34.

WEIGHTED MEANS, GENERALISED HAUSDORFF MATRICES AND THE BOREL PROPERTY

By
D. BORWEIN and F. P. CASS (London, Ontario) *

Introduction

Let $A=\{a_{nk}\}$ $(n, k=0, 1, 2, \dots)$ be a (summability) matrix and let $\{s_k\}$ be a sequence. Let

$$t_n = \sum_{k=0}^{\infty} a_{nk} s_k. \tag{1}$$

The sequence $\{s_k\}$ is said to be *A-convergent* to the value s if t_n exists for $n=0, 1, 2, \dots$ and tends to s. In this case we write $s_n \to s(A)$ and call s the *A-limit* of $\{s_k\}$. The matrix A is said to be *regular* if $s_k \to s(A)$ whenever $\{s_k\}$ converges to s. Necessary and sufficient conditions that A be regular are

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty, \tag{2}$$

$$\lim_{n\to\infty} a_{nk} = 0, \quad k = 0, 1, 2, \dots, \tag{3}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1. \tag{4}$$

In order that $s_n \to 0(A)$ whenever $\{s_k\}$ converges to zero, it is necessary and sufficient that (2) and (3) hold. We will have occasion to use the notation

$$A_n = \sum_{k=0}^{\infty} a_{nk}^2. \tag{5}$$

In this paper we are concerned with the cases where A is the matrix associated with either a weighted mean method of summability, or a generalised Hausdorff method of summability.

Weighted means. Let $\{d_n\}$ be a sequence of non-negative numbers (weights) with $d_0 > 0$. Let $D_n = \sum_{k=0}^{n} d_k$ and define

$$a_{nk} = \begin{cases} \dfrac{d_k}{D_n} & \text{for} \quad 0 \leqq k \leqq n, \\ 0 & \text{for} \quad k > n. \end{cases}$$

*This research was supported in part by Grants A—2983 and A—4806 of the Natural Sciences and Engineering Research Council of Canada.

The resulting matrix $A=\{a_{nk}\}$ is called a *weighted mean matrix* and is denoted by (M, d_n). It is a regular matrix if and only if $D_n\to\infty$.

Generalised Hausdorff matrices. Suppose

$$0=\lambda_0<\lambda_1<\ldots<\lambda_n,\quad \lambda_n\to\infty,\quad\text{and}\quad \sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty. \tag{6}$$

Let $\{\mu_n\}$ be a sequence of real numbers. Define the divided difference $[\mu_k, \ldots, \mu_n]$ inductively by $[\mu_k]=\mu_k$,

$$[\mu_k, \ldots, \mu_n]=\frac{[\mu_k, \ldots, \mu_{n-1}]-[\mu_{k+1}, \ldots, \mu_n]}{\lambda_n-\lambda_k}\quad\text{for}\quad 0\leqq k<n.$$

Let

$$a_{nk}=\begin{cases}\lambda_{k+1}\ldots\lambda_n[\mu_k, \ldots, \mu_n] & \text{for}\quad 0\leqq k\leqq n,\\ 0 & \text{for}\quad k>n,\end{cases} \tag{7}$$

with the convention that products such as $\lambda_{k+1}\ldots\lambda_n=1$ when $k=n$. The resulting matrix $A=\{a_{nk}\}$ is called a *generalised Hausdorff matrix*. This matrix is regular if and only if there is a function χ of bounded variation over the interval $[0, 1]$ such that

$$\mu_n=\int_0^1 t^{\lambda_n}\,d\chi(t), \tag{8}$$

$$\chi(0)=\chi(0+), \tag{9}$$

$$\mu_0=\chi(1)-\chi(0)=1. \tag{10}$$

See [4].

If we take $\lambda_n=n$ we obtain the familiar Hausdorff matrices; see [3] or [2, Chapter XI]. If we take $\chi(t)=t$, then (7) yields the weighted mean matrix (M, d_n) with

$$d_0=1\quad\text{and}\quad d_n=\frac{1}{\lambda_n+1}\left(1+\frac{1}{\lambda_1}\right)\ldots\left(1+\frac{1}{\lambda_n}\right)\quad\text{for}\quad n>0. \tag{11}$$

Note that when $\lambda_n=n$, (M, d_n) reduces to the Cesàro matrix of order one.

For the proof of Theorem 2 we need the following: For $0\leqq k\leqq n$ and $0\leqq t\leqq 1$ let

$$a_{nk}(t)=\lambda_{k+1}\ldots\lambda_n[t^{\lambda_k}, \ldots, t^{\lambda_n}]. \tag{12}$$

Then, with a_{nk} given by (7) and μ_n by (8), we have

$$\mu_0=\sum_{k=0}^{n}a_{nk},\quad [4, (5)] \tag{13}$$

$$0\leqq a_{ns}(t)\leqq\sum_{k=0}^{n}a_{nk}(t)=1\quad\text{for}\quad 0\leqq t\leqq 1,\ 0\leqq s\leqq n.\quad [4, \text{p. } 288] \tag{14}$$

$$\int_0^1 a_{nk}(t)\,dt=\frac{d_k}{D_n}\quad\text{for}\quad 0\leqq k\leqq n,\quad [4, \text{p. } 294] \tag{15}$$

The Borel property

Let S be the set of all sequences $\{s_k\}$ where s_k is either 0 or 1, being 1 for infinitely many k. Let I denote the interval $(0, 1]$ of real numbers. There is a bijection $\Phi\colon S\to I$ obtained by defining $\Phi(\{s_k\})$ to be the dyadic fraction $0.\, s_0s_1s_2\dots$. By a phrase such as "a set $E\subseteqq S$ consists of almost all sequences of zeros and ones" we mean that $\Phi(E)$ has Lebesgue measure one.

The matrix A is said to have the *Borel property* if almost all sequences of zeros and ones are A-convergent to $\frac{1}{2}$. In this case we write $A\in(\mathrm{BP})$.

HILL [7] showed that in order that $A=\{a_{nk}\}\in(\mathrm{BP})$, it is necessary that (3) and (4) hold, that $A_n<\infty$ for $n\geqq 0$ and that

$$\lim_{n\to\infty} A_n = 0. \tag{16}$$

See [7, Theorem 1.1]. In the case of the weighted mean matrix (M, d_n) these conditions reduce to

$$D_n \to\infty, \tag{17}$$

and

$$\frac{d_n}{D_n} \to 0. \tag{18}$$

See [7, 2.8].

Hill posed the question as to whether (17) and (18) were sufficient for (M, d_n) to have the Borel property. We give below a weighted mean matrix which shows that in fact they are not sufficient for the Borel property to hold. This matrix also serves as an alternate and simpler example than the one due to ERDŐS [6, p. 404] to show that (3), (4), and (16) are not sufficient for a matrix A to have the Borel property.

It was also shown in [7] that a regular Hausdorff matrix $(\lambda_n=n)$ has the Borel property if and only if $\mu_n\to 0$. LIU and RHOADES [8] have shown that for a limited generalisation of Hausdorff matrices, viz., $\lambda_n=n+\alpha$, the condition $\mu_n\to 0$ survives as both necessary and sufficient for the Borel property. Our example destroys the hope of any such simple criterion for a generalised Hausdorff matrix to have the Borel property. On the other hand we do give some sufficient conditions (Corollary to Theorem 2) for a generalised Hausdorff matrix to have the Borel property. These are natural generalisations of conditions given in [7, Theorem 2.9].

A bounded sequence $\{s_k\}$ is said to be *almost convergent* to s if

$$\lim_{p\to\infty} (s_{n+1}+\dots+s_{n+p})/p = s \quad \text{uniformly in } n.$$

A matrix A is said to be *strongly regular* if whenever $\{s_k\}$ is almost convergent to s, then $s_n\to s(A)$. LORENTZ [9] has shown that in order for a regular matrix $A=\{a_{nk}\}$ to be strongly regular it is necessary and sufficient that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{nk}-a_{n,k+1}| = 0. \tag{19}$$

HILL proved in [7] that a Hausdorff matrix has the Borel property if and only if it is strongly regular. LIU and RHOADES [8] have shown that the same is true when $\lambda_n=n+\alpha$. Our weighted mean matrix below is a generalised Hausdorff matrix, is strongly regular, but does not have the Borel property. We also give an example of a weighted mean matrix having the Borel property but which fails to be strongly regular. Thus for weighted mean matrices and for generalised Hausdorff matrices, there appears to be no simple relation between having the Borel property and being strongly regular.

Examples

We require the following result due to BOREL [1, pp. 37—47]. We give the less general version quoted by HILL [6] which is adequate for our purposes.

BOREL'S LEMMA. *Let $\{\varkappa_n\}$ be a sequence of positive integers, and let the positive integers $\{n_j\}$ be such that $n_j\geqq n_{j-1}+\varkappa_{j-1}$ $(j=1,2,3,\ldots)$. Then in order that almost all dyadic fractions $y=0.\,s_0s_1s_2\ldots$ have the property that for infinitely many j, s_{n_j} is followed by $\varkappa_j$ zeros and for infinitely many j, by $\varkappa_j$ ones, it is necessary and sufficient that $\sum_{n=0}^{\infty}2^{-\varkappa_n}=\infty$.*

THEOREM 1. *Let d_n be given by* (11) *with $\lambda_n=c\log(n+1)$, $n=0,1,\ldots,0<c<$ $<1/\log 4$. Then the weighted mean matrix (M,d_n) is strongly regular and satisfies* (17) *and* (18), *but does not have the Borel property.*

PROOF. Let $\varkappa_n=[\log(n+3)]$, $n=0,1,\ldots$. Let $\{s_k\}\in S$ and let $t_n=(1/D_n)\sum_{k=0}^{n}d_ks_k$. If s_{n^2} is followed by $\varkappa_n$ zeros, we have

$$0\leqq t_{n^2+\varkappa_n}\leqq\frac{D_{n^2}}{D_{n^2+\varkappa_n}}=\prod_{k=n^2+1}^{n^2+\varkappa_n}\left(1+\frac{1}{\lambda_k}\right)^{-1}\leqq\left(1+\frac{1}{\lambda_{(n+1)^2}}\right)^{-\varkappa_n}=$$
$$=\left(1+\frac{1}{2c\log(n+1)}\right)^{-\varkappa_n}\to e^{-1/2c}<\frac{1}{2}.$$

On the other hand, if s_{n^2} is followed by $\varkappa_n$ ones, we have

$$0<1-t_{n^2+\varkappa_n}\leqq\frac{D_{n^2}}{D_{n^2+\varkappa_n}}\leqq\left(1+\frac{1}{\lambda_{(n+1)^2}}\right)^{-\varkappa_n}\to e^{-1/2c}<\frac{1}{2}.$$

It follows, by Borel's lemma, that the subset of S consisting of (M,d_n)-convergent sequences has measure 0, so that $(M,d_n)\notin(\mathrm{BP})$. On the other hand $D_n\to\infty$ since $D_n=\left(1+\frac{1}{\lambda_1}\right)\ldots\left(1+\frac{1}{\lambda_n}\right)$ and $\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty$, and $\frac{d_n}{D_n}=\frac{1}{1+\lambda_n}\to 0$ (monotonically); i.e., (M,d_n) satisfies (17) and (18).

To show that (M,d_n) is strongly regular we note that it is regular since $D_n\to\infty$, and then verify (19) which in this case reduces to the condition

$$\frac{1}{D_m}\sum_{k=0}^{m-1}|d_k-d_{k+1}|+\frac{d_m}{D_m}=o(1).$$

Since

$$|d_k-d_{k+1}|=\left|\frac{D_k}{1+\lambda_k}-\frac{D_k}{\lambda_k}\right|=\frac{d_k}{\lambda_k}|\lambda_{k+1}-\lambda_k-1|=O\left(\frac{d_k}{\lambda_k}\right),$$

it follows that

$$\frac{1}{D_m}\sum_{k=0}^{m-1}|d_k-d_{k+1}|=O\left(\frac{1}{D_m}\sum_{k=0}^{m-1}\frac{d_k}{\lambda_k}\right)=o(1),$$

which together with (18) implies that (M, d_n) is strongly regular.

This completes the proof of the theorem.

REMARKS. The above matrix (M, d_n) is a generalised Hausdorff matrix with $\chi(t)=t$, $\mu_n=\frac{d_n}{D_n}\to 0$. Further, it shows that (17) and (18) with $\frac{d_n}{D_n}$ monotonic are not sufficient for $(M, d_n)\in(\mathrm{BP})$.

For an example of a weighted mean matrix with the Borel property which is not strongly regular, take $d_{2k}=1$ and $d_{2k+1}=0$ for $k\geqq 0$. Then (19) fails, but $(M, d_n)\in(\mathrm{BP})$ in view of [7, Theorem 2.7]. To make the weights positive we can define $d_0^*=1$ and $d_k^*=d_k+\varepsilon_k$ for $k\geqq 1$ where $\varepsilon_k>0$ and $\sum_{k=1}^{\infty}\varepsilon_k<\infty$. It is easy to see that this does not change the bounded summability field and hence does not affect the method's having the Borel property or failing to be strongly regular.

Generalised Hausdorff matrices and the Borel property

In this section we give some sufficient conditions for a generalised Hausdorff matrix to have the Borel property. We begin by citing a result of HILL [7, Theorem 1.7].

HILL'S THEOREM. *In order that a matrix* $A=\{a_{nk}\}\in(\mathrm{BP})$ *it is sufficient that* (4) *hold,* $A_n<\infty$, *and* $A_n=o\,(1/\log n)$.

Suppose $\{\lambda_n\}$ satisfies (6), $\mu_n=\int_0^1 t^{\lambda_n}\beta(t)\,dt$, a_{nk} is defined by (7) and d_n is given by (11). Then we have the following theorem.

THEOREM 2. (i) *If* $\beta\in L^2[0, 1]$, *then*

$$A_n\leqq\max_{0\leqq k\leqq n}d_k\cdot\frac{1}{D_n}\|\beta\|_2^2. \tag{22}$$

(ii) *If* $\beta\in L^\infty[0, 1]$, *then*

$$A_n\leqq\frac{\|\beta\|_\infty^2}{D_n^2}\sum_{k=0}^{n}d_k^2. \tag{23}$$

PROOF. (i) Applying Schwarz' inequality and using (12) and (15) we obtain

$$a_{nk}^2\leqq\int_0^1 a_{nk}(t)\beta(t)^2\,dt\int_0^1 a_{nk}(t)\,dt=\frac{d_k}{D_n}\int_0^1 a_{nk}(t)\beta(t)^2\,dt.$$

Summing and using (14) we have

$$A_n \leqq \frac{1}{D_n} \int_0^1 \beta(t)^2 \sum_{k=0}^{n} d_k a_{nk}(t)\, dt \leqq \max_{0 \leqq k \leqq n} d_k \cdot \frac{1}{D_n} \|\beta\|_2^2.$$

(ii) Now

$$a_{nk}^2 \leqq \|\beta\|_\infty^2 \Big\{ \int_0^1 a_{nk}(t)\, dt \Big\}^2 = \|\beta\|_\infty^2 \left(\frac{d_k}{D_n}\right)^2.$$

Hence $A_n \leqq \frac{\|\beta\|_\infty^2}{D_n^2} \sum_{k=0}^{n} d_k^2$.

Combining Hill's Theorem with Theorem 2 we obtain the following:

COROLLARY. *Suppose* $\mu_0 = 1$.

(i) *If* $\beta \in L^2[0, 1]$ *and*

$$\max_{0 \leqq k \leqq n} d_k \cdot \frac{\log n}{D_n} = o(1), \tag{24}$$

then $A \in (\mathrm{BP})$.

(ii) *If* $\beta \in L^\infty[0, 1]$ *and*

$$\frac{\log n}{D_n^2} \sum_{k=0}^{n} d_k^2 = o(1), \tag{25}$$

then $A \in (\mathrm{BP})$.

Note that (24) implies (25). This corollary should be compared with [7, Theorem 2.9].

References

[1] E. BOREL, *Traité du calcul des probabilités et de ses applications*, Vol. II, *Part* 1, Gauthier-Villars (Paris, 1926).
[2] G. H. HARDY, *Divergent Series* (Oxford, 1949).
[3] F. HAUSDORFF, Summationsmethoden und Momentfolgen I, *Math. Z.*, **9** (1921), 74—109.
[4] F. HAUSDORFF, Summationsmethoden und Momentfolgen II, *Math. Z.*, **9** (1921), 280—299.
[5] J. D. HILL, Summability of sequences of 0's and 1's, *Annals of Math.*, **46**, **4** (1945), 556—562.
[6] J. D. HILL, The Borel property of summability methods, *Pacific J. Math.*, **1** (1951), 399—409.
[7] J. D. HILL, Remarks on the Borel property, *Pacific J. Math.*, **4** (1954), 227—242.
[8] M. LIU and B. E. RHOADES, Some properties of generalised Hausdorff matrices, *Houston Journal of Math.*, **2** (1976), 239—256.
[9] G. G. LORENTZ, A contribution to the theory of divergent sequences, *Acta Math.*, **80** (1948), 167—190.

(Received August 24, 1979; revised May 27, 1980)

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA N6A 5B7