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# ON THE RELATION BETWEEN THE LOGARITHMIC AND BOREL-TYPE SUMMABILITY METHODS

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1. **Introduction.** Suppose throughout that $\{s_n\}$ is a sequence of real numbers, $\lambda>-1, \alpha>0$, and $\beta$ is real. Let $N$ be any non-negative integer such that $\alpha N+\beta>1$.

We are concerned primarily with the logarithmic summability method $L$ and the Borel-type method $(B, \alpha, \beta)$. Some known results involve the Abel-type summability method $A_\lambda$. The methods are defined as follows. Let

$$L(x)=\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1},$$

$$S(x)=\alpha e^{-x}\sum_{n=N}^{\infty}\frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)},$$

$$\sigma(x)=(1-x)^{\lambda+1}\sum_{n=0}^{\infty}s_n\binom{n+\lambda}{n}x^n.$$

If $L(x)(\sigma(x))$ exists for $|x|<1$ and tends to $s$ as $x\to 1-$, then we say that $\{s_n\}$ is $L$-convergent ($A_\lambda$-convergent) to $s$ and write $s_n\to s(L)(s_n\to s(A_\lambda))$.

If $S(x)$ exists for $x\geq 0$ and tends to $s$ as $x\to\infty$, then we say that $\{s_n\}$ is $(B, \alpha, \beta)$-convergent to $s$ and write $s_n\to s(B, \alpha, \beta)$.

The methods $A_0$ and $(B, 1, 1)$ are the ordinary Abel and Borel exponential methods respectively.

A summability method $P$ is said to be *regular* if $s_n\to s(P)$ whenever $s_n\to s$. The summability methods $L$, $A_\lambda$, and $(B, \alpha, \beta)$ are all regular. In addition, the following propositions are known.

PROPOSITION 1. *If* $s_n\to s(B, \alpha, \beta)$ *and* $\sum_{n=0}^{\infty} s_n x^n$ *converges for* $|x|<1$, *then* $s_n\to s(A_\lambda)$.

PROPOSITION 2. *If* $s_n\to s(A_\lambda)$, *then* $s_n\to s(L)$.

The first of these propositions was proved by Shawyer and Yang in [6], and the second by Borwein in [1]. The converse of each of the above propositions is false.

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Propositions 1 and 2 yield:

PROPOSITION 3. *If* $s_n \to s(B, \alpha, \beta)$ *and* $\sum_{n=0}^{\infty} s_n x^n$ *converges for* $|x|<1$, *then* $s_n \to s(L)$.

The purpose of this paper is to investigate the reverse problem. That is, assuming the $L$-convergence of a sequence, what Tauberian condition will imply its $(B, \alpha, \beta)$-convergence?

2. **The main theorem.** Suppose that $\phi$ is a continuous and unboundedly increasing function on $[a, \infty)$.

A real-valued function $f$ on $[a, \infty)$ is said to be *slowly decreasing with respect to* $\phi$ if $\lim\inf(f(y)-f(x)) \geq 0$ as $y > x \to \infty$ and $\phi(y)-\phi(x) \to 0$, i.e. if, for each $\varepsilon > 0$, there exist positive numbers $\delta$ and $M$ such that $f(y)-f(x) > -\varepsilon$ whenever $y > x \geq M$ and $\phi(y)-\phi(x) < \delta$.

For the $A_\lambda$ and $(B, \alpha, \beta)$ methods, Shawyer and Yang established the following Tauberian result in [7].

PROPOSITION 4. *If* $s_n \to s(A_\lambda)$ *and* $S(x)$ *is slowly decreasing with respect to* $\log x$, *then* $s_n \to s(B, \alpha, \beta)$.

We established the following result in [4] for the $L$ and $A_\lambda$ methods.

PROPOSITION 5. *If* $s_n \to s(L)$ *and* $\sigma(x)$ *is slowly decreasing with respect to* $\log\log x$, *then* $s_n \to s(A_\lambda)$.

In the present paper we prove the following Tauberian theorem for the $L$ and $(B, \alpha, \beta)$ methods.

THEOREM 1. *If* $s_n \to s(L)$ *and* $S(x)$ *is slowly decreasing with respect to* $\log\log x$, *then* $s_n \to s(B, \alpha, \beta)$.

3. **Preliminary results.**

LEMMA 1. $s_n \to s(L)$ *if and only if* $\dfrac{\alpha(n+1)}{\alpha n+\beta-1} s_n \to s(L)$.

This result is a simple consequence of Lemma 1 in [2].

Let

$$J(t)=\frac{1}{\log t}\int_a^{\infty} \frac{e^{-u/t}}{u} S(u)\,du \quad \text{for} \quad t \geq a > 1.$$

LEMMA 2. (i) *If* $s_n \to s(L)$, *then* $J(t) \to s$ *as* $t \to \infty$.
(ii) *If* $L(x)$ *exists for* $|x|<1$ *and* $J(t) \to s$ *as* $t \to \infty$, *then* $s_n \to s(L)$.



**Proof.** Suppose that $L(x)$ exists for $|x| \leq 1$. *Then* $s_n = O(c^n)$ for $c > 1$, and hence $S(x)$ exists for all $x \geq 0$.

Let

$$I(t)=\frac{1}{\log t}\int_0^{a} \frac{e^{-u/t}}{u} S(u)\,du \quad \text{for} \quad t \geq a.$$

Then

$$|I(t)| \leq \frac{1}{\log t}\int_0^{a} \left|\frac{S(u)}{u}\right| du \to 0 \quad \text{as} \quad t \to \infty,$$

since $S(u)=O(u^{\alpha N+\beta-1})$ in $(0, a)$ and $\alpha N+\beta-1>0$.

Next we have, for $t \geq a$,

$$\begin{aligned}
I(t)+J(t) &= \frac{1}{\log t}\int_0^{\infty} \frac{e^{-u/t}}{u}\, \alpha e^{-u} \sum_{n=N}^{\infty} \frac{s_n u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\,du \\
&= \frac{1}{\log t}\sum_{n=N}^{\infty} \frac{\alpha s_n}{(\alpha n+\beta-1)\Gamma(\alpha n+\beta-1)} \\
&\quad \times \int_0^{\infty} e^{-u(1+t)/t} u^{\alpha n+\beta-2}\,du \\
&= \frac{1}{\log t}\sum_{n=N}^{\infty} \frac{\alpha(n+1)s_n}{(\alpha n+\beta-1)(n+1)}\left(\frac{t}{1+t}\right)^{\alpha n+\beta-1} \\
&= \left(\frac{t}{1+t}\right)^{\beta-1-\alpha} \frac{-\log(1-T)}{\log t} \cdot \frac{-1}{\log(1-T)} \\
&\quad \times \sum_{n=N}^{\infty} \frac{\alpha(n+1)}{\alpha n+\beta-1} \frac{s_n}{n+1} T^{n+1}
\end{aligned}$$

where $T=[t/(1+t)]^{\alpha}$, the inversion being justified since the final series is absolutely convergent. Also $(t/1+t)^{\beta-1-\alpha}$ and $-(\log(1-T)/\log t)$ tend to 1 as $t \to \infty$. In view of Lemma 1, the desired results follow.

LEMMA 3. *Let* $\gamma > 1, t > 1, a > 0$. *Then*

(i) $$\frac{1}{\log t}\int_t^{\infty} \frac{e^{-u/t}}{u}\,du \to 0 \quad \text{as} \quad t \to \infty,$$

(ii) $$\frac{1}{\log t}\int_a^{t} \frac{e^{-u/t}}{u}\,du \to 1 \quad \text{as} \quad t \to \infty,$$

(iii) $$0 < \int_a^{t} (e^{-u/t^{\gamma}} - e^{-u/t})\frac{du}{u} < 1,$$

and

(iv) $$\frac{1}{\log t}\int_t^{t^{\gamma}} \frac{e^{-u/t^{\gamma}}}{u}\,du \to \gamma-1 \quad \text{as} \quad t \to \infty.$$

**Proof.** (i) $0<\frac{1}{\log t}\int_t^\infty \frac{e^{-u/t}}{u}\,du=\frac{1}{\log t}\int_1^\infty \frac{e^{-v}}{v}\,dv\to 0$ as $t\to\infty$.

(ii)
$$\frac{1}{\log t}\int_a^t \frac{e^{-u/t}}{u}\,du=\frac{1}{\log t}\int_{a/t}^1 \frac{e^{-v}}{v}\,dv=e^{-a/t}\left(1-\frac{\log a}{\log t}\right)+\frac{1}{\log t}\times\int_{a/t}^1 e^{-v}\log v\,dv\to 1 \quad\text{as}\quad t\to\infty.$$

(iii) By the mean-value theorem,

$$0<\int_a^t (e^{-u/t^\gamma}-e^{-u/t})\frac{du}{u}\le\int_a^t\left(\frac{u}{t}-\frac{u}{t^\gamma}\right)\frac{du}{u}<\left(\frac{1}{t}-\frac{1}{t^\gamma}\right)\int_0^t du=1-t^{1-\gamma}<1.$$

(iv) By parts (ii) and (iii),

$$\frac{1}{\log t}\int_t^{t^\gamma}\frac{e^{-u/t^\gamma}}{u}\,du=\frac{\gamma}{\log t^\gamma}\int_a^{t^\gamma}\frac{e^{-u/t^\gamma}}{u}\,du-\frac{1}{\log t}\int_a^t\frac{e^{-u/t}}{u}\,du-\frac{1}{\log t}\times\int_a^t (e^{-u/t^\gamma}-e^{-u/t})\frac{du}{u}\to\gamma-1 \quad\text{as}\quad t\to\infty.$$

## 4. A general Tauberian result.

THEOREM 2. *Suppose that the following conditions hold:*

(1) $K(t,u)$ *is defined, real-valued, and non-negative for* $t>a$, $u\ge a$; *moreover,* $\int_a^\infty K(t,u)\,du$ *exists in the sense of Lebesgue for each* $t>a$,

(2) $\int_a^\infty K(t,u)\,du\to 1$ *as* $t\to\infty$,

(3) $f$ *is real-valued and continuous on* $[a,\infty)$,

(4) $F(t)=\int_a^\infty K(t,u)f(u)\,du$ *exists in the Cauchy-Lebesgue sense for each* $t>a$,

(5) $f$ *is slowly decreasing with respect to* $\phi$,

(6) $\phi(t)-\phi(t-1)\to 0$ *as* $t\to\infty$,

(7) $\int_a^x K(t,u)\,du\to 0$ *whenever* $t\ge x\to\infty$ *and* $\phi(t)-\phi(x)\to\infty$,

(8) $\int_x^\infty K(t,u)(\phi(u)-\phi(x))\,du\to 0$ *whenever* $x\ge t\to\infty$ *and* $\phi(x)-\phi(t)\to\infty$, *and*

(9) $F(t)=O(1)$ *for* $t>a$.

*Then* $f(u)=O(1)$ *for* $u>a$.

This result was established in [3].



## 5. Proof of Theorem 1.

Set $a=1+e^e$,

$$K(t,u)=\begin{cases}\dfrac{1}{\log t}\dfrac{e^{-u/t}}{u} & \text{for } t\ge a,\ u\ge a,\\ 0 & \text{otherwise,}\end{cases}$$
$$\phi(t)=\log\log t \quad\text{for } t\ge a,$$
$$f(u)=S(u) \quad\text{for } u\ge a.$$

Then

$$\int_a^\infty K(t,u)f(u)\,du=J(t)\quad\text{for}\quad t\ge a.$$

We first show that the conditions of Theorem 1 imply that $S(u)=O(1)$ for $u>a$.

Conditions (1), (3), (5), and (6) clearly hold, and $\int_a^\infty K(t,u)\,du\to 1$ as $t\to\infty$ by parts (i) and (ii) of Lemma 3. Furthermore, the $L$-convergence of $\{s_n\}$ and Lemma 2 guarantee that $F(t)$ exists and is bounded for $t>a$. In view of Theorem 2, to establish the boundedness of $S(u)$ in $(a,\infty)$ it suffices to prove that (7) and (8) hold.

To show that (7) holds, we observe that

$$\int_a^\infty K(t,u)\,du\le\frac{1}{\log t}\int_a^x\frac{du}{u}=\frac{\log x-\log a}{\log t}\to 0\quad\text{as}\quad t\ge x\to\infty$$

and

$$\log\log t-\log\log x\to\infty.$$

To show that (8) holds, we note that

$$\int_x^\infty K(t,u)(\phi(u)-\phi(x))\,du=\frac{1}{\log t}\int_x^\infty\frac{e^{-u/t}}{u}(\log\log u-\log\log x)\,du\le\frac{1}{\log t}\int_x^\infty\frac{e^{-u/t}}{u}\left(\frac{u-x}{x\log x}\right)du\le\frac{1}{x\log x\log t}\int_x^\infty e^{-u/t}\,du=\frac{te^{-x/t}}{x\log x\log t}\to 0\quad\text{as}\quad x\ge t\to\infty.$$

Suppose, as we may without loss of generality, that $s_n\to 0(L)$. Then, by Lemma 2, $J(t)\to 0$ as $t\to\infty$. It remains to show that $S(u)\to 0$ as $u\to\infty$.

Assign $\varepsilon>0$. Since $S(u)$ is slowly decreasing with respect to $\phi$, there exist numbers $x\ge a$ and $\delta>0$ such that $S(u)-S(t)>-\varepsilon$ whenever $u>t\ge x$ and $\log\log u-\log\log t<\delta$. Equivalently, setting $\gamma=e^\delta$,

(10) $$S(t)-\varepsilon<S(u)\quad\text{whenever}\quad x<t<u<t^\gamma.$$

Relation (10) implies that, for $t > x$,

$$I_1 = \frac{1}{\log t} \int_t^{t^\gamma} \frac{e^{-u/t^\gamma}}{u} (S(t) - \varepsilon)\, du$$
$$\le \frac{1}{\log t} \int_t^{t^\gamma} \frac{e^{-u/t^\gamma}}{u} S(u)\, du = I_2.$$

Now, by Lemma 3, and the fact that $S(u) = O(1)$,

$$I_2 = \gamma J(t^\gamma) - J(t) - \frac{\gamma}{\log t^\gamma} \int_{t^\gamma}^{\infty} \frac{e^{-u/t^\gamma}}{u} S(u)\, du$$
$$- \frac{1}{\log t} \int_a^t \frac{e^{-u/t^\gamma} - e^{-u/t}}{u} S(u)\, du + \frac{1}{\log t} \int_t^\infty \frac{e^{-u/t}}{u} S(u)\, du$$
$$= o(1) \quad \text{as} \quad t \to \infty.$$

Further, by part (iv) of Lemma 3,

$$I_1 = (S(t) - \varepsilon)(\gamma - 1 + o(1)).$$

Hence

$$S(t) - \varepsilon \le \frac{I_2}{\gamma - 1 + o(1)} = o(1),$$

and therefore

(11) $$\limsup_{t \to \infty} S(t) \le \varepsilon.$$

Rewriting (10) we get

(12) $$S(u) < S(t) + \varepsilon \quad \text{whenever} \quad x < t^{1/\gamma} < u < t.$$

Relation (12) implies that, for $t^{1/\gamma} \ge x$,

$$I_3 = \frac{1}{\log t} \int_{t^{1/\gamma}}^{t} \frac{e^{-u/t}}{u} S(u)\, du$$
$$\le \frac{1}{\log t} \int_{t^{1/\gamma}}^{t} \frac{e^{-u/t}}{u} (S(t) + \varepsilon)\, du = I_4.$$

By Lemma 3 (with $t$ replaced by $t^{1/\gamma}$) and the fact that $S(u) = O(1)$,

$$I_3 = J(t) - \frac{1}{\gamma} J(t^{1/\gamma}) - \frac{1}{\log t} \int_t^\infty \frac{e^{-u/t}}{u} S(u)\, du$$
$$- \frac{1}{\log t} \int_a^{t^{1/\gamma}} \frac{e^{-u/t} - e^{-u/t^{1/\gamma}}}{u} S(u)\, du + \frac{1}{\log t} \int_{t^{1/\gamma}}^{\infty} \frac{e^{-u/t^{1/\gamma}}}{u} S(u)\, du$$
$$= o(1) \quad \text{as} \quad t \to \infty.$$



Also

$$I_4 = (S(t) + \varepsilon)\left(1 - \frac{1}{\gamma} + o(1)\right).$$

Hence

$$S(t) + \varepsilon \ge \frac{I_3}{1 - 1/\gamma + o(1)} = o(1),$$

and therefore

(13) $$\liminf_{t \to \infty} S(t) \ge -\varepsilon.$$

It follows from (11) and (13) that $S(t) \to 0$ as $t \to \infty$, and this completes the proof.

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