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MATRIX OPERATORS ON ℓ^p

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Introduction. Suppose throughout that $A = (a_{nk})$ $(n, k = 0, 1, \cdots)$ is an infinite matrix of complex numbers, and that

$$p \geqq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

Let ℓ^p be the normed linear space of all complex sequences $x = \{x_n\}$ $(n = 0, 1, \cdots)$ with finite norm $\|x\|_p$, where

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \text{when } 1 \leqq p < \infty$$

and

$$\|x\|_\infty = \sup_{n \geqq 0} |x_n|.$$

Let $B(\ell^p)$ be the normed linear space of all bounded linear operators on ℓ^p into ℓ^p; so that $A \in B(\ell^p)$ if and only if, for every $x \in \ell^p$, $y_n = (Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$ is defined for $n = 0, 1, \cdots$, and $y = \{y_n\} \in \ell^p$. The norm $\|A\|$ of a matrix A in $B(\ell^p)$ is given by

$$\|A\| = \sup_{\|x\|_p \leqq 1} \|Ax\|_p.$$

It is known (see [8, p. 164]) that, for $1 \leqq p < \infty$, every operator in $B(\ell^p)$ has a matrix representation. Matrices in $B(\ell^p)$ have been characterized in terms of their elements only for $p = 1, 2, \infty$. Crone [1] characterized matrices in $B(\ell^2)$ by means of rather complicated conditions that are difficult to apply. The following are characterizations of $B(\ell^1)$ and $B(\ell^\infty)$ (see [8, p. 167 and p. 174]): $A \in B(\ell^1)$ if and only if

$$(\mathrm{C}_1) \qquad \sup_{k \geqq 0} \sum_{n=0}^{\infty} |a_{nk}| < \infty.$$

$A \in B(\ell^\infty)$ if and only if

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$$(C_2) \qquad \sup_{n \geqq 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$

With regard to sufficient conditions for $A \in B(\ell^p)$, it is known (see [8, Theorem 9, p. 174]) that if both (C_1) and (C_2) hold then $A \in B(\ell^p)$ for every $P \geqq 1$. It is also known (see [5, p. 354]) that, for $1 < p < \infty$, $A \in B(\ell^p)$ if

$$(C_3) \qquad \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} |a_{nk}|^q \right)^{p/q} < \infty.$$

Further, it is known (see [3, p. 346]) that, for $1 < p < \infty$, a matrix is in $B(\ell^p)$ if and only if its transpose is in $B(\ell^q)$. Hence, for $1 < p < \infty$, $A \in B(\ell^p)$ if

$$(C_4) \qquad \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nk}|^p \right)^{q/p} < \infty.$$

In § 2 of this paper we establish theorems concerning other conditions for $A \in B(\ell^p)$, and most of the rest of the paper is concerned with applications of these theorems. The main applications are in § 5 where simple necessary and sufficient conditions are obtained for certain weighted generalized Hausdorff matrices to be in $B(\ell^p)$. In some cases the norms of such matrices are easily computed. In all that follows suppose that $1 < p < \infty$.

2. Bounded operators on ℓ^p.

THEOREM 1. *If* $b_{nk} > 0$ *for* $n, k = 0, 1, 2, \cdots$, *and if*

$$\sup_{n \geqq 0} \sum_{k=0}^{\infty} |a_{nk}|(b_{nk})^{1/p} = M_1 < \infty$$

and

$$\sup_{k \geqq 0} \sum_{n=0}^{\infty} |a_{nk}|(b_{nk})^{-1/q} = M_2 < \infty,$$

then $A \in B(\ell^p)$ *and* $\|A\| \leqq M_1^{1/q} M_2^{1/p}$.

PROOF. Let $y_n = \sum_{k=0}^{\infty} a_{nk} x_k$ where $x = \{x_k\} \in \ell^p$. Then, by Hölder's inequality,

$$|y_n|^p \leqq \left(\sum_{k=0}^{\infty} |a_{nk}|(b_{nk})^{1/p} \right)^{p-1} \sum_{k=0}^{\infty} |a_{nk}|(b_{nk})^{-1/q}|x_k|^p$$

$$\leqq M_1^{p-1} \sum_{k=0}^{\infty} |a_{nk}|(b_{nk})^{-1/q}|x_k|^p,$$

and hence

$$\sum_{n=0}^{\infty} |y_n|^p \leqq M_1^{p-1} \sum_{k=0}^{\infty} |x_k|^p \sum_{n=0}^{\infty} |a_{nk}|(b_{nk})^{-1/q}$$

$$\leqq M_1^{p-1} M_2 \sum_{k=0}^{\infty} |x_k|^p.$$

The desired conclusions follow.

As an immediate corrollary we have:

THEOREM 2. *If* $a_{nk} \geqq 0$ *for* $0 \leqq k \leqq n$, $a_{nk} = 0$ *for* $k > n$; *if* $b_n > 0$ *for* $n = 0, 1, \cdots$; *and if*

(1) $$\sup_{n \geqq 0} \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n} \right)^{1/p} = M_1 < \infty$$

and

(2) $$\sup_{k \geqq 0} \sum_{n=k}^{\infty} a_{nk} \left(\frac{b_n}{b_k} \right)^{1/q} = M_2 < \infty,$$

then $A \in B(\ell^p)$ *and* $\|A\| \leqq M_1^{1/q} M_2^{1/p}$.

The next theorem shows that in certain circumstances (2) implies (1).

THEOREM 3. *If* $a_{nk} \geqq 0$ *for* $0 \leqq k \leqq n$, $a_{nk} = 0$ *for* $k > n$; *if* $b_n > 0$ *for* $n = 0, 1, \cdots$, *and* $\sum_{n=0}^{\infty} b_n = \infty$; *and if, as* $n \to \infty$,

(3) $$\sigma_n = \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n} \right)^{1/p} \to \sigma \text{ (finite or infinite)},$$

then (2) *implies* (1) *with* $M_1 = \sup_{n \geqq 0} \sigma_n$.

PROOF. Suppose (2) holds. Then

$$\sum_{n=0}^{m} b_n\sigma_n = \sum_{k=0}^{m} b_k \sum_{n=k}^{m} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/q} \leqq M_2B_m$$

where $B_m = \sum_{k=0}^{m} b_k$. But a simple consequence of (3) is that

$$\frac{1}{B_m}\sum_{n=0}^{m} b_n\sigma_n \to \sigma \text{ as } m \to \infty.$$

Hence $0 \leqq \sigma \leqq M_2 < \infty$, and so (1) holds with $M_1 = \sup_{n\geqq 0}\sigma_n < \infty$.

The following theorem shows that under certain conditions (1) is necessary for $A \in B(\ell^p)$.

THEOREM 4. *Suppose that* $a_{nk} \geqq 0$ *for* $0 \leqq k \leqq n$, $a_{nk} = 0$ *for* $k > n$; *that* $b_n = bd_n/D_n$ *where* $b > 0$, $d_n > 0$ *for* $n = 0, 1, \cdots,$ *and* $D_n = \sum_{k=0}^{n} d_k \to \infty$; *and that* (3) *holds. If* $A \in B(\ell^p)$ *then* (1) *holds and* $\|A\| \geqq \sigma$.

PROOF. Suppose without loss in generality that $\sigma > 0$ and let $\sigma < \mu < \lambda < \sigma$. Let

$$y_n = \sum_{k=0}^{n} a_{nk}x_k \text{ where } x_k = \left(\frac{b_k}{D_k^{\epsilon}}\right)^{1/p}, \epsilon > 0.$$

Then there is an integer N independent of ϵ such that for $n \geqq N$

$$y_n = x_n \sum_{k=0}^{n} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p}\left(\frac{D_n}{D_k}\right)^{\epsilon/p}$$

$$\geqq x_n \sum_{k=0}^{n} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p} \geqq \lambda x_n.$$

Now choose ϵ so small that

$$\sum_{n=N}^{\infty} x_n^{\,p} = b\sum_{n=N}^{\infty}\frac{d_n}{D_n^{\,1+\epsilon}} \geqq \left(\frac{\mu}{\lambda}\right)^p \sum_{n=0}^{\infty} x_n^{\,p}.$$

Then

$$\sum_{n=0}^{\infty} y_n^{\,p} \geqq \lambda^p \sum_{n=N}^{\infty} x_n^{\,p} \geqq \mu^p \sum_{n=0}^{\infty} x_n^{\,p}.$$

Therefore $\|A\| \geqq \mu$ and, since μ is an arbitrary number in the interval $(0, \sigma)$, it follows that $\|A\| \geqq \sigma$. This implies that σ is finite and hence that (1) holds with $M_1 = \sup_{n\geqq 0}\sigma_n$.

3. **Remarks.**

(a) Theorem 4 can be used to show that certain matrices are not in $B(\ell^p)$. Consider for example the matrix A given by

$$a_{nk} = \frac{1}{p(n+1)^{1/p}\log(n+2)} \cdot \frac{\log(k+2)}{(k+1)^{1/q}} \text{ for } 0 \leqq k \leqq n;$$

$$a_{nk} = 0 \text{ for } k > n.$$

This matrix is readily shown to be regular, i.e., $(Ax)_n \to \xi$ whenever $x_n \to \xi$. It also satisfies the conditions

$$\sup_{k \geqq 0} \sum_{n=0}^{\infty} |a_{nk}|^p < \infty; \qquad \sup_{n \geqq 0} \sum_{k=0}^{\infty} |a_{nk}|^q < \infty,$$

which are evidently necessary for $A \in B(\ell^p)$. Take $b_n = 1/(n+1)\log(n+2)$. Then

$$\sum_{k=0}^{n} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p} = \frac{1}{p(\log(n+2))^{1/q}} \sum_{k=0}^{n} \frac{(\log(k+2))^{1/q}}{k+1}$$

$$\to \infty \text{ as } n \to \infty.$$

Thus (3) holds, and so by Theorem 4, A is not in $B(\ell^p)$.

(b) Consider the matrix A given by

$$a_{nk} = \frac{1}{(n+1)^{1/p}\log(n+2)} \frac{1}{(k+1)^{1/q}} \text{ for } 0 \leqq k \leqq n,$$

$$a_{nk} = 0 \text{ for } k > n.$$

Taking $b_n = 1/(n+1)$, we find that

$$\sum_{k=0}^{n} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p} = \frac{1}{\log(n+2)} \sum_{k=0}^{n} \frac{1}{k+1} \to 1,$$

whereas

$$\sum_{n=k}^{\infty} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/q} = \sum_{n=k}^{\infty} \frac{1}{(n+1)\log(n+2)} = \infty.$$

This is inconclusive as a test for whether A is in $B(\ell^p)$ or not, but it shows that (2) may fail to hold when both (1) and (3) hold. It is readily shown, however, that the same a_{nk} satisfies both (1) and (2) with $b_n = 1/(n+1)\log(n+2)$. Thus, by Theorem 2, $A \in B(\ell^p)$. A straightforward calculation shows, however, that neither (C_3) nor (C_4) holds.

(c) Consider now the matrix A given by

$$a_{nk} = \frac{1}{(n+1)^{1/2}\log(n+2)(\log\log(n+3))^{5/4}} \cdot \left(\frac{\log\log(k+3)}{k+1}\right)^{1/2} \text{for } 0 \leqq k \leqq n,$$

$$a_{nk} = 0 \text{ for } n > k.$$

It is readily shown that in this case (C_3) holds with $p = q = 2$, and so $A \in B(\ell^2)$. On the other hand, it can be shown without difficulty that, for $p = q = 2$, (2) fails to hold with $b_n = 1/(n+1)$, whereas both (1) and (2) hold with $b_n = 1/(n+1)\log(n+2)$.

The following are open questions:

(i) If $a_{nk} \geqq 0$ for $0 \leqq k \leqq n$, $a_{nk} = 0$ for $k > n$, and (C_3) holds, is there always a positive sequence $\{b_n\}$ for which both (1) and (2) hold?

(ii) The same as (i), but with "(C_3) holds" replaced by "$A \in B(\ell^p)$".

4. **Operators associated with weighted means.** For $n = 0, 1, \cdots$, let

$$a_n > 0,\ A_n = \sum_{k=0}^{n} a_k.$$

The weighted or $(\overline{N}, a_n)$ means of a sequence $\{s_n\}$ are given by

$$\sum_{k=o}^{n} \frac{a_k}{A_n} s_k.$$

We consider a matrix $A = (a_{nk})$, associated with such means, defined as follows:

Let

$$\lambda_0 \geqq 0,\ \lambda_n = \frac{A_{n-1}}{a_n} \text{ for } n \geqq 1,$$

and let

$$a_{nk} = \begin{cases} \dfrac{a_k}{A_n}\left(\dfrac{\lambda_k}{\lambda_n}\right)^{1/p} & 0 \leqq k \leqq n,\ n \geqq 1, \\ 1 & k = n = 0, \\ 0 & n > k. \end{cases}$$

Let

$$b_n = \frac{1}{\lambda_n} \text{for } n \geqq 1,$$

and let

$$b_0 = \begin{cases} \dfrac{1}{\lambda_0} \text{ if } \lambda_0 > 0, \\ \dfrac{1}{\lambda_1} + 1 \text{ if } \lambda_0 = 0. \end{cases}$$

Then, for $n \geqq 0$,

$$\frac{1}{A_n} \sum_{k=0}^{n} a_k - \frac{a_0}{A_n} = 1 - \frac{a_0}{A_n} \leqq \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n} \right)^{1/p}$$

$$\leqq \frac{1}{A_n} \sum_{k=0}^{n} a_k = 1;$$

and, for $k \geqq 0$,

$$\sum_{n=k}^{\infty} a_{nk} \left(\frac{b_n}{b_k} \right)^{1/q} = \sum_{n=k}^{\infty} \frac{a_k \lambda_k}{A_n \lambda_n}$$

$$= \frac{a_k}{A_k} + a_k \lambda_k \sum_{n=k+1}^{\infty} \left(\frac{1}{A_{n-1}} - \frac{1}{A_n} \right)$$

$$\leqq \frac{a_k}{A_k} (1 + \lambda_k) \leqq 1 + \lambda_0.$$

Hence, by Theorem 2, $A \in B(\ell^p)$ and $\|A\| \leqq (1 + \lambda_0)^{1/p}$.

Suppose in addition that $a_n = O(A_{n-1})$, *i.e., that* $b_n = O(1)$, *and that* $A_n \to \infty$. Let $b = 1 + \sup_{n \geqq 0} b_n$, let $D_{-1} = 0$, and for $n \geqq 0$, let

$$\frac{1}{D_n} = \left(1 - \frac{b_0}{b} \right) \left(1 - \frac{b_1}{b} \right) \cdots \left(1 - \frac{b_n}{b} \right),$$

$$d_n = D_n - D_{n-1}.$$

Then $D_n \to \infty$, since $\sum_{n=1}^{\infty} b_n \geqq \sum_{n=1}^{\infty} a_n / A_n = \infty$; and, for $n \geqq 0$,

$$b \frac{d_n}{D_n} = b \left(1 - \frac{D_{n-1}}{D_n} \right) = b_n.$$

Thus, by Theorem 4, $\|A\| \geqq 1$, i.e., in this case we have

$$(1 + \lambda_0)^{1/p} \geqq \|A\| \geqq 1$$

and in particular, if $\lambda_0 = 0$, $\|A\| = 1$.

5. **Generalized Hausdorff matrices.** Suppose in what follows that

$$0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n, \quad \lambda_n \to \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

Let $\{\mu_n\}$ $(n \geqq 0)$ be a sequence of real numbers. The divided difference $[\mu_n, \cdots, \mu_m]$ is defined inductively by $[\mu_n] = \mu_n$,

$$[\mu_n, \cdots, \mu_m] = \frac{[\mu_n, \cdots, \mu_{m-1}] - [\mu_{n+1}, \cdots, \mu_m]}{\lambda_m - \lambda_n}$$

for $m > n \geqq 0$.

Let

$$\lambda_{nk} = \begin{cases} \lambda_{k+1} \cdots \lambda_n [\mu_k, \cdots, \mu_n] & 0 \leqq k < n, \\ \mu_n & k = n, \\ 0 & k > n, \end{cases}$$

and let

$$\lambda^*_{nk} = \lambda_{nk} \frac{\lambda_k}{\lambda_n} \text{ for } 0 \leqq k \leqq n,\ n \geqq 1;\ \lambda^*_{00} = \lambda_{00} = \mu_0.$$

We require three lemmas, the first of which is known. (See Hausdorff [2] and Leviatan [6, Theorem 2.1; 7, p. 227–228]; and the references given in the latter two papers.)

LEMMA 1. *The following three conditions are equivalent*:

(4) $$\mu_n = \int_0^1 t^{\lambda_n}\, d\alpha(t) \textit{ for } n = 0, 1, 2, \cdots,$$

where $\alpha \in BV[0, 1]$,

(5) $$\sup_{n \geqq 0} \sum_{k=0}^{n} |\lambda_{nk}| = L < \infty,$$

(6) $$\sup_{k \geqq 0} \sum_{n=k}^{\infty} |\lambda^*_{nk}| = L^* < \infty,$$

Moreover, when the conditions hold

$$\max(L, L^*) \leqq \int_0^1 |d\alpha(t)|.$$

LEMMA 2. *If* $L_n = \sum_{k=0}^{n} |\lambda_{nk}|$, $M_n = \sum_{k=1}^{n} |\lambda_{n,k}|$, *then for* $n \geqq 0$, $L_{n+1} \geqq L_n$ *and* $M_{n+2} \geqq M_{n+1}$.

PROOF. We have, for $0 \leqq k \leqq n$,

$$\lambda_{n+1,k} = \lambda_{k+1} \cdots \lambda_{n+1} \frac{[\mu_k, \cdots, \mu_n] - [\mu_{k+1}, \cdots, \mu_{n+1}]}{\lambda_{n+1} - \lambda_k}$$

$$= \frac{\lambda_{n+1}}{\lambda_{n+1} - \lambda_k} \lambda_{nk} - \frac{\lambda_{k+1}}{\lambda_{n+1} - \lambda_k} \lambda_{n+1,k+1},$$

and so

$$\lambda_{nk} = \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}} \lambda_{n+1,k} + \frac{\lambda_{k+1}}{\lambda_{n+1}} \lambda_{n+1,k+1}.$$

It follows that

$$L_{n+1} - L_n - |\lambda_{n+1,0}| = \sum_{k=o}^{n} (|\lambda_{n+1,k+1}| - |\lambda_{n,k}|)$$

$$\geqq \sum_{k=0}^{n} \left(|\lambda_{n+1,k+1}| - \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}} |\lambda_{n+1,k}| - \frac{\lambda_{k+1}}{\lambda_{n+1}} |\lambda_{n+1,k+1}| \right)$$

$$= \sum_{k=0}^{n} \left(|\lambda_{n+1,k+1}| \frac{\lambda_{n+1} - \lambda_{k+1}}{\lambda_{n+1}} - |\lambda_{n+1,k}| \frac{\lambda_{n+1} - \lambda_k}{\lambda_{n+1}} \right)$$

$$= - |\lambda_{n+1,0}| \frac{\lambda_{n+1} - \lambda_0}{\lambda_{n+1}},$$

and hence

$$L_{n+1} - L_n \geqq \frac{\lambda_0}{\lambda_{n+1}} |\lambda_{n+1,0}| \geqq 0.$$

To complete the proof, let

$$\lambda_n' = \lambda_{n+1}, \mu_n' = \mu_{n+1} \text{ for } n \geqq 0.$$

Then, for $n > k \geqq 1$,

$$\lambda_{nk} = \lambda_k' \cdots \lambda_{n-1}'[\mu_{k-1}', \cdots, \mu_{n-1}']$$

$$= \lambda_{n-1,k-1}',$$

and for $n \geqq 1$,

$$\lambda_{nn} = \mu_n = \lambda_{n-1,n-1}'.$$

Hence, for $n \geqq 1$,

$$M_n = \sum_{k=0}^{n-1} |\lambda'_{n-1,k}|,$$

and so, by the part already proved, $M_n \leqq M_{n+1}$.

A function $\alpha \in BV[0, 1]$ is said to be normalized if $\alpha(0) = 0$ and $2\alpha(t) = \alpha(t+) + \alpha(t-)$ for $0 < t < 1$.

LEMMA 3. *Suppose* (4) *holds with* α *normalized.*

(i) *If* $\lambda_0 = 0$, *then* $\lim_{n\to\infty} \lambda_{n0} = \alpha(0+)$ *and*

$$\lim_{n\to\infty} \sum_{k=0}^{n} |\lambda_{nk}| = \int_0^1 |d\alpha(t)|.$$

(ii) *If* $\lambda_0 > 0$, *then* $\lim_{n\to\infty} \sum_{k=0}^{n} |\lambda_{nk}| = \int_0^1 |d\alpha(t)| - |\alpha(0+)|.$

PROOF. (i) The first conclusion in (i) is known (see Hausdorff [2, (25) p. 287). To establish the second, define $\alpha_n(t)$ for $0 \leqq t \leqq 1$, $n = 1, 2, \cdots$, by setting

$$\alpha_n(0) = 0; \; \alpha_n(t) = \sum_{t_{nk} \leqq t} \lambda_{nk} \text{ for } 0 < t \leqq 1$$

where

$$t_{nk} = \left(1 - \frac{\lambda_1}{\lambda_{k+1}}\right) \cdots \left(1 - \frac{\lambda_1}{\lambda_n}\right).$$

Then by Lemma 1,

$$\int_0^1 |d\alpha_n(t)| = \sum_{k=0}^{n} |\lambda_{nk}| \leqq \int_0^1 |d\alpha(t)|. \tag{7}$$

Further, Schoenberg [9, p. 607] (see also Leviatan [6, p. 102]) has shown that (4) is sufficient for

$$\lim_{n\to\infty} \int_0^1 t^{\lambda_s} \, d\alpha_n(t) = \int_0^1 t^{\lambda_s} \, d\alpha(t) = \mu_s \text{ for } s = 0, 1, 2, \cdots. \tag{8}$$

It follows from (7) by Helly's Theorem (see [10, Theorem 16.3, p. 29]) and the Helly-Bray theorem (see [10, Theorem 16.4 and Corollary 16.4, pp. 31–32]) that there is a strictly increasing sequence $\{n_i\}$ of positive integers and a normalized function $\gamma \in BV[0, 1]$ such that

$$(9)\qquad \lim_{i\to\infty} \int_0^1 t^{\lambda_s}\, d\alpha_{n_i}(t) = \int_0^1 t^{\lambda_s}\, d\gamma(t) \text{ for } s = 0, 1, \cdots$$

and

$$\int_0^1 |d\gamma(t)| \leqq \liminf_{i\to\infty} \int_0^1 |d\alpha_{n_i}(t)|.$$

But (8) and (9) imply that $\gamma(t) = \alpha(t)$ for $0 \leqq t \leqq 1$ (see Schoenberg [9, Corollary 8.1, p. 609]). Hence, by (7) and Lemma 2,

$$\int_0^1 |d\alpha(t)| \leqq \liminf_{i\to\infty} \sum_{k=0}^{n_i} |\lambda_{n_i,k}|$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n} |\lambda_{nk}| \leqq \int_0^1 |d\alpha(t)|.$$

(ii) Define sequences $\{\lambda_n'\}$, $\{\mu_n'\}$ by

$$\lambda_0' = 0,\ \mu_0' = \alpha(1) - \alpha(0);$$

$$\lambda_n' = \lambda_{n-1},\ \mu_n' = \mu_{n-1} \textit{ for } n \geqq 1.$$

Then

$$\mu_n' = \int_0^1 t^{\lambda_n'}\, d\alpha(t) \text{ for } n = 0, 1, \cdots$$

Further, for $n > k \geqq 1$,

$$\lambda'_{nk} = \lambda'_{k+1} \cdots \lambda_n'[\mu_k', \cdots, \mu_n']$$

$$= \lambda_k \cdots \lambda_{n-1}[\mu_{k-1}, \cdots, \mu_{n-1}] = \lambda_{n-1,k-1};$$

and for $n \geqq 1$, $\lambda'_{n,n} = \mu_n' = \lambda_{n-1,n-1}$.
Hence, by part (i),

$$\sum_{k=0}^{n-1} |\lambda_{n-1,k}| = \sum_{k=1}^{n} |\lambda_{n-1,k-1}|$$

$$= \sum_{k=0}^{n} |\lambda'_{nk}| - |\lambda'_{n0}|$$

$$\to \int_0^1 |d\alpha(t)| - |\alpha(0+)| \text{ as } n \to \infty.$$

This completes the proof of Lemma 3.

Now let $H = (h_{nk})$ be the "generalized weighted Hausdorff" matrix given by

$$h_{nk} = \begin{cases} \lambda_{nk} \left(\dfrac{\lambda_k}{\lambda_n} \right)^{1/p} & 0 \leqq k \leqq n,\ n \geqq 1, \\ \lambda_{00} & k = n = 0, \\ 0 & k > n, \end{cases}$$

and let $\tilde{H}$ be the matrix $(|h_{nk}|)$.

Theorem 5. (i) *If* (4) *holds with* α *normalized, then* $H, \tilde{H} \in B(\ell^p)$, $\|H\| \leqq \|\tilde{H}\|$ *and*

$$\int_0^1 |d\alpha(t)| - |\alpha(0+)| \leqq \|\tilde{H}\| \leqq \int_0^1 |d\alpha(t)|.$$

(ii) *If* $\tilde{H} \in B(\ell^p)$ *then* (4) *holds.*

Proof. As in §4, let $b_n = 1/\lambda_n$ for $n \geqq 1$, and let

$$b_0 = \begin{cases} \dfrac{1}{\lambda_0} & \text{if } \lambda_0 > 0, \\ \dfrac{1}{\lambda_1} + 1 & \text{if } \lambda_0 = 0. \end{cases}$$

Let $b = 1 + \sup_{n \geqq 0} b_n$, let $D_{-1} = 0$, and, for $n \geqq 0$, let

$$\frac{1}{D_n} = \left(1 - \frac{b_0}{b}\right) \left(1 - \frac{b_1}{b}\right) \cdots \left(1 - \frac{b_n}{b}\right),$$

$$d_n = D_n - D_{n-1}.$$

Then $D_n \to \infty$, since $\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty 1/\lambda_n = \infty$; and, for $n \geqq 0$,

$$b\,\frac{d_n}{D_n} = b\left(1 - \frac{D_{n-1}}{D_n}\right) = b_n.$$

Let

$$\sigma_n = \sum_{k=0}^n |h_{nk}| \left(\frac{b_k}{\beta_n} \right)^{1/p} \quad \text{for } n \geqq 0.$$

Then

$$\sigma_n = \begin{cases} \sum_{k=0}^{n} |\lambda_{nk}| & \text{when } \lambda_0 > 0,\ n \geqq 0, \\ \sum_{k=1}^{n} |\lambda_{nk}| & \text{when } \lambda_0 = 0,\ n \geqq 1. \end{cases}$$

(i) Suppose (4) holds with α normalized. Then, by Lemma 1, we have

$$\sigma_n \leqq \int_0^1 |d\alpha(t)| \quad \text{for } n \geqq 0$$

and

$$\sum_{n=k}^{\infty} |h_{nk}| \left(\frac{b_n}{b_k} \right)^{1/q} = \sum_{n=k}^{\infty} |\lambda^*_{nk}|$$
$$\leqq \int_0^1 |d\alpha(t)| \quad \text{for } k \geqq 0.$$

Hence, by Theorem 2, $\tilde{H} \in B(l^p)$ and $\|\tilde{H}\| \leqq \int_0^1 |d\alpha(t)|$; and this implies that $H \in B(l^p)$ and $\|H\| \leqq \|\tilde{H}\|$.

Next, by Lemma 3 and Theorem 4,

$$\sigma_n \to \int_0^1 |d\alpha(t)| - |\alpha(0+)| \leqq \|\tilde{H}\|.$$

(ii) Suppose $\tilde{H} \in B(l^p)$. By Lemma 2, $\sigma_n \to \sigma$ and, by Theorem 3, $\sigma < \infty$. Further, Hausdorff [2, (7) p. 282] has shown that, if $\lambda_0 = 0$, then

$$\sum_{k=0}^{n} \lambda_{nk} = \mu_0,$$

and so

$$|\lambda_{n0}| \leqq \sum_{k=1}^{n} |\lambda_{nk}| + |\mu_0| \quad \text{for } n \geqq 1.$$

It follows that

$$\sup_{n \geqq 0} \sum_{k=0}^{n} |\lambda_{nk}| \leqq 2 \sup_{n \geqq 0} \sigma_n + |\mu_0| < \infty$$

and therefore, by Lemma 1, that (4) holds.

This completes the proof of Theorem 5.

EXAMPLE. Let $\delta + 1/p \geqq 0$ and let $\lambda_n = n + \delta + 1/p$. Then, it is readily shown that

$$\lambda_{nk} = \binom{n + \delta + 1/p}{n - k} \Delta^{n-k}\mu_k \quad \text{for } 0 \leqq k \leqq n$$

where $\Delta^0\mu_k = \mu_k$, $\Delta^n\mu_k = \Delta^{n-1}\mu_k - \Delta^{n-1}\mu_{k+1}$. The associated h_{nk} is given by

$$\begin{aligned} h_{nk} &= \lambda_{nk}\left(\frac{\lambda_k}{\lambda_n}\right)^{1/p} \\ &= \binom{n + \delta + 1/p}{n - k}\left(\frac{k + \delta + 1/p}{n + \delta + 1/p}\right)^{1/p} \Delta^{n-k}\mu_k \\ &\qquad \text{for } 0 \leqq k \leqq n,\ n \geqq 1, \\ h_{00} &= \mu_0. \end{aligned}$$

By Theorem 5, we have that $\tilde{H} \in B(\ell^p)$ if and only if $\mu_n = \int_0^1 t^{n+\delta+1/p}\, d\gamma(t)$ for $n \geqq 0$, where $\gamma \in BV[0, 1]$. Furthermore, if γ is normalized and $\gamma(0+) = 0$, then $\|\tilde{H}\| = \int_0^1 |d\gamma(t)|$. The condition $\gamma(0+) = 0$ involves no loss in generality when $\delta + 1/p > 0$, and when $\delta + 1/p = 0$ it only affects the value of μ_0. This is similar to results of Jakimovski, Rhoades and Tzimbalario [4, Theorems 1 and 2], the main parts of which we can deduce from the above result. Let $H' = (h'_{nk})$ be the matrix given by

$$h'_{nk} = \begin{cases} \binom{n + \delta}{n - k} \Delta^{n-k}\mu_k & 0 \leqq k \leqq n, \\ 0 & k > n, \end{cases}$$

and let $\tilde{H}' = (|h'_{nk}|)$. We have that

$$\frac{\binom{n + \delta + 1/p}{n - k}}{\binom{n + \delta}{n - k}} \left(\frac{k + \delta + 1/p}{n + \delta + 1/p}\right)^{1/p} = \frac{w_n}{w_k},$$

where

$$w_n = \binom{n + \delta + 1/p}{1/p} (n + \delta + 1/p)^{-1/p} \rightarrow \frac{1}{\Gamma(1 + 1/p)}$$

as $n \to \infty$, and $w_n > 0$ for $n \geqq 1$. It follows that there are positive constants c_1, c_2 such that

$$c_1|h_{nk}| \leqq |h'_{nk}| \leqq c_2|h_{nk}| \text{ for } 0 \leqq k \leqq n.$$

Hence $\tilde{H}' \in B(l^p)$ if and only if $\tilde{H} \in B(l^p)$ and so, by the result proved above, $\tilde{H}' \in B(l^p)$ if and only if $\mu_n = \int_0^1 t^{n+\delta+1/p}\, d\gamma(t)$ for $n \geqq 0$, $\delta + 1/p \geqq 0$, where $\gamma \in BV[0, 1]$. Jakimovski, Rhoades and Tzimbalario proved this only for $\delta \geqq 0$, but they also showed that in this case $|\tilde{H}'| = \int_0^1 |d\gamma(t)|$ provided γ is normalized. This we cannot deduce from the results established in the present paper.

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