Tauberian theorems on a scale of Abel-type summability methods

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1. Introduction

Suppose throughout that $\{s_n\}$ is a sequence of real numbers, λ is real, $\varepsilon_0^\lambda = 1$, and $\varepsilon_n^\lambda = \binom{n+\lambda}{n}$ for $n = 1, 2, 3, \ldots$.

We are concerned with the methods of summability A_λ, introduced and studied by Borwein [1], and defined as follows. If

$$(1) \qquad \sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n$$

converges for $y > 0$, and tends to s as $y \to \infty$, then we say that the sequence $\{s_n\}$ is A_λ-convergent to s and write $s_n \to s(A_\lambda)$. The method A_0 is the ordinary Abel method.

Borwein has proved (in [1]) the following basic results.

Lemma 1. *For* $\lambda > -1$, *if* $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n$ *converges for* $y > 0$, *then for* $\varepsilon > 0$,

$$(2) \qquad \sigma_\lambda(y) = \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\,\Gamma(\varepsilon)} \frac{1}{y} \int_0^y \left(1 - \frac{t}{y}\right)^{\varepsilon-1} \left(\frac{t}{y}\right)^\lambda \sigma_{\lambda+\varepsilon}(t)\,dt.$$

Lemma 2. A_λ *is regular for* $\lambda > -1$. [That is, $s_n \to s$ implies $s_n \to s(A_\lambda)$.]

Lemma 3. $A_{\lambda+\varepsilon} \subset A_\lambda$ *for* $\lambda > -1$, $\varepsilon > 0$. [That is, $s_n \to s(A_{\lambda+\varepsilon})$ implies $s_n \to s(A_\lambda)$ and there exists a sequence $\{s_n\}$, depending on λ and ε, such that $\{s_n\}$ is A_λ-convergent but not $A_{\lambda+\varepsilon}$-convergent.]

The real-valued function f is said to be *slowly decreasing* if $\liminf\{f(y) - f(x)\} \geqq 0$ whenever $y \geqq x \to \infty$ and $\ln \frac{y}{x} \to 0$; i.e., if for each $\varepsilon > 0$, there exist positive numbers δ and M such that $f(y) - f(x) \geqq -\varepsilon$ whenever $y \geqq x \geqq M$ and $0 \leqq \ln \frac{y}{x} < \delta$.

The object of this paper is to prove the following two Tauberian theorems.

Theorem 1. *For* $\lambda > -1$ *and* $\varepsilon > 0$, *if* $s_n \to s(A_\lambda)$, *and* $\sigma_{\lambda+\varepsilon}(t)$ *is slowly decreasing, then* $s_n \to s(A_{\lambda+\varepsilon})$.

Theorem 2. *For* $\lambda > -1$ *and* $\varepsilon > 0$, *if* $s_n \to s(A_\lambda)$, *and* $\sigma_{\lambda+\varepsilon}(t) = O(1)$ *for* $t > 0$, *then* $s_n \to s(A_{\lambda+\delta})$ *for* $0 < \delta < \varepsilon$.

2. A general Tauberian result

Throughout this section we assume the following four initial hypotheses.

(i) $K(u, v)$ *is defined, real-valued, and nonnegative for* $u > 0, v \geqq 0$. *Moreover* $\int_0^\infty K(u, v)\,dv$ *exists in the Lebesgue sense for each* $u > 0$.

(ii) $\int_0^\infty K(u, v)\,dv \to 1$ *as* $u \to \infty$.

(iii) $f(v)$ *is real-valued and continuous for* $v \geqq 0$.

(iv) $F(u) = \int_0^\infty K(u, v) f(v)\,dv$ *exists in the Cauchy-Lebesgue sense for each* $u > 0$.

Theorem 3. *Suppose the following conditions hold*:

(3) Φ *is a real-valued, nonnegative, increasing, continuous function defined on* $[0, \infty)$ *such that* $\Phi(x) \to \infty$ *as* $x \to \infty$;

(4) $\liminf\{f(y) - f(x)\} \geqq 0$ *whenever* $y \geqq x \to \infty$ *and* $\Phi(y) - \Phi(x) \to 0$;

(5) $\Phi(x) - \Phi(x-1) \to 0$ *as* $x \to \infty$;

(6) $\int_0^x K(u, v)\,dv \to 0$ *whenever* $u > x \to \infty$ *and* $\Phi(u) - \Phi(x) \to \infty$;

(7) $\int_x^\infty K(u, v)\,\{\Phi(v) - \Phi(x)\}\,dv \to 0$ *whenever* $x > u \to \infty$ *and* $\Phi(x) - \Phi(u) \to \infty$;

and

(8) $F(u) = O(1)$ *for* $u > 0$.

Then $f(v) = O(1)$ *for* $v > 0$.

This result is the integral analogue, with slightly weakened hypotheses, of a theorem originally given by Vijayaraghavan [4]. A proof patterned on the one given by Hardy can easily be constructed using the following four lemmas. We omit the details.

Lemma 4. *If* $\int_0^M K(u, v)\,dv \to 0$ *as* $u \to \infty$ *for each* $M > 0$, *then*

$$\liminf_{v \to \infty} f(v) \leqq \liminf_{u \to \infty} F(u) \leqq \limsup_{u \to \infty} F(u) \leqq \limsup_{v \to \infty} f(v)\,.$$

Lemma 4 is the integral analogue of Theorem 9 in [2], and is proved by an argument of standard type.

Lemma 5. *If* (3) *and* (6) *hold, and if* $f(v) \to s$ *as* $v \to \infty$, *then* $F(u) \to s$ *as* $u \to \infty$, *where* s *may be finite or infinite.*

Proof. By Lemma 4, it suffices to show that, for every fixed $M > 0$,

$$\int_0^M K(u, v)\,dv \to 0 \quad \text{as } u \to \infty .$$

Let ε, M be given positive numbers. By (6) there exist an $X \geqq M > 0$ and an $R > 0$ such that $\int_0^X K(u, v)\,dv < \varepsilon$ whenever $u > X$ and $\Phi(u) - \Phi(X) \geqq R$. Let U be the positive number such that $\Phi(U) = R + \Phi(X)$. Then for $u \geqq U$, $\int_0^M K(u, v)\,dv \leqq \int_0^X K(u, v)\,dv < \varepsilon$. This completes the proof.

Lemma 6. *If* (3) *and* (4) *hold, then there exist positive constants* M_1, M_2 *such that*

$$f(y) - f(x) > -M_1\{\Phi(y) - \Phi(x)\} - M_2$$

for $y \geqq x \geqq 0$.

Proof. By (4) there exist positive numbers X and δ such that $f(y) - f(x) > -1$ whenever $y \geqq x \geqq X$ and $\Phi(y) - \Phi(x) \leqq \delta$.

If $X \geqq y \geqq x \geqq 0$, then by the continuity of f, there exists a positive constant N_1 such that $f(y) - f(x) > -N_1$.

If $y \geqq X \geqq x \geqq 0$ and $\Phi(y) - \Phi(x) \leqq \delta$, then $\Phi(y) - \Phi(X) \leqq \Phi(y) - \Phi(x) \leqq \delta$ and since Φ is increasing to infinity, y must be bounded above, so that $f(y) - f(x) > -N_2$ for some positive constant N_2.

It follows that $f(y) - f(x) > -M_2$ whenever $y \geqq x \geqq 0$ and $\Phi(y) - \Phi(x) \leqq \delta$, where $M_2 = \max(1, N_1, N_2)$.

Suppose now that $y > x \geqq 0$. Define an increasing sequence $\{x_r\}$ so that $x_0 = x$ and $\Phi(x_r) = \Phi(x_{r-1}) + \delta$ for $r = 1, 2, \ldots$. Since $\Phi(x_r) = \Phi(x_0) + r\delta$ we have $x_r \to \infty$. Hence, there exists an integer m such that $x_m \leqq y < x_{m+1}$. Therefore

$$\text{(9)} \qquad f(y) - f(x) = \sum_{r=0}^{m-1} [f(x_{r+1}) - f(x_r)] + f(y) - f(x_m) > -mM_2 - M_2 .$$

Since $m\delta = \Phi(x_m) - \Phi(x_0) \leqq \Phi(y) - \Phi(x)$ it follows from (9) that

$$f(y) - f(x) > -\frac{M_2}{\delta}\{\Phi(y) - \Phi(x)\} - M_2 .$$

The desired result follows.

Lemma 7. *If* (3) *and* (7) *hold, then*

$$\int_x^\infty K(u, v)\,dv \to 0$$

whenever $x > u \to \infty$ *and* $\Phi(x) - \Phi(u) \to \infty$.

Proof. Assign $\varepsilon > 0$. By (7), there exist positive numbers X and R such that $R > 1$ and

$$\int_x^\infty K(u, v)\,\{\Phi(v) - \Phi(x)\}\,dv < \varepsilon$$

whenever $x > u > X$ and $\Phi(x) - \Phi(u) \geqq R$.

Suppose now that $x > u > X$ and $\Phi(x) - \Phi(u) \geqq R + 1$. Since Φ is continuous and increasing, there exists a w satisfying $u < w < x$ and $\Phi(x) - \Phi(w) = 1$. Now

$$\Phi(w) - \Phi(u) = \Phi(w) - \Phi(x) + \Phi(x) - \Phi(u) \geqq -1 + R + 1 = R\,.$$

Hence,

$$\begin{aligned}\int_x^\infty K(u, v)\,dv &= \int_x^\infty K(u, v)\,\{\Phi(x) - \Phi(w)\}\,dv \\ &\leqq \int_x^\infty K(u, v)\,\{\Phi(v) - \Phi(w)\}\,dv \\ &\leqq \int_w^\infty K(u, v)\,\{\Phi(v) - \Phi(w)\}\,dv < \varepsilon\,.\end{aligned}$$

This completes the proof.

3. A Tauberian theorem of Wiener

In this section we state a version of a Tauberian theorem of Wiener (see [2], Theorem 233).

Theorem 4. *If*

(10) $g \in L(0, \infty)$;

(11) $\int_0^\infty g(t)\,t^{-ix}\,dt \neq 0$ *for any real* x;

(12) f *is bounded and measurable over* $(0, \infty)$;

(13) f *is slowly decreasing*;

(14) $\lim_{y \to \infty} \frac{1}{y} \int_0^\infty g\left(\frac{t}{y}\right) f(t)\,dt = s \int_0^\infty g(t)\,dt$;

then $f(t) \to s$ *as* $t \to \infty$.

4. Proof of Theorem 1

Let

$$f(t) = \sigma_{\lambda+\varepsilon}(t),$$

$$g(t) = \begin{cases} \dfrac{\Gamma(\lambda + \varepsilon + 1)}{\Gamma(\lambda + 1)\,\Gamma(\varepsilon)}\, t^\lambda (1 - t)^{\varepsilon - 1} & 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the function g satisfies conditions (10) and (11) and that

$$\int_0^\infty g(t)\,dt = 1 .$$

Further, the hypotheses of Theorem 1 together with identity (2) ensure that

$$\sigma_\lambda(y) = \frac{1}{y}\int_0^\infty g\left(\frac{t}{y}\right) f(t)\,dt$$

and that f satisfies conditions (13) and (14). In view of Theorem 4, it therefore suffices to prove that f is bounded on $(0, \infty)$.

Let

$$K(u, v) = \begin{cases} \dfrac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\,\Gamma(\varepsilon)}\, \dfrac{1}{u} \left(\dfrac{v}{u}\right)^\lambda \left(1-\dfrac{v}{u}\right)^{\varepsilon-1} & 0 < v < u \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(x) = \begin{cases} \dfrac{x}{e} & 0 \leqq x \leqq e \\ \ln x & e < x < \infty . \end{cases}$$

Now, $K(u, v) \geqq 0$ and $\int_0^\infty K(u, v)\,dv = 1$ for $u > 0$. Moreover, since f is continuous, $\int_0^\infty K(u, v)\, f(v)\,dv$ exists for each $u > 0$.

It is clear that the function Φ satisfies conditions (3) and (5), and the hypotheses of Theorem 1 guarantee that (4) and (8) hold. Condition (7) is satisfied since $K(u, v) = 0$ whenever $v \geqq u > 0$. For (6), when $u > x$ we have

$$\begin{aligned} \int_0^x K(u, v)\,dv &= \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\,\Gamma(\varepsilon)}\, \frac{1}{u} \int_0^x \left(\frac{v}{u}\right)^\lambda \left(1-\frac{v}{u}\right)^{\varepsilon-1} dv \\ &= \frac{\Gamma(\lambda+\varepsilon+1)}{\Gamma(\lambda+1)\,\Gamma(\varepsilon)} \int_0^{x/u} t^\lambda (1-t)^{\varepsilon-1}\,dt \to 0 \text{ as } \frac{x}{u} \to 0 , \end{aligned}$$

and hence as $\ln \dfrac{u}{x} \to \infty$. As a result of Theorem 3, the proof of Theorem 1 is complete.

5. Additional lemmas

In order to establish Theorem 2 we require two additional lemmas.

Lemma 8. *If $f \in L(-\infty, \infty)$ then*

$$\int_{-\infty}^\infty |f(rx) - f(x)|\,dx \to 0 \textit{ as } r \to 1 .$$

The proof of Lemma 8 is straightforward and not unlike that of Theorem 248 in [3] (for example). We omit the details.

Lemma 9. *If*

(15) $h \in L(0, 1)$, *and*

(16) $f(t)$ *is measurable and bounded for* $t \geqq 0$, *then the function* F, *defined for* $y > 0$ *by*

$$F(y) = \frac{1}{y} \int_0^y h\left(\frac{t}{y}\right) f(t)\,dt\,,$$

is slowly decreasing.

Proof. It suffices to show that, for $y > x > 0$, $F(y) - F(x) \to 0$ as $\frac{y}{x} \to 1$.

For some fixed positive constant M we have

$$\begin{aligned} |F(y) - F(x)| &= \left| \frac{1}{y} \int_0^y h\left(\frac{t}{y}\right) f(t)\,dt - \frac{1}{x} \int_0^x h\left(\frac{t}{x}\right) f(t)\,dt \right| \\ &= \left| \frac{1}{y} \int_0^x h\left(\frac{t}{y}\right) f(t)\,dt - \frac{1}{x} \int_0^x \left\{ h\left(\frac{t}{y}\right) + h\left(\frac{t}{x}\right) - h\left(\frac{t}{y}\right) \right\} f(t)\,dt \right. \\ &\quad \left. + \frac{1}{y} \int_x^y h\left(\frac{t}{y}\right) f(t)\,dt \right| \\ &\leqq \frac{M(y-x)}{xy} \int_0^x \left| h\left(\frac{t}{y}\right) \right| dt + \frac{M}{x} \int_0^x \left| h\left(\frac{t}{y}\right) - h\left(\frac{t}{x}\right) \right| dt + \frac{M}{y} \int_x^y \left| h\left(\frac{t}{y}\right) \right| dt \\ &= M\left(\frac{y}{x} - 1\right) \int_0^{x/y} |h(t)|\,dt + M \int_0^1 \left| h\left(\frac{x}{y}t\right) - h(t) \right| dt + M \int_{x/y}^1 |h(t)|\,dt \to 0 \end{aligned}$$

as $\frac{x}{y} \to 1$ by Lemma 8 and (15). This establishes the result.

6. Proof of Theorem 2

Let

$$h(t) = \begin{cases} \dfrac{\Gamma(\lambda + \varepsilon + 1)}{\Gamma(\lambda + \delta + 1)\,\Gamma(\varepsilon - \delta)}\, t^{\lambda + \delta}(1 - t)^{\varepsilon - \delta - 1} & 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $h(t) \in L(0, 1)$, and by identity (2) we have

$$\sigma_{\lambda+\delta}(y) = \frac{1}{y} \int_0^y h\left(\frac{t}{y}\right) \sigma_{\lambda+\varepsilon}(t)\,dt\,.$$

By hypothesis, $\sigma_{\lambda+\varepsilon}(t)$ is bounded for $t \geqq 0$. Hence, by Lemma 9, $\sigma_{\lambda+\delta}(t)$ is slowly decreasing. The result follows by Theorem 1.

References

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