# ON THE ABSOLUTE SUMMABILITY OF STIELTJES INTEGRALS

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1. It is supposed throughout that $\lambda \geqslant 0$ and that $x(t)$ is a real function having bounded variation in every finite sub-interval of $[1, \infty)$.

Bosanquet has shown‡ that, when $\lambda$ is an integer, the conditions

(i) $k(t)$ *is continuous for* $t \geqslant 1$,

(ii) $\int_1^\infty t^{-1} |k(t)| \, dt < \infty$,

(iii)′ $\int_1^\infty t^\lambda |dk^{(\lambda)}(t)| < \infty$,

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‡ Bosanquet [3], Theorem B.

are sufficient to ensure the truth of the proposition:

$\int_1^\infty k(t)\,dx(t)$ *is summable* $|C, \lambda+1|$ *whenever* $\int_1^w dx(t)$ *is bounded* $(C, \lambda)$ *in* $(1, \infty)$.

My object in this paper is to prove that, for any $\lambda$, conditions (i), (ii) and the condition

(iii) *there is a number* $c$ $(\geqslant 1)$ *and a function* $h(u)$ *such that*

$$k(t) = \frac{1}{\Gamma(\lambda+1)} \int_t^\infty (u-t)^\lambda\, dh(u)$$

*for* $t \geqslant c$, *and* $\int_c^\infty u^\lambda |dh(u)| < \infty,$

are both necessary and sufficient for the above proposition to be true.

Series analogues involving integral orders of summability have been established by Fekete and Bosanquet*.

Given any function $g(t)$ integrable $L$ in every finite sub-interval of $(1, \infty)$, we shall write

$$g_0(t) = g(t), \quad g_\mu(t) = \frac{1}{\Gamma(\mu)} \int_1^t (t-u)^{\mu-1} g(u)\, du \quad (t \geqslant 1,\ \mu > 0);$$

we shall also use this notation with $x$ in place of $g$.

It is well known† that if

$$g(t) = \int_1^t k(u)\, dx(u) \quad (t \geqslant 1),$$

where $k(u)$ is continuous for $u \geqslant 1$ and $x(1) = 0$, then, for $\mu \geqslant 0$, $t \geqslant 1$,

$$g_\mu(t) = \frac{1}{\Gamma(\mu+1)} \int_1^t (t-u)^\mu k(u)\, dx(u).$$

2. LEMMA‡. *If* $p \geqslant 0$, $q \geqslant 0$, $r \geqslant -p$ *and* $\int_1^w dx(t) = O(w^r)$ $(C, p)$ *in*

* Fekete [5]; Bosanquet [2], Theorem 3. I have been informed by Dr. Bosanquet that H. C. Chow has recently obtained results of a similar character for series involving ractional orders of summability. (See preceding paper.)

† For this result and for the meaning of the summability notation used see Bosanquet [3].

‡ *Cf.* Sargent [7], Lemma 6.



$(1, \infty)$, *then there are numbers* $H$ *and* $K$, *independent of* $v$ *and* $w$, *such that*

$$\left| \int_1^w (w-t)^p (v-t)^q\, dx(t) \right| < H w^{p+r} v^q,$$

$$\left| \int_1^w (w-t)^p \{(v-t)^q - (v-w)^q\}\, dx(t) \right| < K w^{p+r+1} (w^{q-1} + v^{q-1}),$$

*whenever* $v \geqslant w \geqslant 1$.

Suppose, without loss in generality, that $x(1) = 0$, and let

$$M = \overline{\text{bound}}_{w \geqslant 1}\, w^{-p-r} |x_p(w)|.$$

Note that, for $v \geqslant w \geqslant 1$,

$$0 \leqslant v^q - (v-w)^q \leqslant \begin{cases} w^q & \text{when } 0 \leqslant q \leqslant 1, \\ q w v^{q-1} & \text{when } q > 1. \end{cases}$$

When $p = 0$*, we have, for $v \geqslant w \geqslant 1$,

$$\left| \int_1^w (v-t)^q\, dx(t) \right| \leqslant v^q \overline{\text{bound}}_{1 \leqslant \xi \leqslant w} \left| \int_1^\xi dx(t) \right| \leqslant M v^q w^r,$$

$$\left| \int_1^w \{(v-t)^q - (v-w)^q\}\, dx(t) \right| \leqslant \{v^q - (v-w)^q\} \overline{\text{bound}}_{1 \leqslant \xi \leqslant w} \left| \int_1^\xi dx(t) \right|$$

$$\leqslant M (w^q + q w v^{q-1}) w^r,$$

from which the required results follow.

Suppose now that $p = n + \delta$, where $n$ is a positive integer or zero and $0 < \delta \leqslant 1$, and that $v \geqslant w > 1$. Integration by parts yields:

$$\int_1^w (w-t)^p (v-t)^q\, dx(t) = (-1)^{n+1} \int_1^w x_n(t) \frac{d^{n+1}}{dt^{n+1}} \{(w-t)^p (v-t)^q\}\, dt$$

$$= \sum_{s=0}^{n+1} c_s J_s,$$

where $c_s$ is independent of $v$ and $w$ and

$$J_s = \int_1^w (w-t)^{\delta+s-1} (v-t)^{q-s} x_n(t)\, dt;$$

$$\int_1^w (w-t)^p \{(v-t)^q - (v-w)^q\}\, dx(t) = c_0 I + \sum_{s=1}^{n+1} c_s J_s,$$

where $c_s$ and $J_s$ are as above and

$$I = \int_1^w (w-t)^{\delta-1} \{(v-t)^q - (v-w)^q\} x_n(t)\, dt.$$

* See Widder [8], 18.

Applying the second mean value theorem and Riesz's mean value theorem, we find that*

$$|J_s| = \left(\frac{w}{v}\right)^s v^q \left| \int_1^{\xi_s} (w-t)^{\delta-1} x_n(t)\,dt \right| \quad (1 < \xi_s < w)$$

$$\leqslant \left(\frac{w}{v}\right)^s v^q \operatorname*{bound}_{1\leqslant\xi\leqslant w} \left| \int_1^{\xi} (\xi-t)^{\delta-1} x_n(t)\,dt \right|$$

$$\leqslant M\Gamma(\delta) \left(\frac{w}{v}\right)^s v^q w^{p+r};$$

$$|I| = \{v^q - (v-w)^q\} \left| \int_1^{\xi} (w-t)^{\delta-1} x_n(t)\,dt \right| \quad (1 < \xi < w)$$

$$\leqslant M\Gamma(\delta)(w^q + qwv^{q-1})\, w^{p+r}.$$

The truth of the lemma is now evident.

3. THEOREM 1. *If*

(i) *$k(t)$ is continuous for $t \geqslant 1$,*

(ii) $\int_1^\infty t^{-1} |k(t)|\,dt < \infty$,

(iii) *for some number $c$ $(\geqslant 1)$,*

$$k(t) = \frac{1}{\Gamma(\lambda+1)} \int_t^\infty (u-t)^\lambda\, dh(u)$$

*whenever $t \geqslant c$, where*

$$\int_c^\infty u^\lambda |dh(u)| < \infty,$$

(iv) $$\int_1^w dx(t) = O(1) \quad (C, \lambda) \text{ in } (1, \infty),$$

*then* $$\int_1^\infty k(t)\,dx(t) \text{ is summable } |C, \lambda+1|.$$

Suppose, without real loss in generality, that $c = 1$ and $x(1) = 0$, and write, for $v \leqslant 1$,

$$P(v) = \int_1^v (v-t)^\lambda t k(t)\,dx(t), \quad Q(v) = k(v) \int_1^v (v-t)^\lambda t\,dx(t).$$

For $w > 1$,

$$\int_1^w \left(1 - \frac{t}{w}\right)^{\lambda+1} k(t)\,dx(t) = (\lambda+1) \int_1^w k(t)\,dx(t) \int_t^w \left(1 - \frac{t}{v}\right)^\lambda t v^{-2}\,dv$$

$$= (\lambda+1) \int_1^w v^{-\lambda-2} P(v)\,dv;$$

* *Cf.* Borwein [1], 312.



from which it follows that the summability $|C, \lambda+1|$ of $\int_1^\infty k(t)\,dx(t)$ is equivalent to the convergence of

$$\int_1^\infty v^{-\lambda-2} |P(v)|\,dv.$$

We shall consider two cases.

A. *Suppose that* $\lambda = 0$. Then, for $v > 1$,

$$P(v) - Q(v) = -\int_1^v dk(t) \int_1^t u\,dx(u).$$

In view of (iv) there is a number $M$ such that, for $t \geqslant 1$,

$$\left| \int_1^t u\,dx(u) \right| = \left| tx(t) - \int_1^t x(u)\,du \right| \leqslant Mt.$$

Consequently

$$\int_1^\infty v^{-2} |P(v) - Q(v)|\,dv \leqslant M \int_1^\infty v^{-2}\,dv \int_1^v u\,|dk(u)|$$

$$= M \int_1^\infty |dk(u)| = M \int_1^\infty |dh(u)| < \infty.$$

Further, $$\int_1^\infty v^{-2} |Q(v)|\,dv \leqslant M \int_1^\infty v^{-1} |k(v)|\,dv < \infty.$$

Hence $$\int_1^\infty v^{-2} |P(v)|\,dv < \infty,$$

and this is the required result.

B. *Suppose that* $\lambda > 0$. It follows from (iii) that

$$\int_t^\infty (u-t)^\lambda |dh(u)| < \infty \quad (t \geqslant 1),$$

and hence, by Fubini's theorem for Lebesgue-Stieltjes integrals, that

$$k(t) = \frac{1}{\Gamma(\lambda)} \int_t^\infty dv \int_v^\infty (w-v)^{\lambda-1}\,dh(w) \quad (t \geqslant 1).$$

Consequently $k(t)$ is absolutely continuous in every finite sub-interval of $[1, \infty)$, $k(t) \to 0$ as $t \to \infty$, and, for almost all $t$ in $(1, \infty)$,

$$k'(t) = -\frac{1}{\Gamma(\lambda)} \int_t^\infty (w-t)^{\lambda-1}\,dh(w).$$

Now, for $v > 1$,

$$P(v)-Q(v) = -\int_1^v k'(t)\,dt \int_1^t (v-u)^\lambda u\,dx(u)$$

$$= \frac{1}{\Gamma(\lambda)} \int_1^v dt \int_t^v (w-t)^{\lambda-1} dh(w) \int_1^t (v-u)^\lambda u\,dx(u)$$

$$+\frac{1}{\Gamma(\lambda)} \int_1^v dt \int_v^\infty (w-t)^{\lambda-1} dh(w) \int_1^t (v-u)^\lambda u\,dx(u)$$

$$= \frac{1}{\Gamma(\lambda+1)} \int_1^v dh(w) \int_1^w (w-u)^\lambda (v-u)^\lambda u\,dx(u)$$

$$+\frac{1}{\Gamma(\lambda+1)} \int_v^\infty dh(w) \int_1^v (v-u)^\lambda \{(w-u)^\lambda-(w-v)^\lambda\} u\,dx(u);$$

the changes in order of integration being easily justified by Fubini's theorem and, where infinite ranges are involved, by the convergence of $\int_1^\infty w^\lambda |dh(w)|$.

Since $\int_1^w dx(t)$ is $O(1)$ $(C, \lambda)$ and, *a fortiori*, $O(1)$ $(C, \lambda+1)$ in $(1, \infty)$, it follows from the Lemma that there is a number $H$ such that, for $v \geqslant w \geqslant 1$,

$$\left| \int_1^w (w-u)^\lambda (v-u)^\lambda u\,dx(u) \right|$$

$$= \left| w \int_1^w (w-u)^\lambda (v-u)^\lambda dx(u) - \int_1^w (w-u)^{\lambda+1} (v-u)^\lambda dx(u) \right|$$

$$\leqslant H\Gamma(\lambda+1)\, w^{\lambda+1} v^\lambda.$$

Similarly, there is a number $K$ such that, for $w \geqslant v \geqslant 1$,

$$\left| \int_1^v (v-u)^\lambda \{(w-u)^\lambda-(w-v)^\lambda\} u\,dx(u) \right| \leqslant K\Gamma(\lambda+1)\, v^{\lambda+2} (v^{\lambda-1}+w^{\lambda-1}).$$

Consequently

$$\int_1^\infty v^{-\lambda-2} |P(v)-Q(v)|\,dv$$

$$\leqslant H \int_1^\infty v^{-2} dv \int_1^v w^{\lambda+1} |dh(w)| + K \int_1^\infty v^{\lambda-1} dv \int_v^\infty |dh(w)|$$

$$+K \int_1^\infty dv \int_v^\infty w^{\lambda-1} |dh(w)|$$

$$\leqslant \{H+K(\lambda^{-1}+1)\} \int_1^\infty w^\lambda |dh(w)| < \infty.$$



Further, in view of (iv), there is a number $M$ such that, for $v \geqslant 1$,

$$|Q(v)| = \left| v\,k(v) \int_1^v (v-t)^\lambda dx(t) - k(v) \int_1^v (v-t)^{\lambda+1} dx(t) \right| \leqslant M v^{\lambda+1} |k(v)|;$$

and so $$\int_1^\infty v^{-\lambda-2} |Q(v)|\,dv \leqslant M \int_1^\infty v^{-1} |k(v)|\,dv < \infty.$$

It follows that $$\int_1^\infty v^{-\lambda-2} |P(v)|\,dv < \infty,$$

and the proof of the theorem is thus completed.

4. THEOREM 2. *If* $\int_1^\infty k(t)\,dx(t)$ *is summable* $|C, \lambda+1|$ *whenever* $\int_1^\infty dx(t)$ *is summable* $(C, \lambda)$, *then**

(i) $k(t)$ *is continuous for* $t \geqslant 1$,

(ii) $\int_1^\infty t^{-1} |k(t)|\,dt < \infty$,

(iii) *there is a number* $c$ $(\geqslant 1)$ *and a function* $h(u)$ *such that*

$$k(t) = \frac{1}{\Gamma(\lambda+1)} \int_t^\infty (u-t)^\lambda dh(u)$$

*for* $t \geqslant c$, *and* $$\int_c^\infty u^\lambda |dh(u)| < \infty.$$

Since, for any $w > 1$, $\int_1^w k(t)\,dx(t)$ exists in the Riemann-Stieltjes sense whenever $x(t)$ is of bounded variation in $[1, w]$, we immediately deduce (i).

It follows from the hypothesis, on putting $x(t) = \int_1^t f(u)\,du$, that $\int_1^\infty k(t) f(t)\,dt$ is summable $|C, \lambda+1|$ whenever $\int_1^\infty f(t)\,dt$ is summable $(C, \lambda)$. Sargent has shown† that in consequence of this there are numbers $c$, $l$ and a function $h(u)$ such that $c \geqslant 1$, $\int_c^\infty u^\lambda |dh(u)| < \infty$ and

$$\theta(t) = \frac{1}{\Gamma(\lambda+1)} \int_t^\infty (u-t)^\lambda dh(u)$$

is equivalent, for $t \geqslant c$, to $k(t)-l$.

* It is to be understood that $\int_1^X k(t)\,dx(t)$ exists as a Riemann-Stieltjes integral for every $X > 1$.

† Sargent [7], Theorem 1.

When $\lambda = 0$, $\theta(t)$ is of bounded variation in $[c, \infty)$ and tends to zero as $t \to \infty$; and, since $k(t)$ is continuous for $t \geqslant c$, it follows easily that*

$$\int_c^\infty |dk(u)| < \infty \quad \text{and} \quad k(t) - l = -\int_t^\infty dk(u) \quad (t \geqslant c).$$

When $\lambda > 0$, $\theta(t)$ is continuous for $t \geqslant c$ and so in this case

$$k(t) - l = \theta(t) \quad (t \geqslant c).$$

We have thus established (iii) with $k(t) - l$ in place of $k(t)$. Since it follows that $k(t) \to l$ as $t \to \infty$, it remains only to prove $\int_t^\infty t^{-1}|k(t)|\,dt$ convergent, for this will ensure that $l = 0$.

Note that in the proof of Theorem 1 no use was made of the convergence of $\int_1^\infty t^{-1}|k(t)|\,dt$ in establishing the convergence of

$$\int_1^\infty v^{-\lambda-2}|P(v) - Q(v)|\,dv.$$

Consequently, we can now deduce that $\int_1^\infty v^{-\lambda-2}\,dv \left| k(v) \int_1^v (v-t)^\lambda t\,dx(t) \right|$ is convergent whenever $\int_1^\infty dx(t)$ is summable $(C, \lambda)$.

It follows, on putting $x(t) = \int_1^t u^{-1} g(u)\,du$, that $\int_1^\infty v^{-\lambda-2}|k(v) g_{\lambda+1}(v)|\,dv$ is convergent whenever $\int_1^\infty u^{-1} g(u)\,du$ is summable $(C, \lambda)$.

Let $(\alpha_n)$ be a sequence of positive numbers decreasing to zero with $\alpha_1 \leqslant 1$, and let $s$ be the integer such that $\lambda < s \leqslant \lambda + 1$. Then there is a function $\phi(t)$ such that $\phi^{(s)}(t)$ is absolutely continuous in every finite sub-interval of $[1, \infty)$,

$$\phi(t) = \begin{cases} 0 \text{ for } 1 \leqslant t \leqslant 2, \\ (-1)^n \alpha_n t^{\lambda+1} \text{ for } n + 1/n < t < n + 1 - 1/(n+1) \quad (n = 2, 3, \ldots), \end{cases}$$

$|\phi(t)| \leqslant t^{\lambda+1}$ for $t \geqslant 1$ and $t^{-\lambda-1}\phi(t) \to 0$ as $t \to \infty$.

Let
$$g(v) = \frac{1}{\Gamma(s-\lambda)} \int_1^v (v-t)^{s-\lambda-1} \phi^{(s+1)}(t)\,dt \quad (v \geqslant 1).$$

Then, for $v \geqslant 1$,

$$g_{\lambda+1}(v) = \frac{1}{s!} \int_1^v (v-t)^s \phi^{(s+1)}(t)\,dt = \phi(v).$$

Now suppose that $w > v > 2$ and that $p$, $q$ are the integers such that $p \leqslant v < p+1$, $q \leqslant w < q+1$. Then

$$\left| \int_v^w t^{-\lambda-2} g_{\lambda+1}(t)\,dt \right| \leqslant \left| \sum_{n=p}^{q} (-1)^n \alpha_n \int_n^{n+1} t^{-1}\,dt \right| + 2 \sum_{n=p}^{q+1} \int_{n-1/n}^{n+1/n} t^{-1}\,dt + \int_p^{p+1} t^{-1}\,dt + \int_q^{q+1} t^{-1}\,dt,$$

* *Cf.* Sargent [6], Lemma 2.



which tends to zero as $w$ and $v$ tend to infinity. Hence

$$\int_1^w t^{-\lambda-2} g_{\lambda+1}(t)\,dt$$

tends to a finite limit as $w \to \infty$. Since $w^{-\lambda-1} g_{\lambda+1}(w) \to 0$ as $w \to \infty$, it follows that

$$\int_1^w t^{-\lambda-1} g_\lambda(t)\,dt = w^{-\lambda-1} g_{\lambda+1}(w) + (\lambda+1) \int_1^w t^{-\lambda-2} g_{\lambda+1}(t)\,dt$$

tends to a finite limit as $w \to \infty$.

Further, when $\lambda > 0$, $w > 1$,

$$\Gamma(\lambda) \int_1^w t^{-\lambda-1} g_\lambda(t)\,dt = \int_1^w t^{-\lambda-1}\,dt \int_1^t (t-u)^{\lambda-1} g(u)\,du$$
$$= \int_1^w g(u)\,du \int_u^w \left(1 - \frac{u}{t}\right)^{\lambda-1} t^{-2}\,dt = \frac{1}{\lambda} \int_1^w \left(1 - \frac{u}{w}\right)^\lambda u^{-1} g(u)\,du.$$

Hence, for $\lambda \geqslant 0$, $\int_1^\infty u^{-1} g(u)\,du$ is summable $(C, \lambda)$ and consequently

$$\int_1^\infty v^{-\lambda-2}|k(v) g_{\lambda+1}(v)|\,dv < \infty.$$

Since $k(v)$ is bounded in $(1, \infty)$, we now deduce that

$$\sum_{n=1}^\infty \alpha_n \int_n^{n+1} v^{-1}|k(v)|\,dv \leqslant \int_1^\infty v^{-\lambda-2}|k(v) g_{\lambda+1}(v)|\,dv + \sum_{n=2}^\infty \int_{n-1/n}^{n+1/n} v^{-1}|k(v)|\,dv + \int_1^2 v^{-1}|k(v)|\,dv < \infty.$$

It follows that

$$\int_1^\infty v^{-1}|k(v)|\,dv = \sum_{n=1}^\infty \int_n^{n+1} v^{-1}|k(v)|\,dv$$

is finite, for if not we could make

$$\sum_{n=1}^\infty \alpha_n \int_n^{n+1} v^{-1}|k(v)|\,dv$$

infinite by putting

$$\alpha_n = 1 \Big/ \left\{1 + \int_1^{n+1} v^{-1}|k(v)|\,dv\right\}.$$

This completes the proof of the theorem.

5. The object of this section is to show that *Theorem* 1 *remains valid if condition* (iii) *is replaced by*

(iii)′ *there is an integer* $n$ $(\geqslant \lambda)$ *and a number* $c$ $(\geqslant 1)$ *such that*

$$\int_c^\infty t^n |dk^{(n)}(t)| < \infty.$$

*Suppose that* (iii)′ *is satisfied and that* $\int_1^\infty t^{-1}|k(t)|\,dt < \infty$. Since $\int_c^\infty dk^{(n)}(u)$ is convergent, there is a number $l$ such that, for $t \geqslant c$,

$$k^{(n)}(t) - l = -\int_t^\infty dk^{(n)}(u) = o(1) \quad \text{as } t \to \infty.$$

If $n = 0$, we have $l = \lim_{t\to\infty} k(t)$; and if $n \geqslant 1$,

$$l = \lim_{t\to\infty} nt^{-n}\int_1^t (t-u)^{n-1} k^{(n)}(u)\,du = \lim_{t\to\infty} n!\,t^{-n}k(t).$$

In either case we deduce from the convergence of $\int_1^\infty t^{-1}|k(t)|\,dt$ that $l = 0$.

Since the result is now evident if $n = \lambda = 0$, we shall suppose that $n \geqslant 1$. We have, for $t \geqslant c$,

$$\int_t^\infty (u-t)^n\,dk^{(n)}(u) = -n\int_t^\infty dk^{(n)}(u)\int_t^u (v-t)^{n-1}\,dv$$

$$= -n\int_t^\infty (v-t)^{n-1}\,dv\int_v^\infty dk^{(n)}(u) = n\int_t^\infty (v-t)^{n-1}\,dk^{(n-1)}(v);$$

the change in order of integration being justified because of the absolute convergence of the first integral.

Further,

$$\int_c^\infty t^{n-1}|dk^{(n-1)}(t)| = \int_c^\infty t^{n-1}|k^{(n)}(t)|\,dt$$

$$\leqslant \int_c^\infty t^{n-1}\,dt\int_t^\infty |dk^{(n)}(u)|$$

$$\leqslant \frac{1}{n}\int_c^\infty u^n|dk^{(n)}(u)| < \infty.$$

The above argument yields, after repetition if necessary,

$$\frac{(-1)^{n+1}}{n!}\int_t^\infty (u-t)^n\,dk^{(n)}(u) = -\int_t^\infty dk(u) = k(t) \quad (t \geqslant c).$$

The required result is now a consequence of the proposition: *Theorem* 1 *remains valid if in condition* (iii) $\lambda$ *is replaced by* $\mu$ $(\geqslant \lambda)$.

That this is a true proposition is evident from the following argument.

*Suppose that* $\mu > \lambda$, $c \geqslant 1$, *and that* $\int_c^\infty u^\mu|dh(u)| < \infty$. Let

$$g(t) = -\frac{1}{\Gamma(\mu-\lambda+1)}\int_t^\infty (u-t)^{\mu-\lambda}\,dh(u) \quad (t \geqslant c).$$



Then, as in the proof of Theorem 1,

$$g(t) = -\frac{1}{\Gamma(\mu-\lambda)}\int_t^\infty du\int_u^\infty (v-u)^{\mu-\lambda-1}\,dh(v) \quad (t \geqslant c).$$

Consequently,

$$\int_c^\infty t^\lambda|dg(t)| = \int_c^\infty t^\lambda|g'(t)|\,dt \leqslant \frac{1}{\Gamma(\mu-\lambda)}\int_c^\infty t^\lambda\,dt\int_t^\infty (v-t)^{\mu-\lambda-1}|dh(v)|$$

$$= \frac{1}{\Gamma(\mu-\lambda)}\int_c^\infty |dh(v)|\int_c^v (v-t)^{\mu-\lambda-1}t^\lambda\,dt \leqslant \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}\int_c^\infty v^\mu|dh(v)| < \infty.$$

Further, for $t \geqslant c$,

$$\frac{1}{\Gamma(\lambda+1)}\int_t^\infty (u-t)^\lambda\,dg(u) = \frac{1}{\Gamma(\lambda+1)\Gamma(\mu-\lambda)}\int_t^\infty (u-t)^\lambda\,du\int_u^\infty (v-u)^{\mu-\lambda-1}\,dh(v)$$

$$= \frac{1}{\Gamma(\lambda+1)\Gamma(\mu-\lambda)}\int_t^\infty dh(v)\int_t^v (v-u)^{\mu-\lambda-1}(u-t)^\lambda\,du$$

$$= \frac{1}{\Gamma(\mu+1)}\int_t^\infty (v-t)^\mu\,dh(v);$$

the change in order of integration being justified because of the absolute convergence of the final integral.

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