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August Leopold Crelle

fortgeführt von
C. W. Borchardt, K. Weierstrass, L. Kronecker,
L. Fuchs, K. Hensel, L. Schlesinger

gegenwärtig herausgegeben von

Helmut Hasse · Hans Rohrbach

unter Mitwirkung von
M. Deuring, P. R. Halmos, O. Haupt,
F. Hirzebruch, M. Kneser, G. Köthe, K. Krickeberg,
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L. Schmetterer, E. Stiefel, B. Volkmann

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A Tauberian theorem for Abelian summability methods

By *D. Borwein* and *B. Watson* at London

Let $\{\lambda_n\}$ be a sequence of real numbers satisfying

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots; \ \lambda_n \to \infty.$$

Let $\{a_n\}$ be a sequence of real numbers, and let

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

The series $\sum_{n=1}^{\infty} a_n$ is said to be summable to s by the abelian summability method (A, λ) if $\sum_{n=1}^{\infty} a_n e^{-y\lambda_n}$ is convergent for $y > 0$, and tends to s as $y \to 0+$.

The purpose of this note is to prove the following general tauberian theorem:

Theorem. *If* $\frac{\lambda_{n+1}}{\lambda_n} \to 1$, *if* $\sum_{n=1}^{\infty} a_n$ *is summable* (A, λ) *to* s, *and if* $\liminf \{s_n - s_m\} \geq 0$ *as* $n \geq m \to \infty$ *and* $\frac{\lambda_n}{\lambda_m} \to 1$, *then* $s_n \to s$.

The only two special cases of this theorem which appear to be known are $\lambda_n = n$ and $\lambda_n = \log n$. The case $\lambda_n = n$ is a familiar result for ordinary Abel summability. The case $\lambda_n = \log n$ was established by Kwee [2] by a method fundamentally different from, and more complicated than, ours. Our proof is based on the following known result (see [1], Theorem 105, p. 164):

Lemma. *If* α *is a function of bounded variation on every interval* $[0, T]$; *if* $\int_0^{\infty} e^{-yt}\, d\alpha(t)$ *is convergent for* $y > 0$ *and tends to* s *as* $y \to 0+$; *and if* $\liminf \{\alpha(y) - \alpha(x)\} \geq 0$ *as* $y \geq x \to \infty$ *and* $\frac{y}{x} \to 1$; *then* $\alpha(y) \to s$ *as* $y \to \infty$.

Proof of the theorem. Set

$$\alpha(t) = \sum_{\lambda_n < t} a_n.$$

Then for $y>0$, we have

$$\int_0^\infty e^{-yt}\,d\alpha(t)=\sum_{n=1}^\infty a_n\,e^{-y\lambda_n}$$

and, by hypothesis, the series converges and its sum tends to s as $y\to 0+$.

Assign $\varepsilon>0$. Then there exist positive numbers M, δ such that $s_n-s_m>-\varepsilon$ whenever $n\ge m\ge M$ and $\frac{\lambda_n}{\lambda_m}\le 1+2\delta$. Choose an integer N such that $\lambda_N>\lambda_{M+1}$ and, for $m+1\ge N$,

$$\frac{\lambda_{m+1}}{\lambda_m}\le\frac{1+2\delta}{1+\delta}.$$

Let $y\ge x\ge\lambda_N$ and $\frac{y}{x}\le 1+\delta$. Then there exist integers n, m such that $\lambda_{n+1}\ge y>\lambda_n$ and $\lambda_{m+1}\ge x>\lambda_m$. Hence $n\ge m\ge M$ and

$$\frac{\lambda_n}{\lambda_m}<\frac{y}{x}\,\frac{\lambda_{m+1}}{\lambda_m}\le(1+\delta)\,\frac{1+2\delta}{1+\delta}=1+2\delta;$$

and therefore

$$\alpha(y)-\alpha(x)=\alpha(\lambda_{n+1})-\alpha(\lambda_{m+1})=s_n-s_m>-\varepsilon.$$

Consequently, $\liminf\{\alpha(y)-\alpha(x)\}\ge 0$ as $y\ge x\to\infty$ and $\frac{y}{x}\to 1$; and so, by the lemma, $s_n\to s$.

References

[1] *G. H. Hardy*, Divergent Series, Oxford 1949.

[2] *B. Kwee*, Some theorems on the (A, log n) method of summation, J. Lond. Math. Soc. **2** (1969), 323—330.

Department of Mathematics, University of Western Ontario, London, Ontario, Canada

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