EQUIVALENCE OF RIESZ METHODS OF SUMMABILITY

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Suppose throughout that $\lambda = \{\lambda_n\}$ is an unbounded strictly increasing sequence with $\lambda_0 = 0$, that p is a non-negative integer, that $0 < \delta \leqslant 1$, that $\kappa \geqslant 0$, and that $\sum_{\nu=0}^{\infty} a_\nu$ is a series of real numbers.

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be summable by the Riesz method (R, λ, κ) to s if

$$\sum_{\lambda_\nu < \omega} \left(1-\frac{\lambda_\nu}{\omega}\right)^\kappa a_\nu \to s \quad \text{as} \quad \omega \to \infty.$$

A summability method Q is said to include a method P, and we write $\mathrm{P} \subseteq \mathrm{Q}$, if every series summable P to s is necessarily summable Q to s. The methods are said to be equivalent, and we write $\mathrm{P} \sim \mathrm{Q}$, if $\mathrm{P} \subseteq \mathrm{Q}$ and $\mathrm{Q} \subseteq \mathrm{P}$.

It is familiar that

$$(R, \lambda, \alpha) \subseteq (R, \lambda, \beta) \; (0 \leqslant \alpha \leqslant \beta). \tag{1}$$

See [3; Theorem 16].

Our object is to prove the following theorem.

THEOREM. *If $\kappa > p$, then a necessary and sufficient condition for $(R, \lambda, \kappa) \sim (R, \lambda, p)$ is*

$$\liminf_{n\to\infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1. \tag{2}$$

The case $p = 0$ of the above theorem is known; see [4] for full references.

The following lemma is essentially part of a theorem we established elsewhere [**1**; Theorem 5]. A proof is given here for completeness.

LEMMA. *Suppose that $a_{n,\nu} \geqslant 0$ and that*

$$t_n = \sum_{\nu=0}^{\infty} a_{n,\nu} s_\nu$$

tends to zero if and only if s_n tends to zero. Then

$$\liminf_{\nu\to\infty} \max_{n\geqslant 0} a_{n,\nu} > 0.$$

Proof. The hypotheses imply that $\lim_{n\to\infty} a_{n,\nu} = 0$ for $\nu = 0, 1, 2, \ldots$, and that $\sup_{n\geqslant 0} \sum_{\nu=0}^{\infty} a_{n,\nu} < \infty$.

Let $\mu_\nu = \max_{n\geqslant 0} a_{n,\nu}$ and assume that $\liminf \mu_\nu = 0$. There is an increasing sequence of integers $\{k_i\}$ such that

$$\mu_{k_i} < 2^{-i}.$$

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Define a divergent sequence $\{s_\nu\}$ by setting $s_\nu = 1$ if $\nu = k_i$ and $s_\nu = 0$ otherwise. The corresponding t_n tends to zero since for each integer m we have

$$t_n = \sum_{i=0}^{\infty} a_{n,k_i} \leqslant \sum_{i=0}^{m} a_{n,k_i} + \sum_{i=m+1}^{\infty} 2^{-i}.$$

We now introduce some notation and state a definition of generalised Cesàro summability given by Bosanquet and Russell [2].

Let

$$h(\omega, \nu) = (\lambda_{\nu+p+1} - \lambda_\nu)\, D^{p+1}(\omega - \lambda_\nu)^{p+\delta} \quad (\nu \leqslant n,\ \lambda_{n+p+1} \leqslant \omega),$$

where D^{p+1} is a divided difference operator defined inductively by

$$D^0 b_\nu = b_\nu,\ D^{m+1} b_\nu = \frac{D^m b_\nu - D^m b_{\nu+1}}{\lambda_{\nu+m+1} - \lambda_\nu} \quad (m = 0, 1, \ldots).$$

Let

$$C_n^0 = \sum_{\nu=0}^{n} a_\nu, \quad C_n^{m+1} = \sum_{\nu=0}^{n} (\lambda_{n+1} - \lambda_\nu) \ldots (\lambda_{n+m+1} - \lambda_\nu)\, a_\nu \quad (m = 0, 1, \ldots)$$

and let

$$C_n^{p+\delta} = \sum_{\nu=0}^{n} h(\lambda_{n+p+1}, \nu)\, C_\nu^p.$$

The definition of $C_n^{p+\delta}$ is unambiguous in the case $\delta = 1$, since (see [2] or [5])

$$C_n^{p+1} = \sum_{\nu=0}^{n} (\lambda_{\nu+p+1} - \lambda_\nu)\, C_\nu^p. \tag{3}$$

Let

$$t_n^\kappa = \frac{C_n^\kappa}{E_n^\kappa},$$

where E_n^κ is the value of C_n^κ obtained from the series with $a_0 = 1$, $a_\nu = 0$ $(\nu > 0)$, i.e.,

$$E_n^0 = 1, \quad E_n^{m+1} = \lambda_{n+1} \ldots \lambda_{n+m+1} \quad (m = 0, 1, \ldots),$$

$$E_n^{p+\delta} = \sum_{\nu=0}^{n} h(\lambda_{n+p+1}, \nu)\, E_\nu^p.$$

The series $\sum_{\nu=0}^{\infty} a_\nu$ is said to be summable by the generalised Cesàro method $(\mathscr{C}^*, \lambda, \kappa)$ to s if $t_n^\kappa \to s$ as $n \to \infty$.

We require the following results established by Bosanquet and Russell [2].

$$(R, \lambda, \kappa) \sim (\mathscr{C}^*, \lambda, \kappa); \tag{4}$$

$$0 < h(\lambda_{n+p+1}, \nu) \leqslant h(\lambda_{\nu+p+1}, \nu) \leqslant \binom{p+\delta}{p} (\lambda_{\nu+p+1} - \lambda_\nu)^\delta \quad (0 \leqslant \nu \leqslant n) \tag{5}$$

[2; Lemma 2];

$$E_n^{p+\delta} \geqslant \binom{p+\delta}{p+1} E_n^p \lambda_{n+p+1}^\delta \quad \text{[2; Lemma 3].} \tag{6}$$

Proof of the Theorem.

Sufficiency. In view of (1), and the fact that $\lambda_{n+p+1}/\lambda_n$ increases with p, it is enough to show that condition (2) implies $(\mathscr{C}^*, \lambda, p+1) \subseteq (\mathscr{C}^*, \lambda, p)$. By (3), we have that

$$t_n^{p+1} - t_{n-1}^{p+1} \frac{\lambda_n}{\lambda_{n+p+1}} = t_n^p \left(1 - \frac{\lambda_n}{\lambda_{n+p+1}}\right).$$

It follows that condition (2) implies that $t_n^p \to s$ whenever $t_n^{p+1} \to s$, and hence that $(\mathscr{C}^*, \lambda, p+1) \subseteq (\mathscr{C}^*, \lambda, p)$.

Necessity. Define $A = \{a_{n,\nu}\}$ to be the matrix such that

$$t_n^{p+\delta} = \sum_{\nu=0}^{n} a_{n,\nu}\, t_n^p, \tag{7}$$

so that $a_{n,\nu} = h(\lambda_{n+p+1}, \nu)\, E_\nu^p / E_n^{p+\delta}$ for $0 \leqslant \nu \leqslant n$, and $a_{n,\nu} = 0$ for $\nu > n$.

By (5) and (6) we have, for $0 \leqslant \nu \leqslant n$,

$$0 < h(\lambda_{n+p+1}, \nu) \leqslant h(\lambda_{\nu+p+1}, \nu) \leqslant \binom{p+\delta}{p} (\lambda_{\nu+p+1} - \lambda_\nu)^\delta$$

and

$$E_n^{p+\delta} \geqslant E_n^p \lambda_{n+p+1}^\delta \binom{p+\delta}{p+1} \geqslant E_\nu^p \lambda_{\nu+p+1}^\delta \binom{p+\delta}{p+1};$$

and therefore

$$0 \leqslant a_{n,\nu} \leqslant \frac{p+1}{\delta} \left(1 - \frac{\lambda_\nu}{\lambda_{\nu+p+1}}\right)^\delta \quad (0 \leqslant \nu \leqslant n). \tag{8}$$

Suppose now that

$$(R, \lambda, p+\delta) \sim (R, \lambda, p). \tag{9}$$

Then, by (4) and (7), the summability method associated with the matrix A is equivalent to convergence, and hence, by the lemma,

$$\liminf_{\nu \to \infty} \max_{n \geqslant 0} a_{n,\nu} > 0. \tag{10}$$

It follows from (8) and (10) that

$$\liminf_{\nu \to \infty} \left(1 - \frac{\lambda_\nu}{\lambda_{\nu+p+1}}\right)^\delta > 0,$$

and hence that

$$\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1.$$

We have thus shown that (9) implies (2) and, in view of (1), the proof is complete.

Remark. Let p, q be integers with $p > q \geqslant 0$, and let the sequence $\lambda = \{\lambda_n\}$ be such that

$$\liminf_{n \to \infty} \frac{\lambda_{n+p+1}}{\lambda_n} > 1 \quad \text{and} \quad \liminf_{n \to \infty} \frac{\lambda_{n+p}}{\lambda_n} = 1.$$

Then, using the above theorem, we find that $(R, \lambda, \alpha) \sim (R, \lambda, p)$ whenever $\alpha > p$, but $(R, \lambda, \beta) \nsim (R, \lambda, q)$ whenever $\beta > q$. An example of such a sequence is given by

$$\lambda_n = 2^m + r \quad \text{for} \quad n = m(p+1)+r \quad \text{and} \quad 0 \leqslant r \leqslant p.$$

References

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