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Generalized strong summability of infinite series

By David Borwein at London and J.H.Rizvi at Karachi

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In this note, we consider the generalized strong Abel-type summability methods $[A_\lambda]_p$ and $[A'_\lambda]_p$, and establish some equivalence and inclusion relations. We also consider the product of Abel-type methods with regular Hausdorff methods.

1. Introduction

We write throughout:

$$\varepsilon_n^\lambda = \binom{n+\lambda}{n} = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}{n!} \qquad \text{for } n = 1, 2, \ldots,$$

$$\varepsilon_0^\lambda = 1,$$

$$S_n = \sum_{r=0}^{n} u_r,$$

$$S_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda S_n \left(\frac{y}{1+y}\right)^n,$$

$$u_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda u_n \left(\frac{y}{1+y}\right)^n,$$

$$v_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=1}^{\infty} \varepsilon_n^\lambda n u_n \left(\frac{y}{1+y}\right)^n,$$

$$U_\lambda(y) = \lambda \int_0^y u_\lambda(t)\,dt.$$

We also use M for a constant, not necessarily having the same value at each occurrence. The Abel-type methods A_λ and A'_λ, introduced in [2] and [3], are defined as follows:

If

$$(1-x)^{\lambda+1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda S_n x^n$$

is convergent for all x in the open interval (0, 1) and tends to a finite limit l as $x \to 1$ in the open interval (0, 1), we say that the sequence $\{S_n\}$ is A_λ-convergent to l and write $S_n \to l(A_\lambda)$.

It is evident that $S_n \to l(A_\lambda)$ if and only if the series defining $S_\lambda(y)$ is convergent for all $y > 0$ and $S_\lambda(y) \to l$ as $y \to \infty$. For $\lambda = 0$, we have the ordinary Abel summability A.

If the series defining $u_\lambda(y)$ is convergent for all $y > 0$ and $U_\lambda(y)$ tends to a finite limit l as $y \to \infty$, we say that the sequence $\{S_n\}$ is A'_λ-convergent to l and write $S_n \to l(A'_\lambda)$.

It is known that the methods A_λ and $A'_{\lambda+1}$ are regular for all $\lambda > -1$ ([2], Theorem 1; [6], Theorem 34).

We now recall the definition of a regular Hausdorff method H_χ and define the product method $A_\lambda H_\chi$.

Let $\chi(t)$ be a real-valued function of bounded variation in [0, 1] such that

$$\chi(0+) = \chi(0) = 0, \quad \text{and} \quad \chi(1) = 1,$$

and let

$$h_n = \sum_{r=0}^{n} \binom{n}{r} S_r \int_0^1 t^r (1-t)^{n-r} d\chi(t). \tag{1.1}$$

If $h_n \to l$ as $n \to \infty$, we say that the sequence $\{S_n\}$ is H_χ-convergent to l and write $S_n \to l(H_\chi)$.

If $h_n \to l(A_\lambda)$, we say that the sequence $\{S_n\}$ is $A_\lambda H_\chi$-convergent to l and write $S_n \to l(A_\lambda H_\chi)$.

2. Definitions of strong summability

Let p be a positive number. The strong Abel-type summability methods $[A_\lambda]_p$ and $[A'_\lambda]_p$ are defined as follows ([5] and [8]):

Strong Abel-type Summability $[A_\lambda]_p$. If

$$\int_0^y |S_{\lambda+1}(t) - l|^p dt = o(y) \tag{2.1}$$

as $y \to \infty$, we say that the sequence $\{S_n\}$ is strongly A_λ-convergent with index p or $[A_\lambda]_p$-convergent to l and write $S_n \to l[A_\lambda]_p$.

Strong Abel-type Summability $[A'_\lambda]_p$. If

$$\int_0^y |U_{\lambda+1}(t) - l|^p dt = o(y) \tag{2.2}$$

as $y \to \infty$, we say that the sequence $\{S_n\}$ is strongly A'_λ-convergent with index p or $[A'_\lambda]_p$-convergent to l and write $S_n \to l[A'_\lambda]_p$.

Strong Product Summability $[A_\lambda H_\chi]_p$.

If $h_n \to l[A_\lambda]_p$, we say that the sequence $\{S_n\}$ is $[A_\lambda H_\chi]_p$-convergent to l and write $S_n \to l[A_\lambda H_\chi]_p$.

3. Main results

We prove the following theorems:

Theorem 1. *If $0 < q < p$, and $S_n \to l[A'_\lambda]_p$, then $S_n \to l[A'_\lambda]_q$.*

Theorem 2. *If $\lambda > 0$, $p > 1$, and $S_n \to l[A'_\lambda]_p$, then $S_n \to l(A'_\lambda)$.*

Theorem 3. *If $\lambda > 0$, and $S_n \to l(A'_\lambda)$, then $S_n \to l[A'_{\lambda-1}]_p$ for every $p > 0$.*

The next theorem gives necessary and sufficient conditions for the $[A'_\lambda]_p$-convergence of the sequence $\{S_n\}$.

Theorem 4. *For $\lambda > 0$, $p > 1$, necessary and sufficient conditions for the $[A'_\lambda]_p$-convergence of the sequence $\{S_n\}$ to l are:*

(3. 1) $$S_n \to l(A'_\lambda)$$

and

(3. 2) $$\int_0^y \left| t \frac{d}{dt} U_\lambda(t) \right|^p dt = o(y) \qquad \text{as } y \to \infty.$$

The following two theorems give relationships between the $[A_\lambda]_p$ and $[A'_\lambda]_p$ methods.

Theorem 5. *If $\lambda > 0$, $p > 1$, then $S_n \to l[A_\lambda]_p$ if and only if $S_n \to l[A'_\lambda]_p$ and $n u_n \to 0[A_{\lambda-1}]_p$.*

Theorem 6. *If $\lambda > 0$, $p > 1$, then $S_n \to l[A'_\lambda]_p$ if and only if $S_n \to l[A_{\lambda-1}]_p$.*

Finally, we have the following theorem about the product method $[A_\lambda H_\chi]_p$.

Theorem 7. *If $\lambda > -1$, $p > 1$, H_χ is a regular Hausdorff method, and $S_n \to l[A_\lambda]_p$, then $S_n \to l[A_\lambda H_\chi]_p$.*

The corresponding results for ordinary summability are established in [2] and [3], for absolute summability in [4] and for strong summability (i. e. the case $p = 1$) in [5].

4. Preliminary results

We require the following results.

Lemma 1. *If $\lambda > \mu > -1$, $y > 0$ and $\sum_{n=0}^{\infty} \varepsilon_n^\lambda S_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

(4. 1) $$S_\mu(y) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu)\Gamma(\mu+1)} y^{-\lambda} \int_0^y (y-t)^{\lambda-\mu-1} t^\mu S_\lambda(t)\, dt.$$

This lemma is proved in [2] (Lemma 2 (i)).

Lemma 2. *If $\lambda > -1$, $y > 0$, and $\sum_{n=0}^{\infty} \varepsilon_n^\lambda S_n \left(\frac{t}{1+t}\right)^n$ is convergent for all $t > 0$, then*

(4. 2) $$u_\lambda(y) = (1+y)^{-1} S_\lambda(y) - \lambda(1+y)^{-\lambda-1} \int_0^y (1+t)^{\lambda-1} S_\lambda(t)\, dt,$$

(4. 3) $$U_\lambda(y) = \lambda(1+y)^{-\lambda} \int_0^y (1+t)^{\lambda-1} S_\lambda(t)\, dt,$$

(4. 4) $$S_\lambda(y) = U_\lambda(y) + (1+y) u_\lambda(y),$$

(4. 5) $$S_\lambda(y) = U_{\lambda+1}(y) + u_\lambda(y),$$

(4. 6) $$y u_\lambda(y) = U_{\lambda+1}(y) - U_\lambda(y),$$

(4. 7) $$y \frac{d}{dy} U_{\lambda+1}(y) = \frac{1}{\lambda+1} [U_{\lambda+2}(y) - U_{\lambda+1}(y)],$$

(4. 8) $$y \frac{d}{dy} S_\lambda(y) = (\lambda+1)[S_{\lambda+1}(y) - S_\lambda(y)] = v_\lambda(y),$$

(4. 9) $$U_\lambda(y) = \lambda y^{-\lambda} \int_0^y t^{\lambda-1} U_{\lambda+1}(t)\, dt.$$

Some of these relations are established in [3]. For complete proofs, see [9].

Lemma 3. *If $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda S_n x^n$ is convergent for $0 \leqq x < 1$ and h_n is defined by (1. 1), then*

(4. 10) $$h_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda h_n \left(\frac{y}{1+y}\right)^n = \int_0^1 S_\lambda(yt)\, d\chi(t).$$

This lemma is proved in [2] (Lemma 5). See also [1], p. 376.

Lemma 4. *For $\lambda > -1$, $p > 1$, necessary and sufficient conditions for the $[A_\lambda]_p$-convergence of the sequence $\{S_n\}$ to l are that*

(i) $$S_n \to l(A_\lambda),$$

and

(ii) $$\int_0^y \left| t \frac{d}{dt} S_\lambda(t) \right|^p dt = o(y) \qquad \text{as } y \to \infty.$$

This lemma is proved by Mishra in [8] (Theorem 4).

5. Proofs of the main results

Theorem 1. We have, by assumption, that (2. 2) holds. Using Hölder's inequality with indices $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$\int_0^y |U_{\lambda+1}(t) - l|^q dt \leqq \left[\int_0^y |U_{\lambda+1}(t) - l|^p dt\right]^{\frac{q}{p}} \left[\int_0^y dt\right]^{1-\frac{q}{p}}$$

$$= o(y^{\frac{q}{p}})\, O(y^{1-\frac{q}{p}}) = o(y) \qquad \text{as } y \to \infty.$$

Theorem 2. In view of Theorem 1, we may assume that $S_n \to l[A'_\lambda]_1$. Now, by (4. 9), we get that

$$|U_\lambda(y) - l| \leqq \lambda y^{-\lambda} \int_0^y t^{\lambda-1} |U_{\lambda+1}(t) - l|\, dt$$

$$= \lambda y^{-\lambda} \left[t^{\lambda-1} \cdot o(t) \Big|_0^y - (\lambda-1) \int_0^y t^{\lambda-2} \cdot o(t)\, dt \right] = o(1) \qquad \text{as } y \to \infty.$$

Theorem 3. The result is a consequence of the regularity of the $(C, 1)$-method.

Theorem 4. *Necessity*: We need only establish (3. 2). By (4. 7), we have that

$$\int_0^y \left| t \frac{d}{dt} U_\lambda(t) \right|^p dt \leqq M \left[\int_0^y |U_{\lambda+1}(t) - l|^p dt + \int_0^y |U_\lambda(t) - l|^p dt \right]$$

$$= o(y) \qquad \text{as } y \to \infty$$

in view of Theorem 3.

Sufficiency: Again, by (4. 7), it follows that

$$\int_0^y |U_{\lambda+1}(t) - l|^p dt \leqq M \left[\int_0^y \left| t \frac{d}{dt} U_\lambda(t) \right|^p dt + \int_0^y |U_\lambda(t) - l|^p dt \right] = o(y) \qquad \text{as } y \to \infty,$$

by Theorem 3 and (3. 2).

Theorem 5. (i) Suppose that $S_n \to l[A_\lambda]_p$, i. e., $\int_0^y |S_{\lambda+1}(t) - l|^p dt = o(y)$ as $y \to \infty$. In view of (4. 3) we have that

$$\int_0^y |U_{\lambda+1}(t) - l|^p dt \leqq M\left[\int_0^y (1+t)^{-p(\lambda+1)} dt \left|\int_0^t (1+z)^\lambda (S_{\lambda+1}(z) - l) dz\right|^p + \int_0^y (1+t)^{-p(\lambda+1)} dt\right]$$

$$= M\int_0^y J_1(t)\,dt + M\int_0^y J_2(t)\,dt.$$

Now $\int_0^y J_2(t) = o(y)$ as $y \to \infty$, since $-p(\lambda+1)+1 < 0$.

Further, using Hölder's inequality with indices p and $\frac{p}{p-1}$, it follows that

$$J_1(t) \leqq (1+t)^{-p(\lambda+1)} \int_0^t |S_{\lambda+1}(z) - l|^p dz \left|\int_0^t (1+z)^{\frac{\lambda p}{p-1}} dz\right|^{p-1}$$

$$\leqq M\,[(1+t)^{-1} + (1+t)^{-p(\lambda+1)}] \int_0^t |S_{\lambda+1}(z) - l|^p dz = o(1) \text{ as } t \to \infty.$$

Hence,

$$\frac{1}{y}\int_0^y J_1(t)\,dt = o(1) \text{ as } y \to \infty.$$

Consequently $\int_0^y |U_{\lambda+1}(t) - l|^p dt = o(y)$ as $y \to \infty$. Thus $S_n \to l[A'_\lambda]_p$. Taking note of Lemma 4 and (4. 8) we get that $nu_n \to 0[A_{\lambda-1}]_p$.

(ii) Suppose that $S_n \to l[A'_\lambda]_p$ and $nu_n \to 0\,[A_{\lambda-1}]_p$. We show first that $S_n \to l[A_{\lambda-1}]_p$. Now, by (4. 4) and (4. 5), we get that

$$S_\lambda(t) - l = \left(1 + \frac{1}{t}\right)(U_{\lambda+1}(t) - l) - \frac{1}{t}\,(U_\lambda(t) - l).$$

Thus, for $y > 1$, we have that

$$\int_1^y |S_\lambda(t) - l|^p dt \leqq M\left[\int_1^y |U_{\lambda+1}(t) - l|^p dt + \int_1^y |U_\lambda(t) - l|^p dt\right] = o(y) \qquad \text{as } y \to \infty.$$

It follows that $S_n \to l[A_{\lambda-1}]_p$. To complete the proof, we note that by (4. 8),

$$\int_0^y |S_{\lambda+1}(t) - l|^p dt \leqq M\left[\int_0^y |v_\lambda(t)|^p dt + \int_0^y |S_\lambda(t) - l|^p dt\right] = o(y) \qquad \text{as } y \to \infty,$$

i. e. $S_n \to l[A_\lambda]_p$.

Theorem 6. We have already proved that $S_n \to l[A_{\lambda-1}]_p$ whenever $S_n \to l[A'_\lambda]_p$. Now, suppose that $S_n \to l[A_{\lambda-1}]_p$, i. e. $\int_0^y |S_\lambda(t) - l|^p dt = o(y)$ as $y \to \infty$. In view of (4. 2) and (4. 5) we have that

$$U_{\lambda+1}(t) - l = S_\lambda(t) - l - (1+t)^{-1}[S_\lambda(t) - l]$$
$$+ \lambda(1+t)^{-\lambda-1}\int_0^t (1+z)^{\lambda-1}[S_\lambda(z) - l]\,dz - l(1+t)^{-\lambda-1},$$

so that

$$|U_{\lambda+1}(t) - l|^p \leqq M\left[|S_\lambda(t) - l|^p + \lambda(1+t)^{-p(\lambda+1)}\left|\int_0^t (1+z)^{\lambda-1}[S_\lambda(z) - l]\,dz\right|^p + (1+t)^{-p(\lambda+1)}\right]$$

$$= I_1(t) + I_2(t) + I_3(t).$$

By assumption $\int_0^y I_1(t)\,dt = o(y)$ as $y \to \infty$; and since $-p(\lambda+1)+1 < 0$, we also have that $\int_0^y I_3(t)\,dt = o(y)$ as $y \to \infty$. Thus we are left with $I_2(t)$. By Hölder's inequality, it follows that

$$I_2(t) \leqq M(1+t)^{-p(\lambda+1)}\int_0^t |S_\lambda(z) - l|^p dz\,[(1+t)^{p\lambda-1} + 1]$$

$$\leqq M[(1+t)^{-1} + (1+t)^{-p(\lambda+1)}]\int_0^t |S_\lambda(z) - l|^p dz = o(1) \text{ as } t \to \infty,$$

and so $\frac{1}{y}\int_0^y I_2(t)\,dt = o(1)$ as $y \to \infty$. This completes the proof of the theorem.

Theorem 7. Suppose that $S_n \to l[A_\lambda]_p$. It follows from the regularity of H_χ and (4. 10) that $h_{\lambda+1}(z) - l = \int_0^1 (S_{\lambda+1}(zt) - l)\,d\chi(t)$. Using Hölder's inequality for Stieltjes integrals ([7], Theorem 210), we get

$$|h_{\lambda+1}(z) - l|^p \leqq M\int_0^1 |S_{\lambda+1}(zt) - l|^p\,|d\chi(t)|.$$

Hence

$$\frac{1}{y}\int_0^y |h_{\lambda+1}(z) - l|^p dz \leqq M\int_0^1 f(yt)\,|d\chi(t)|,$$

where

$$f(t) = \frac{1}{t}\int_0^t |S_{\lambda+1}(x) - l|^p dx = o(1) \qquad \text{as } t \to \infty.$$

The theorem follows now from a standard argument (cf. [6], proof of Theorem 217), since $\chi(t)$ is continuous at 0.

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