INTEGRATION BY PARTS OF CESÀRO SUMMABLE INTEGRALS

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PART 1.

1. We shall suppose throughout that κ is a non-negative integer and shall denote by Ω_r^{κ} $(r=1$ or $2)$ the class of functions $\phi(x)$ which have absolutely continuous‡ κ-th derivatives, and which, for all $x \geqslant 1$ and some pair of constants m, M, satisfy conditions (r) below:

$$\left.\begin{array}{ll}\text{(i)} & 0<|\phi(x)|<m, \\ \text{(ii)} & \displaystyle\int_x^{\infty} t^{\kappa}\,|\phi^{(\kappa+1)}(t)|\,dt<M\,|\phi(x)|.\end{array}\right\} \qquad (1)$$

$$\left.\begin{array}{ll}\text{(i)} & |\phi(x)|>m>0, \\ \text{(ii)} & \displaystyle\int_1^{x} |\phi'(t)|\,dt<M\,|\phi(x)|, \\ \text{(iii)} & \displaystyle\int_1^{x} t^{\kappa}\,|\phi^{(\kappa+1)}(t)|\,dt<M\,|\phi(x)|.\end{array}\right\} \qquad (2)$$

We shall write, for $x \geqslant 1$,

$$\psi(x)=1/\phi(x).$$

We shall suppose that $f(x)$ denotes a function which is Lebesgue integrable in every finite interval in $(1, \infty)$, and shall use the notation:

$$f_0(x)=f(x), \quad f_{\mu}(x)=\frac{1}{\Gamma(\mu)}\int_1^x (x-t)^{\mu-1} f(t)\,dt \quad (x \geqslant 1,\ \mu>0).$$

2. The object of Part 1 of this paper is to prove the following theorem§.

THEOREM 1. *Let $\kappa \geqslant \lambda \geqslant 1$ and let*

$$\int_1^{\infty} f(t)\,\phi(t)\,dt \qquad (3)$$

be summable (C, λ) to sum A.

† Part 1 received 20 August, 1952; read 20 November, 1952; revised with Part 2 added 28 September, 1953.

‡ Where no interval of absolute continuity is specified it is to be understood that the property pertains to every finite interval in $[1, \infty)$.

§ For results of greater generality involving the particular function $\phi(x)=x^{\rho}$, see Borwein (1).

(i) *If*† $\phi \in \Omega_1^{\kappa}-\Omega_2^{\kappa}$, *then, for every value of s,*

$$\int_1^{\infty} \{f_1(t)-s\}\phi'(t)\,dt \qquad (4)$$

is summable $(C, \lambda-1)$ to sum $s\phi(1)-A$.

(ii) *If $\phi \in \Omega_2^{\kappa}-\Omega_1^{\kappa}$, the above conclusion is true only when $s=s_0$, where s_0 is the sum (C, λ) of $\int_1^{\infty} f(t)\,dt$; for other values of s, (4) is strictly divergent.*

(iii) *If $\phi \in \Omega_1^{\kappa} \cap \Omega_2^{\kappa}$, then for every value of s, (4) is summable $(C, \lambda-1)$ to sum $(s_0-s)\,\phi(\infty)+s\phi(1)-A$; s_0 being as in* (ii)‡.

In Part 2 we shall investigate the necessity of the conditions on ϕ; more explicitly, we shall show that if $\phi(x)$ is a real function which satisfies either (1)(i) or (2)(i) and has an absolutely continuous κ-th derivative, and if, for every λ in the range $1, 2, \ldots, \kappa$, the summability (C, λ) of (3) implies the summability $(C, \lambda-1)$, for some value of s, of (4), then $\phi \in \Omega_1^{\kappa} \cup \Omega_2^{\kappa}$.

For $r=1$ or 2 let $\Omega_r^{*\kappa}$ be a class of functions defined as follows: $\phi^* \in \Omega_r^{*\kappa}$ if and only if $\phi^*(x)$ is absolutely continuous and there is a ϕ in Ω_r^{κ} and a constant $c \geqslant 1$ such that $\phi^*(x)=\phi(x)$ for all $x \geqslant c$.

We can replace Ω_r^{κ} by $\Omega_r^{*\kappa}$ in Theorem 1 to obtain a slight and easily verified generalization.

Hardy and Littlewood have proved§ that Theorem 1 (ii) holds for integral λ when $\phi(x)$ is an absolutely continuous L-function‖ satisfying, for some $\Delta>0$, the relation $1 \prec \phi \prec x^{\Delta}$ [*e.g.* $\phi(x)=x^{\alpha}\{\log(1+x)\}^{\beta}$ where, either $\alpha>0$ and β is real, or $\alpha=0$, $\beta>0$]. It can readily be shown¶ that such a ϕ is in $\Omega_2^{*\kappa}-\Omega_1^{*\kappa}$. Parts (i) and (iii) of Theorem 1 generalize the Hardy-Littlewood result in a different way, for it may be shown†† that if $\phi(x)$ is an absolutely continuous L-function then (i) $\phi \in \Omega_1^{*\kappa}-\Omega_2^{*\kappa}$ when $x^{-\Delta} \prec \phi \prec 1$ for some $\Delta>0$ [*e.g.* $\phi(x)=x^{-\alpha}\log(1+x)^{\beta}$ where, either $\alpha>0$ and β is real, or $\alpha=0$, $\beta<0$], (ii) $\phi \in \Omega_1^{*\kappa} \cap \Omega_2^{*\kappa}$ when $\phi \asymp 1$ [*e.g.* $\phi(x)=\log(2-1/x)$].

It is interesting to note in passing that when $\phi(x)$ is a positive unboundedly increasing function for $x>0$ with an absolutely continuous

† It will appear later (Lemma 4) that "$\phi \in \Omega_1^{\kappa}-\Omega_2^{\kappa}$" and "$\phi \in \Omega_2^{\kappa}-\Omega_1^{\kappa}$" are respectively equivalent to "$\phi \in \Omega_1^{\kappa}$ and $\lim_{x\to\infty} \phi(x)=0$" and "$\phi \in \Omega_2^{\kappa}$ and $\lim_{x\to\infty} |\phi(x)|=\infty$".

‡ $s_0=\lim_{x\to\infty} \Gamma(\lambda+1)\,x^{-\lambda} f_{\lambda+1}(x)$ and $\phi(\infty)=\lim_{x\to\infty} \phi(x)$; these limits will be shown to exist under the appropriate hypotheses.

§ Hardy and Littlewood (7).

‖ For properties of L-functions (or logarithmico exponential functions) see Hardy (5).

¶ See for instance Hirst (9), Theorem 3.

†† These properties follow from the result concerning the range $1 \prec \phi \prec x^{\Delta}$ and Lemma 5.

κ-th derivative, then a necessary and sufficient condition for every series summable (R, λ_n, κ) to also be summable $\left(R, \phi(\lambda_n), \kappa\right)$ is that $\phi \in \Omega_2^{\kappa}$†.

3. We require the following lemmas.

LEMMA 1. *If $\kappa \geqslant 1$ and $\phi \in \Omega_1^{\kappa}$, then, for $n = 1, 2, \ldots, \kappa$ and $x \geqslant 1$,*

$$\text{(i)} \int_x^{\infty} t^{n-1} |\phi^{(n)}(t)|\, dt = O\{|\phi(x)|\},$$

$$\text{(ii)}\ x^n \phi^{(n)}(x) = O\{|\phi(x)|\}.$$

The hypothesis ensures the convergence of $\int_x^{\infty} \phi^{(\kappa+1)}(t)\, dt$, and so there is a finite constant C such that, for $x \geqslant 1$,

$$\phi^{(\kappa)}(x) - C = -\int_x^{\infty} \phi^{(\kappa+1)}(t)\, dt.$$

Consequently

$$C = \lim_{x\to\infty} \phi^{(\kappa)}(x) = \lim_{x\to\infty} \kappa x^{-\kappa} \int_1^x (x-t)^{\kappa-1} \phi^{(\kappa)}(t)\, dt$$
$$= \lim_{x\to\infty} \{\kappa!\, x^{-\kappa} \phi(x) + O(x^{-1})\}.$$

Since $\phi(x)$ is bounded in $(1, \infty)$ and $\kappa > 0$, it follows that $C = 0$, and hence that

$$\phi^{(\kappa)}(x) = -\int_x^{\infty} \phi^{(\kappa+1)}(t)\, dt.$$

Therefore, for $x \geqslant 1$,

$$\int_x^{\infty} t^{\kappa-1} |\phi^{(\kappa)}(t)|\, dt \leqslant \int_x^{\infty} t^{\kappa-1}\, dt \int_t^{\infty} |\phi^{(\kappa+1)}(u)|\, du$$
$$\leqslant \int_x^{\infty} |\phi^{(\kappa+1)}(u)|\, du \int_0^u t^{\kappa-1}\, dt$$
$$= \frac{1}{\kappa} \int_x^{\infty} u^{\kappa} |\phi^{(\kappa+1)}(u)|\, du < \frac{M}{\kappa} |\phi(x)|,$$

and

$$x^{\kappa} |\phi^{(\kappa)}(x)| \leqslant x^{\kappa} \int_x^{\infty} |\phi^{(\kappa+1)}(t)|\, dt \leqslant \int_x^{\infty} t^{\kappa} |\phi^{(\kappa+1)}(t)|\, dt < M|\phi(x)|.$$

Conclusions (i) and (ii) are thus true in the case $n = \kappa$. Repetition yields the full result.

LEMMA 2. *If $\kappa \geqslant 1$ and $\phi \in \Omega_2^{\kappa}$, then, for $n = 1, 2, \ldots, \kappa$ and $x \geqslant 1$,*

$$\text{(i)} \int_1^x t^{n-1} |\phi^{(n)}(t)|\, dt = O\{|\phi(x)|\},$$

$$\text{(ii)}\ x^n \phi^{(n)}(x) = O\{|\phi(x)|\}.$$

† Kuttner (**10**), and Hirst (**9**).

Conclusion (i) is due to Kuttner†. Conclusion (ii) follows from (i) since, for $n = 1, 2, \ldots, \kappa$ and $x \geqslant 1$,

$$x^n \phi^{(n)}(x) = \int_1^x t^n \phi^{(n+1)}(t)\, dt + n \int_1^x t^{n-1} \phi^{(n)}(t)\, dt + \phi^{(n)}(1).$$

LEMMA 3. *If $\phi \in \Omega_1^{\kappa} \cup \Omega_2^{\kappa}$, then a necessary and sufficient condition that $\phi \in \Omega_1^{\kappa} \cap \Omega_2^{\kappa}$ is that there should be constants m_0, m such that*

$$0 < m_0 < |\phi(x)| < m$$

for all $x \geqslant 1$.

This follows easily from the definitions of the classes concerned.

LEMMA 4. (i) *If $\phi \in \Omega_1^{\kappa} - \Omega_2^{\kappa}$ then $\phi(x) \to 0$ as $x \to \infty$.*

(ii) *If $\phi \in \Omega_2^{\kappa} - \Omega_1^{\kappa}$ then $|\phi(x)| \to \infty$ as $x \to \infty$.*

(iii) *If $\phi \in \Omega_1^{\kappa} \cap \Omega_2^{\kappa}$ then $\phi(x)$ tends to a finite, non-zero limit as $x \to \infty$.*

Suppose that $\phi \in \Omega_1^{\kappa}$. By Lemma 1 (i), $\int_1^{\infty} |\phi'(t)|\, dt < \infty$ and so $\phi(x)$ tends to a finite limit as $x \to \infty$. In view of Lemma 3, results (i) and (iii) follow.

Suppose now that $\phi \in \Omega_2^{\kappa} - \Omega_1^{\kappa}$. In view of (2)(ii), we have, for all $x \geqslant 1$ and some positive constant M,

$$|\phi(x)| \leqslant |\phi(1)| + \int_1^x |\phi'(t)|\, dt < |\phi(1)| + M|\phi(x)|.$$

Since $\int_1^x |\phi'(t)|\, dt$ increases and, by (2)(i) and Lemma 3, $\phi(x)$ is unbounded in $(1, \infty)$, it follows that $|\phi(x)| \to \infty$ as $x \to \infty$.

LEMMA 5. (i) *If $\phi \in \Omega_1^{\kappa}$ then $\psi \in \Omega_2^{\kappa}$.*

(ii) *If $\phi \in \Omega_2^{\kappa}$ then $\psi \in \Omega_1^{\kappa}$.*

Let p be any integer such that $1 \leqslant p \leqslant \kappa+1$. By a particular case of a theorem of Faa di Bruno on the differentiation of a function of a function‡

$$\psi^{(p)}(t) = \sum_{r=1}^{q} \frac{a_r}{\phi(t)} \prod_{s=1}^{p} \left\{\frac{\phi^{(s)}(t)}{\phi(t)}\right\}^{b_{r,s}},$$

where the a's are constants and the b's are non-negative integers such that

$$\sum_{s=1}^{p} s b_{r,s} = p \quad (r = 1, 2, \ldots, q).$$

† Kuttner (**10**), Lemma 2.
‡ *Cf.* Hirst (**9**), Lemma A.

Further, if $p \geqslant 2$, we have, under either hypothesis by Lemmas 1 (ii) and 2 (ii), that

$$\frac{\phi^{(s)}(t)}{\phi(t)} = O(t^{-s}) \quad (s = 1, 2, \ldots, p-1;\ t \geqslant 1).$$

Now let s_r be the largest value of s for which $b_{r,s} \neq 0$. Since $b_{r,p} \leqslant 1$, we can immediately deduce that, for $t \geqslant 1$,

$$t^{p-1}\psi^{(p)}(t) = O\left\{\sum_{r=1}^{q} t^{s_r-1}\frac{|\phi^{(s_r)}(t)|}{|\phi(t)|^2}\right\}. \tag{5}$$

(i) *Suppose now that* $\phi \in \Omega_1^{\kappa} - \Omega_2^{\kappa}$, and write, for $t \geqslant 1$,

$$\chi(t) = \int_t^{\infty} |\phi'(u)|\, du.$$

Then, by Lemmas 1 (i) and 4 (i),

$$0 < |\phi(t)| \leqslant \chi(t) \leqslant G|\phi(t)|$$

for all $t \geqslant 1$ and some constant G. Since, in addition, $\chi(t)$ is continuous and non-increasing for $t \geqslant 1$, we can proceed as follows.

Let $x > 1$ and let $N = N(x)$ be the positive integer for which

$$2^{N-1}\chi(x) \leqslant \chi(1) < 2^N\chi(x).$$

Put $x_0 = x$, $x_N = 1$ and, when $N \geqslant 2$, choose $x_1, x_2, \ldots, x_{N-1}$ such that, for $n = 1, 2, \ldots, N-1$,

$$x_{n-1} > x_n \geqslant 1$$

and

$$\chi(x_n) = 2^n\chi(x).$$

Then, for $r = 1, 2, \ldots, q$,

$$\begin{aligned}\int_1^x t^{s_r-1}\frac{|\phi^{(s_r)}(t)|}{|\phi(t)|^2}\,dt &\leqslant G^2\sum_{n=0}^{N-1}\int_{x_{n+1}}^{x_n} t^{s_r-1}\frac{|\phi^{(s_r)}(t)|}{\{\chi(t)\}^2}\,dt \\ &\leqslant G^2\sum_{n=0}^{N-1}\frac{1}{\{\chi(x_n)\}^2}\int_{x_{n+1}}^{\infty} t^{s_r-1}|\phi^{(s_r)}(t)|\,dt \\ &\leqslant H\sum_{n=0}^{N-1}\frac{\chi(x_{n+1})}{\{\chi(x_n)\}^2} \leqslant H\sum_{n=0}^{N-1}\frac{2^{1-n}}{\chi(x)} \\ &\leqslant 4H|\psi(x)|,\end{aligned}$$

where, in virtue of Lemma 1 (i), H is a constant independent of x and r. Hence, by (5),

$$\int_1^x t^{p-1}|\psi^{(p)}(t)|\,dt = O\{|\psi(x)|\}$$

for $x \geqslant 1$, and so $\psi \in \Omega_2^{\kappa}$.

(ii) *Suppose that* $\phi \in \Omega_2^{\kappa} - \Omega_1^{\kappa}$. Note that, in view of Lemma 3, $\phi(t)$ is unbounded in $(1, \infty)$, and write, for $t \geqslant 1$,

$$\chi(t) = |\phi(1)| + \int_1^t |\phi'(u)|\,du.$$

Then, by (2) (i) and (2) (ii), there is a constant G such that, for $t \geqslant 1$,

$$0 < |\phi(t)| \leqslant \chi(t) \leqslant G|\phi(t)|,$$

and so $\chi(t)$ is positive, continuous, non-decreasing and, by Lemma 4 (ii), unbounded for $t \geqslant 1$.

Let $x > 1$ and let $\{x_n\}$ be an increasing and unbounded sequence such that $x_0 = x$ and, for $n = 1, 2, \ldots$,

$$\chi(x_n) = 2^n\chi(x).$$

Then, for $r = 1, 2, \ldots, q$,

$$\begin{aligned}\int_x^{\infty} t^{s_r-1}\frac{|\phi^{(s_r)}(t)|}{|\phi(t)|^2}\,dt &\leqslant G^2\sum_{n=0}^{\infty}\int_{x_n}^{x_{n+1}} t^{s_r-1}\frac{|\phi^{(s_r)}(t)|}{\{\chi(t)\}^2}\,dt \\ &\leqslant G^2\sum_{n=0}^{\infty}\frac{1}{\{\chi(x_n)\}^2}\int_1^{x_{n+1}} t^{s_r-1}|\phi^{(s_r)}(t)|\,dt \\ &\leqslant H\sum_{n=0}^{\infty}\frac{\chi(x_{n+1})}{\{\chi(x_n)\}^2} = H\sum_{n=0}^{\infty}\frac{2^{1-n}}{\chi(x)} \\ &\leqslant 4H|\psi(x)|,\end{aligned}$$

where, in view of Lemma 2 (i), H is a constant independent of x and r. Hence, by (5), $\psi \in \Omega_1^{\kappa}$.

(iii) *Suppose that* $\phi \in \Omega_1^{\kappa} \cap \Omega_2^{\kappa}$. Then there are constants m_0, m such that, for $t \geqslant 1$, $0 < m_0 < |\phi(t)| < m$. Hence, in view of (5) and Lemma 2 (i), both

$$\int_1^x t^{p-1}|\psi^{(p)}(t)|\,dt \quad \text{and} \quad \int_x^{\infty} t^{p-1}|\psi^{(p)}(t)|\,dt$$

are

$$O\left\{\sum_{r=1}^{q}\int_1^{\infty} t^{s_r-1}|\phi^{(s_r)}(t)|\,dt\right\} = O(1) = O\{|\psi(x)|\} \text{ in } (1, \infty).$$

It follows that $\psi \in \Omega_1^{\kappa} \cap \Omega_2^{\kappa}$ and the proof of the lemma is thus completed.

LEMMA 6. *If* $\kappa \geqslant 1$ *and* $\phi \in \Omega_1^{\kappa} \cup \Omega_2^{\kappa}$, *then, for* $x \geqslant 1$, $n = 1, 2, \ldots, \kappa$ *and* $r = 0, 1, \ldots, \kappa - n$,

$$(d/dx)^r\{\psi^{(n)}(x)\,\phi(x)\} = O(x^{-r-n}).$$

By Lemma 5, $\psi \in \Omega_1^{\kappa} \cup \Omega_2^{\kappa}$ and so, in view of Lemmas 1 (ii) and 2 (ii), we have, for $x \geqslant 1$, $s = 0, 1, \ldots, \kappa$,

$$\psi^{(s)}(x)\,\phi(x) = O(x^{-s}) \quad \text{and} \quad \phi^{(s)}(x)\,\psi(x) = O(x^{-s}).$$

The result follows since

$$(d/dx)^r\{\psi^{(n)}(x)\,\phi(x)\} = \sum_{s=0}^{r} \binom{r}{s} \psi^{(n+r-s)}(x)\,\phi^{(s)}(x).$$

LEMMA 7. *Let $\mu > 0$, $p > -\mu-1$, $p+q > -1$, and let $g^{(n)}(t) = O(t^{q-n})$ in $(1, \infty)$ for $n = 0, 1, \ldots, m$, where $m \geqslant \mu$.*

(i) *If $f(t) = o(t^p)$ (C, μ) as $t \to \infty$†, then*

$$f(t)\,g(t) = o(t^{p+q})\ (C, \mu) \text{ as } t \to \infty.$$

(ii) *If $f(t) = O(t^p)$ (C, μ) in $(1, \infty)$, then*

$$f(t)\,g(t) = O(t^{p+q})\ (C, \mu) \text{ in } (1, \infty).$$

Part (ii) is due to Bosanquet‡; his proof is easily adapted to (i).

4. *Proof of Theorem* 1. Suppose that§ $\kappa \geqslant \lambda > \kappa-1$. Write, for $t \geqslant 1$,

$$v(t) = \int_1^t f(u)\,\phi(u)\,du - A,$$

and note that the first hypothesis of the theorem is equivalent to

$$v(t) = o(1)\ (C, \lambda) \text{ as } t \to \infty. \tag{6}$$

We suppose to start with that $\phi \in \Omega_1^{\kappa} \cup \Omega_2^{\kappa}$, and that s is an arbitrary constant. Then, for $x \geqslant 1$,

$$\int_1^x \{f_1(t)-s\}\phi'(t)\,dt + s\{\phi(x)-\phi(1)\}$$
$$= \int_1^x f(t)\{\phi(x)-\phi(t)\}\,dt$$
$$= A\{\phi(x)\,\psi(1)-1\} - \phi(x)\int_1^x v(t)\,\psi'(t)\,dt, \tag{7}$$

and so

$$\int_1^x \{f_1(t)-s\}\,\phi'(t)\,dt + A - s\phi(1)$$
$$= -\phi(x)\int_1^x v(t)\,\psi'(t)\,dt + \phi(x)\{A\psi(1)-s\}$$
$$= \sum_{n=1}^{\kappa} (-1)^n v_n(x)\,\psi^{(n)}(x)\,\phi(x)$$
$$+ (-1)^{\kappa+1}\phi(x)\int_1^x v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt + \phi(x)\{A\psi(1)-s\}. \tag{8}$$

† *I.e.* $f_\mu(t) = o(t^{p+\mu})$ as $t \to \infty$.

‡ Bosanquet (3), Lemma 1. See also Bosanquet (4), Theorem 2.

§ Because of Lemmas 1(i) and 2(i) this leads to no loss in generality.

It follows from (6) that, as $x \to \infty$,

$$v_n(x) = o(x^n)\ (C, \lambda-n) \quad (n = 0, 1, \ldots, \kappa-1), \tag{9}$$
$$v_\kappa(x) = o(x^\kappa); \tag{10}$$

whence, in view of Lemmas 6 and 7(i),

$$\sum_{n=1}^{\kappa} (-1)^n v_n(x)\,\psi^{(n)}(x)\,\phi(x) = o(1)\ (C, \lambda-1) \text{ as } x \to \infty. \tag{11}$$

We now consider two cases.

Case 1. *Suppose that* $\phi \in \Omega_1^{\kappa} - \Omega_2^{\kappa}$. Then, by Lemma 4(i),

$$\phi(x) \to 0 \text{ as } x \to \infty,$$

and, in view of Lemma 5(i),

$$\phi(x)\int_1^x t^\kappa\,|\psi^{(\kappa+1)}(t)|\,dt = O(1) \text{ in } (1, \infty);$$

and it is well known and easily demonstrated that these relations together with (10) ensure that

$$(-1)^{\kappa+1}\phi(x)\int_1^x v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt + \phi(x)\{A\psi(1)-s\} = o(1) \text{ as } x \to \infty. \tag{12}$$

Part (i) of the theorem now follows from (8), (11) and (12).

Case 2. *Suppose that* $\phi \in \Omega_2^{\kappa}$. Then, by Lemma 5(ii),

$$\phi(x)\int_x^\infty t^\kappa\,|\psi^{(\kappa+1)}(t)|\,dt = O(1) \text{ in } (1, \infty). \tag{13}$$

Put

$$s_0 = A\psi(1) + (-1)^{\kappa+1}\int_1^\infty v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt, \tag{14}$$

the integral being convergent in view of (10) and (13).

We deduce from (10), (13) and (14) that

$$\phi(x)\int_1^x v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt + (-1)^{\kappa+1}\phi(x)\{A\psi(1)-s_0\} = -\phi(x)\int_x^\infty v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt$$
$$= o(1) \text{ as } x \to \infty. \tag{15}$$

It follows now from (8), (11) and (15) that

$$\int_1^\infty \{f_1(t)-s_0\}\,\phi'(t)\,dt = s_0\,\phi(1) - A \quad (C, \lambda-1). \tag{16}$$

We show next that s_0 is the sum (C, λ) of $\int_1^\infty f(t)\,dt$. We have, for $x \geqslant 1$,

$$\int_1^x f(t)\,dt = \int_1^x f(t)\,\phi(t)\,\psi(t)\,dt$$
$$= A\psi(1) + v(x)\,\psi(x) - \int_1^x v(t)\,\psi'(t)\,dt$$
$$= A\psi(1) + \sum_{n=0}^{\kappa} (-1)^n v_n(x)\,\psi^{(n)}(x) + (-1)^{\kappa+1}\int_1^x v_\kappa(t)\,\psi^{(\kappa+1)}(t)\,dt. \tag{17}$$

Further, since $\psi(x) = O(1)$ in $(1, \infty)$, we have by Lemmas 1(ii) and 5(ii) that, for $n = 0, 1, \ldots, \kappa$,

$$\psi^{(n)}(x) = O(x^{-n}) \text{ in } (1, \infty);$$

whence, in view of (9), (10) and Lemma 7(i),

$$\sum_{n=0}^{\kappa} (-1)^n v_n(x) \psi^{(n)}(x) = o(1)\ (C, \lambda) \text{ as } x \to \infty. \tag{18}$$

It follows from (14), (17) and (18) that $\int_1^\infty f(t)\,dt = s_0\ (C, \lambda)$.

In view now of parts (ii) and (iii) of Lemma 4, parts (ii) and (iii) of the theorem can be easily deduced from (16).

PART 2.

5. In what follows we shall suppose that all functions are real, that κ is a positive integer and that $\phi(x)$ has an absolutely continuous κ-th derivative. Our object now is to prove

THEOREM 2. *Let $\phi(x)$ be such that, for every λ in the range $1, 2, \ldots, \kappa$, $\int_1^x \{f_1(t) - s\}\phi'(t)\,dt$ is bounded $(C, \lambda-1)$ in $(1, \infty)$, for some value of s, whenever $\int_1^\infty f(t)\phi(t)\,dt$ is summable (C, λ).*

(i) *If $0 < \phi(x) < m$, for all $x \geqslant 1$ and some constant m, then $\phi \in \Omega_1^{\kappa}$.*

(ii) *If $\phi(x) > m > 0$, for all $x \geqslant 1$ and some constant m, then $\phi \in \Omega_2^{\kappa}$.*

We require additional lemmas.

6. We shall denote by S_p^q (p, q non-negative integers) the class of functions $v(t)$ such that $v^{(q)}(t)$ is absolutely continuous and $v_p(t) = o(t^p)$ as $t \to \infty$.

LEMMA† 8. *If*

$$\overline{\lim_{x\to\infty}} \left| \int_1^\infty v_p(t) f(t, x)\,dt \right| < \infty$$

and the integral is convergent‡ for all $x \geqslant 1$ whenever $v \in S_p^q$ (p, q fixed), and if, for each $y > 1$,

$$\overline{\lim_{x\to\infty}} \int_1^y t^p |f(t, x)|\,dt < \infty, \tag{19}$$

then

$$\overline{\lim_{x\to\infty}} \int_1^\infty t^p |f(t, x)|\,dt < \infty. \tag{20}$$

† *Cf.* Hardy (6), chapter 3; and Hill (8).

‡ *I.e.* $\int_1^y v_p(t) f(t, x)\,dt$ exists in the Lebesgue sense for all $y > 1$ and tends to a finite limit as $y \to \infty$.

A. We shall prove first that, for $x \geqslant 1$, $a > 1$,

$$\int_a^\infty t^p |f(t, x)|\,dt < \infty. \tag{21}$$

Note that, for $y > a > 1$, there is a function $v(t)$ in S_p^q such that $v_p(t) = t^p$ for $a \leqslant t \leqslant y$, and so, for $x \geqslant 1$,

$$\int_a^y t^p |f(t, x)|\,dt < \infty.$$

Assume that for $x = x_0 \geqslant 1$, $a = a_0 > 1$, (21) is false. Then there is an increasing unbounded sequence $\{y_n\}$ such that $y_1 = a_0$ and

$$\infty > \int_{y_n}^{y_{n+1}} t^p |f(t, x_0)|\,dt > n+4. \tag{22}$$

We can now find† a step function $g(t)$, with a finite number of steps in each interval $[y_n, y_{n+1}]$, such that

$$\int_{y_n}^{y_{n+1}} t^p |g(t) - f(t, x_0)|\,dt < 1. \tag{23}$$

It follows from (22) and (23) that

$$\int_{y_n}^{y_{n+1}} t^p |g(t)|\,dt > n+3. \tag{24}$$

Let Y_n be the union of a finite number of non-intersecting closed intervals lying in the interior of (y_n, y_{n+1}), but not containing any points at which $g(t)$ is discontinuous, such that

$$\int_{Y_n} t^p |g(t)|\,dt > \int_{y_n}^{y_{n+1}} t^p |g(t)|\,dt - 1. \tag{25}$$

Clearly there is a function $v(t)$ in S_p^q such that

$$v_p(t) = n^{-1} t^p \operatorname{sgn} g(t) \quad (t \in Y_n) \tag{26}$$

and

$$|v_p(t)| \leqslant n^{-1} t^p \quad (y_n \leqslant t < y_{n+1}). \tag{27}$$

It follows from (24), (25), (26) and (27) that

$$\int_{y_n}^{y_{n+1}} v_p(t) g(t)\,dt > \frac{n+1}{n}; \tag{28}$$

and then from (23), (27) and (28) that

$$\int_{y_n}^{y_{n+1}} v_p(t) f(t, x_0)\,dt > 1.$$

† *Cf.* Kuttner (10), 109.

Hence, in contradiction to one of the hypotheses, $\int_1^\infty v_p(t) f(t, x_0)\,dt$ is not convergent, and so (21) must be true.

B. *Assume now that* (20) *is false.* Then we can define two increasing unbounded sequences $\{x_n\}$, $\{y_n\}$ as follows†.

Let $x_0 = y_1 = 1$ and suppose that $x_0, x_1, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n$ have been determined. Let

$$\delta_n = \overline{\lim_{x\to\infty}} \int_1^{y_n} t^p |f(t, x)|\,dt,$$

and note that, by (19), δ_n is finite. Choose x_n so that $x_n > x_{n-1}+1$,

$$\int_1^{y_n} t^p |f(t, x_n)|\,dt < \delta_n + 1 \tag{29}$$

and

$$\int_1^\infty t^p |f(t, x_n)|\,dt > n^2+2n+(n+1)\delta_n+6.$$

Since (21) is true, we can now choose y_{n+1} so that $y_{n+1} > y_n + 1$ and

$$\int_{y_{n+1}}^\infty t^p |f(t, x_n)|\,dt < 1. \tag{30}$$

It follows that

$$\int_{y_n}^{y_{n+1}} t^p |f(t, x_n)|\,dt > n^2+2n+n\delta_n+4.$$

Proceeding now as in A we can find a function $v(t)$ in $S_p{}^q$ such that

$$|v_p(t)| \leqslant n^{-1} t^p \leqslant t^p \quad (y_n \leqslant t < y_{n+1}) \tag{31}$$

and

$$\int_{y_n}^{y_{n+1}} v_p(t) f(t, x_n)\,dt > n+2+\delta_n. \tag{32}$$

We deduce from (29), (30), (31) and (32) that

$$\int_1^\infty v_p(t) f(t, x_n)\,dt > n;$$

and, since this is inconsistent with the main hypothesis, the lemma is proved.

LEMMA 9. *If $\kappa \geqslant \lambda \geqslant 1$, $\phi \,\varepsilon\, \Omega_1{}^\kappa$ and $v(t) = O(1)$ (C, λ) in $(1, \infty)$, then*

$$\phi(x) \int_1^x v(t) \psi'(t)\,dt = O(1) \ \ (C, \lambda-1) \ \ in \ \ (1, \infty).$$

In view of Lemma 7(ii) the argument used in establishing part (i) of Theorem 1 is easily adapted to the present requirements.

† *Cf.* Hardy (6), 45.

LEMMA 10. *If, for some positive constant m and all $x \geqslant 1$,*

$$\text{(i)} \ \int_x^\infty |\phi'(t)|\,dt < m\phi(x), \quad \text{(ii)} \ x|\phi'(x)| < m\phi(x),$$

and if, for some fixed $\mu > 0$,

$$\text{(iii)} \ \phi(x) \int_1^x v(t)\psi'(t)\,dt = O(1) \ (C, \mu) \ in \ (1, \infty) \ whenever \ v \,\varepsilon\, S_\kappa{}^0, \ then$$

$\phi \,\varepsilon\, \Omega_1{}^\kappa$.

Suppose, without loss in generality, that $\mu \geqslant \kappa$. Note that, in virtue of hypothesis (i), $\phi(\infty)$ exists and $0 \leqslant \phi(\infty) < \infty$; and write, for $x \geqslant 1$,

$$\chi(x) = \int_x^\infty |\phi'(t)|\,dt + \phi(\infty).$$

Then, for $x \geqslant 1$, $\chi(x)$ is non-increasing and there is a positive constant G such that

$$0 < \phi(x) \leqslant \chi(x) \leqslant G\phi(x).$$

A further consequence of hypothesis (i) is, by Lemma 5(i), that $\psi \,\varepsilon\, \Omega_2{}^0$. *Assume now that $\psi \,\varepsilon\, \Omega_2^{p-1}$ where p is an integer such that $1 \leqslant p \leqslant \kappa$.* Let $v \,\varepsilon\, S_p{}^0$. Then, by hypothesis (iii), we obtain after p integrations by parts

$$\begin{aligned}
&\frac{1}{x^\mu}\int_1^x (x-u)^{\mu-1}\phi(u)\,du \int_1^u v(t)\psi'(t)\,dt \\
&\quad= \frac{1}{x^\mu}\int_1^x v(t)\psi'(t)\,dt \int_t^x (x-u)^{\mu-1}\phi(u)\,du \\
&\quad= \frac{(-1)^p}{x^\mu}\int_1^x v_p(t)\psi^{(p+1)}(t)\,dt \int_t^x (x-u)^{\mu-1}\phi(u)\,du \\
&\qquad\qquad - \frac{(-1)^p p}{x^\mu}\int_1^x (x-t)^{\mu-1} v_p(t)\psi^{(p)}(t)\phi(t)\,dt \\
&\qquad\qquad + \sum_{r=2}^{p}\sum_{s=1}^{r}\frac{c_{r,s}}{x^\mu}\int_1^x (x-t)^{\mu-s} v_p(t)\psi^{(p+1-r)}(t)\phi^{(r-s)}(t)\,dt \\
&\quad= O(1) \text{ in } (1, \infty),
\end{aligned}$$

where the c's are constants (the double sum does not occur when $p = 1$).

In view of hypothesis (ii) and the assumption, we have, for $x \geqslant 1$,

$$\begin{aligned}
&\frac{1}{x^\mu}\int_1^x (x-t)^{\mu-1} v_p(t)\psi^{(p)}(t)\phi(t)\,dt \\
&\quad= \frac{\phi(x)}{x^\mu}\int_1^x (x-t)^{\mu-1} v_p(t)\psi^{(p)}(t)\,dt - \frac{1}{x^\mu}\int_1^x \phi'(t)\,dt \int_1^t (x-u)^{\mu-1} v_p(u)\psi^{(p)}(u)\,du \\
&\quad= O\left\{\phi(x)\int_1^x t^{p-1}|\psi^{(p)}(t)|\,dt\right\} + O\left\{\frac{1}{x^\mu}\int_1^x \frac{\phi(t)}{t}\,dt \int_1^t (x-u)^{\mu-1} u^p |\psi^{(p)}(u)|\,du\right\} \\
&\quad= O(1) + O\left\{\frac{1}{x}\int_1^x \phi(t)\,dt \int_1^x u^{p-1}|\psi^{(p)}(u)|\,du\right\} = O(1).
\end{aligned}$$

Further, when $p \geqslant 2$, it follows from the assumption, by Lemmas 1(ii), 2(ii) and 5(ii), that, for $r = 2, 3, \ldots, p$, $s = 1, 2, \ldots, r$, $t \geqslant 1$,

$$\psi^{(p+1-r)}(t)\,\phi^{(r-s)}(t) = O(t^{s-p-1}).$$

Hence the above double sum is

$$O\left\{\sum_{r=2}^{p}\sum_{s=1}^{r}\frac{1}{x^\mu}\int_1^x (x-t)^{\mu-s}t^{s-1}\,dt\right\} = O(1) \text{ in } (1,\infty).$$

Consequently

$$\frac{1}{x^\mu}\int_1^x v_p(t)\,\psi^{(p+1)}(t)\,dt\int_t^x (x-u)^{\mu-1}\phi(u)\,du = O(1) \text{ in } (1,\infty).$$

Since $\phi(u)$ is positive and bounded, we have for $y>1$,

$$\varlimsup_{x\to\infty}\frac{1}{x^\mu}\int_1^y t^p|\psi^{(p+1)}(t)|\,dt\int_t^x (x-u)^{\mu-1}\phi(u)\,du < \infty.$$

It follows, by Lemma 8, that there is a positive constant H such that, for $x\geqslant 1$,

$$H > \frac{1}{(4x)^\mu}\int_1^{4x} t^p|\psi^{(p+1)}(t)|\,dt\int_t^{4x}(4x-u)^{\mu-1}\phi(u)\,du$$

$$= \frac{1}{(4x)^\mu}\int_1^{4x}(4x-u)^{\mu-1}\phi(u)\,du\int_1^u t^p|\psi^{(p+1)}(t)|\,dt$$

$$\geqslant \frac{1}{2^{\mu+1}xG}\int_x^{2x}\chi(u)\,du\int_1^u t^p|\psi^{(p+1)}(t)|\,dt$$

$$\geqslant \frac{\chi(2x)}{2^{\mu+1}G}\int_1^x t^p|\psi^{(p+1)}(t)|\,dt.$$

Now, for $x\geqslant 1$,

$$0\leqslant \log\left\{\frac{\chi(x)}{\chi(2x)}\right\} = \log\chi(x)-\log\chi(2x) = -x\frac{\chi'(\xi)}{\chi(\xi)} = x\frac{|\phi'(\xi)|}{\chi(\xi)} \leqslant \xi\frac{|\phi'(\xi)|}{\phi(\xi)},$$

where $x<\xi<2x$; and so, in view of hypothesis (ii),

$$\chi(2x)\geqslant e^{-m}\chi(x)\geqslant e^{-m}\phi(x).$$

Consequently

$$\phi(x)\int_1^x t^p|\psi^{(p+1)}(t)|\,dt = O(1) \text{ in } (1,\infty);$$

and, since $\psi\in\Omega_2^0$, it follows that $\psi\in\Omega_2^p$. Hence, by induction and Lemma 5(ii), $\phi\in\Omega_1^\kappa$.

LEMMA† 11. *If $0<\phi(x)<m$, for all $x\geqslant 1$ and some constant m, and if $\phi(x)\int_1^x v(t)\psi'(t)\,dt$ is bounded in $(1,\infty)$ whenever $v\in S_1^0$, then $\phi\in\Omega_1^1$.*

† *Cf.* Bosanquet (2), 279, I.

We shall first prove that†

$$x\phi(x)\,\psi'(x) = O(1) \text{ in } (1,\infty). \tag{33}$$

Assume (33) *to be false.* We shall define a strictly increasing unbounded sequence $\{x_n\}$ as follows.

Let $x_0=1$, suppose $x_0, x_1, \ldots, x_{2n-2}$ to have been determined and put

$$c_n = \int_1^{x_{2n-2}+1} t|\psi''(t)|\,dt.$$

Choose x_{2n} so that $x_{2n}>x_{2n-2}+1$ and

$$x_{2n}\,\phi(x_{2n})\,|\psi'(x_{2n})| > n(n+mc_n);$$

and then choose x_{2n-1} so that $x_{2n}>x_{2n-1}>x_{2n-2}+1$ and

$$\phi(x_{2n})\int_{x_{2n-1}}^{x_{2n}} t|\psi''(t)|\,dt < n.$$

Now it is clear that there is a function $v(t)$ in S_1^0 such that

$$v_1(x_{2n}) = \frac{x_{2n}}{n},\quad |v_1(t)|\leqslant\frac{t}{n} \quad\text{for}\quad x_{2n-1}<t<x_{2n}+1 \quad (n=1,2,\ldots)$$

and $v_1(t)=0$ for all other values of t.

It follows that

$$\left|\phi(x_{2n})\int_0^{x_{2n}} v(t)\,\psi'(t)\,dt\right| \geqslant v_1(x_{2n})\,\phi(x_{2n})\,|\psi'(x_{2n})|$$

$$-\phi(x_{2n})\int_1^{x_{2n-2}+1}|v_1(t)\,\psi''(t)|\,dt-\phi(x_{2n})\int_{x_{2n-1}}^{x_{2n}}|v_1(t)\,\psi''(t)|\,dt$$

$$\geqslant n+mc_n-mc_n-1 = n-1.$$

This contradicts the main hypothesis and so (33) must be true.

Note that, for $y>1$,

$$\varlimsup_{x\to\infty}\phi(x)\int_1^y|\psi'(t)|\,dt<\infty,\quad \varlimsup_{x\to\infty}\phi(x)\int_1^y t|\psi''(t)|\,dt<\infty.$$

Since $S_0^0\subset S_1^0$, it follows from the main hypothesis, by Lemma 8, that

$$\phi(x)\int_1^x|\psi'(t)|\,dt = O(1) \text{ in } (1,\infty).$$

Further, in virtue of (33),

$$\phi(x)\int_1^x v_1(t)\,\psi''(t)\,dt = v_1(x)\,\phi(x)\,\psi'(x)-\phi(x)\int_1^x v(t)\,\psi'(t)\,dt$$

† *Cf.* Kuttner (10), 110.

is bounded in $(1, \infty)$ whenever $v \in S_1^0$. Hence, by Lemma 8,

$$\phi(x) \int_1^x t|\psi''(t)|\,dt = O(1) \text{ in } (1, \infty).$$

In view of Lemma 5(ii), the result follows.

LEMMA 12. *If* $0 < \phi(x) < m$, *for all* $x \geqslant 1$ *and some constant* m, *and if* $\int_1^x f_1(t)\phi'(t)\,dt$ *is* (i) *bounded in* $(1, \infty)$ *whenever* $\int_1^\infty f(t)\phi(t)\,dt$ *is summable* $(C, 1)$, (ii) *bounded* (C, μ) *in* $(1, \infty)$ *for some fixed* $\mu > 0$, *whenever* $\int_1^\infty f(t)\phi(t)\,dt$ *is summable* (C, κ), *then* $\phi \in \Omega_1^\kappa$.

Referring back to (7) we note that, for $x \geqslant 1$,

$$\phi(x) \int_1^x v(t)\psi'(t)\,dt = A\psi(1)\phi(x) - A - \int_1^x f_1(t)\phi'(t)\,dt,$$

where
$$v(t) = \int_1^t f(u)\phi(u)\,du - A. \tag{34}$$

Given an absolutely continuous $v(t)$ we can find $f(u)$ and A to satisfy (34). Since $\phi(x)$ is bounded, we can deduce that, for $x \geqslant 1$,

$$\phi(x) \int_1^x v(t)\psi'(t)\,dt$$

is (i)′ bounded whenever $v \in S_1^0$, (ii)′ bounded (C, μ) whenever $v \in S_\kappa^0$.

It follows from (i)′, by Lemma 11, that $\phi \in \Omega_1^1$. Hence, in view of Lemma 1 and (ii)′, ϕ satisfies the hypotheses of Lemma 10 and so $\phi \in \Omega_1^\kappa$.

7. Part (i) of Theorem 2 is included in Lemma 12 and part (ii) can be established as follows.

Proof of Theorem 2 (ii).

A. *Suppose that* $v \in S_0^0$ and put $f(u) = v'(u)\psi(u)$ $(u \geqslant 1)$, $A = -v(1)$. Then

$$v(t) = \int_1^t f(u)\phi(u)\,du - A = o(1) \text{ as } t \to \infty.$$

Referring now to (7) we note that, by hypothesis, there is a constant s such that

$$\int_1^x \{f_1(t) - s\}\phi'(t)\,dt + A - s\phi(1) = \phi(x)\left\{A\psi(1) - s - \int_1^x v(t)\psi'(t)\,dt\right\}$$
$$= O(1) \text{ in } (1, \infty).$$

Since $\psi(x)$ is bounded, it follows that

$$\int_1^x v(t)\psi'(t)\,dt = O(1) \text{ in } (1, \infty),$$

and hence, by Lemma 8, that

$$\int_1^\infty |\psi'(t)|\,dt < \infty.$$

Consequently $\int_1^\infty v(t)\psi'(t)\,dt$ is convergent. If $\phi(x)$ is unbounded the value of this integral is necessarily $A\psi(1) - s$ and thus

$$\phi(x) \int_x^\infty v(t)\psi'(t)\,dt = O(1) \text{ in } (1, \infty).$$

If $\phi(x)$ is bounded this result follows immediately from the convergence of $\int_1^\infty v(t)\psi'(t)\,dt$. Hence, by Lemma 8,

$$\phi(x) \int_x^\infty |\psi'(t)|\,dt = O(1) \text{ in } (1, \infty),$$

and so $\psi \in \Omega_1^0$.

B. *Assume now that*

$$\psi \in \Omega_1^{p-1}, \tag{35}$$

where p *is an integer such that* $1 \leqslant p \leqslant \kappa$. Let q be any integer in the range $1, 2, \ldots, p$ and suppose that $\int_1^\infty g(t)\psi(t)\,dt$ is summable (C, q). Put

$$f(t) = g(t)\{\psi(t)\}^2 \quad (t \geqslant 1).$$

Then $\int_1^\infty f(t)\phi(t)\,dt$ is summable (C, q) and so, by hypothesis, there is an indefinite integral $F(t)$ of $f(t)$ such that

$$h(x) = \int_1^x F(t)\phi'(t)\,dt = O(1)\ (C, q-1) \text{ in } (1, \infty). \tag{36}$$

Further, for $x \geqslant 1$,

$$\int_1^x g_1(t)\psi'(t)\,dt = \int_1^x \psi'(t)\,dt \int_1^t f(u)\{\phi(u)\}^2\,du$$
$$= \int_1^x f(u)\{\phi(u)\}^2\,du \int_u^x \psi'(t)\,dt$$
$$= -\int_1^x f(u)\phi(u)\,du + \psi(x)\int_1^x f(u)\{\phi(u)\}^2\,du$$
$$= -\int_1^x f(u)\phi(u)\,du + F(x)\phi(x) - c\psi(x)$$
$$-2\psi(x)\int_1^x h'(u)\phi(u)\,du$$
$$= c\{\psi(1) - \psi(x)\} - h(x) + 2\psi(x)\int_1^x h(u)\phi'(u)\,du, \tag{37}$$

where $c = F(1)\{\phi(1)\}^2$.