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Strict inclusion between strong and ordinary methods of summability

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1. Introduction

Let $Q = \{q_{n,\nu}\}$ $(n, \nu = 0, 1, 2, \ldots)$ be a (summability) matrix and let $\{s_\nu\}$ be a sequence. Let

$$\sigma_n = \sum_{\nu=0}^{\infty} q_{n,\nu} s_\nu. \tag{1}$$

The sequence $\{s_\nu\}$ is said to be *Q-convergent* to the value s if σ_n exists for $n = 0, 1, 2, \ldots$ and tends to s.

In this case we write $s_n \to s(Q)$ and call s the *Q-limit* of $\{s_\nu\}$. We denote the set of all Q-convergent sequences by c_Q.

The symbol P will be reserved for matrices $\{p_{n,\nu}\}$ with

$$p_{n,\nu} \geqq 0 \qquad (n, \nu = 0, 1, 2, \ldots).$$

Necessary and sufficient conditions for every null sequence to be P-convergent to zero are:

$$\sup_{n\geqq 0} \sum_{\nu=0}^{\infty} p_{n,\nu} < \infty, \tag{2}$$

$$\lim_{n\to\infty} p_{n,\nu} = 0 \qquad \text{for } \nu = 0, 1, 2, \ldots. \tag{3}$$

The matrix P is regular if and only if, in addition to (2) and (3), it satisfies

$$\lim_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} = 1. \tag{4}$$

Throughout this paper λ is a positive number. As in [2] we define the strong summability method $[P, Q]_\lambda$ as follows. We write $s_n \to s[P, Q]_\lambda$ if

$$\tau_n = \sum_{\nu=0}^{\infty} p_{n,\nu} \,|\, \sigma_\nu - s \,|^{\lambda} \tag{5}$$

exists for $n = 0, 1, 2, \ldots$ and tends to zero. We call s the *$[P, Q]_\lambda$-limit* of $\{s_\nu\}$ and say that the sequence is *$[P, Q]_\lambda$-convergent* to s. We denote the set of all sequences which are $[P, Q]_\lambda$-convergent by $[c_{P,Q}]_\lambda$.

We denote the set of all convergent sequences by c, the identity matrix by I, and write $[c_P]_\lambda$ instead of $[c_{P,I}]_\lambda$, and $[P]_\lambda$ instead of $[P, I]_\lambda$.

If V and W are summability methods of either of the above two types, we shall say that W *includes* V, and use the notation $V \Rightarrow W$, if any sequence V-convergent to s is necessarily W-convergent to s. If W includes V but V does not include W, the inclusion $V \Rightarrow W$ is said to be *strict*. If both $V \Rightarrow W$ and $W \Rightarrow V$, we say that V and W are *equivalent*.

The sequence $\{s_\nu\}$ is said to be *Q-bounded* if $\sup_{n\geqq 0} |\,\sigma_n\,| < \infty$, where σ_n is defined by (1). The set of all Q-bounded sequences is denoted by m_Q.

The sequence $\{s_\nu\}$ is said to be *$[P, Q]_\lambda$-bounded* if there is a number M such that

$$\sum_{\nu=0}^{\infty} p_{n,\nu} \,|\, \sigma_\nu \,|^{\lambda} < M \qquad \text{for } n = 0, 1, 2, \ldots.$$

We denote the set of all $[P, Q]_\lambda$-bounded sequences by $[m_{P,Q}]_\lambda$, and we write $[m_P]_\lambda$ instead of $[m_{P,I}]_\lambda$.

2. Simple inclusion theorems

We state now some simple results, of which all but parts (i) and (ii) of Theorem 1 are proved in [2]. The proof of part (iii) of Theorem 1 can easily be adapted to establish parts (i) and (ii).

Theorem 1. *If P satisfies* (2), *and* $\lambda > \mu > 0$, *then*

(i) $$[c_{P,Q}]_\lambda \subset [m_{P,Q}]_\lambda;$$

(ii) $$[m_{P,Q}]_\lambda \subset [m_{P,Q}]_\mu;$$

(iii) $$[P, Q]_\lambda \Rightarrow [P, Q]_\mu.$$

In particular the conclusions hold if $\lambda > \mu > 0$ and P is regular.

Theorem 2. (i) *If P satisfies* (2) *and* (3), *then*

$$Q \Rightarrow [P, Q]_\lambda \qquad \textit{for } \lambda > 0.$$

(ii) *If P is regular* (*i. e. it satisfies* (2), (3) *and* (4)), *then*

$$[P, Q]_\lambda \Rightarrow PQ \qquad \textit{for } \lambda \geqq 1.$$

The summability method PQ referred to above, is defined by '$\{s_n\}$ is PQ-convergent to s if $\{\sigma_n\}$ as given by (1), is P-convergent to s.'

Both Theorem 2(i) and its converse are established in [4] for the case $Q = I$.

It is our principal purpose to investigate general conditions on the matrix P under which the inclusion relations in Theorems 1 and 2 are strict. In the next section we give some basic properties of strong summability and strong boundedness which will be useful in our investigation.

3. Basic properties

Theorem 3. *Let P satisfy* (3) *and*

$$\limsup_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} > 0. \tag{6}$$

(i) *If $s_n \to s[P]_\lambda$, then s is a limit point of $\{s_n\}$.*

(ii) *If $s_n \in [m_P]_\lambda$, then $\{s_n\}$ has a limit point.*

Proof. (i) Suppose that s is not a limit point of $\{s_\nu\}$. Then there is a positive integer N and a positive number δ such that $|s_\nu - s|^\lambda \geqq \delta$ for $\nu \geqq N$. Thus by (3) and (6) we have

$$\limsup_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} |s_\nu - s|^\lambda \geqq \delta \limsup_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} > 0,$$

and (i) follows.

(ii) Suppose that $\{s_\nu\}$ has no limit point. Then given a positive number K, there is a positive integer N such that $|s_\nu|^\lambda \geqq K$ for $\nu \geqq N$. As above we have

$$\limsup_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} |s_\nu|^\lambda \geqq K \limsup_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} > 0.$$

Part (ii) follows.

When (6) does not hold, we must have

$$(7) \qquad \lim_{n\to\infty} \sum_{\nu=0}^{\infty} p_{n,\nu} = 0;$$

and in this case we have the following theorem.

Theorem 4. *Let P satisfy (3) and (7), and let $\{\xi_m\}$ be any unbounded sequence of positive numbers. Then there is an index sequence*) $\{q_m\}$ such that the sequence $\{s_\nu\}$, defined by*

$$(8) \qquad s_\nu = \xi_m \quad \text{for } q_m \leqq \nu < q_{m+1},$$

is $[P]_\lambda$-convergent to zero.

Proof. Since $p_{n,\nu} \geqq 0$, it follows from (3) and (7) that the series $\sum_{\nu=0}^{\infty} p_{n,\nu}$ is uniformly convergent. Thus we can choose $\{q_m\}$ so that $q_0 = 0$, $q_{m+1} > q_m$ for $m = 0, 1, 2, \ldots$ and

$$\sum_{\nu=q_m}^{\infty} p_{n,\nu} < \frac{2^{-m}}{\xi_m^\lambda}, \quad m, n = 0, 1, 2, \ldots.$$

Now for $\{s_n\}$ satisfying (8), we have

$$\sum_{\nu=0}^{\infty} p_{n,\nu} |s_\nu|^\lambda = \sum_{m=0}^{\infty} \sum_{\nu=q_m}^{q_{m+1}-1} p_{n,\nu} |s_\nu|^\lambda = \sum_{m=0}^{\infty} \xi_m^\lambda \sum_{\nu=q_m}^{q_{m+1}-1} p_{n,\nu} = \sum_{m=0}^{\infty} \xi_m^\lambda Q_{n,m},$$

and $0 \leqq \xi_m^\lambda Q_{n,m} \leqq 2^{-m}$ for all n and m. The latter series is thus uniformly convergent, and since $Q_{n,m} \to 0$ as $n \to \infty$ for each m, our result follows.

4. Some theorems on strict inclusion

Theorem 5. *Let P satisfy (3).*

(i) *If $0 \leqq L \leqq \infty$ and*

$$(9) \qquad \liminf_{\nu\to\infty} \max_{n\geqq 0} p_{n,\nu} = 0,$$

then there is a sequence $\{s_\nu\}$ of non-negative numbers, with $\limsup s_\nu = L$, which is P-convergent to zero.

(ii) *If there is a sequence $\{s_\nu\}$ of non-negative numbers, with $\limsup s_\nu > 0$, which is P-convergent to zero, then (9) holds.*

*) A strictly increasing sequence of non-negative integers.

Proof. (i) By (9), there is an index sequence $\{\nu_k\}$ such that

$$\mu_k = \max_{n\geqq 0} p_{n,\nu_k} \to 0 \text{ as } k \to \infty.$$

Let $\{\lambda_k\}$ be any sequence of positive numbers such that $\lambda_k \to L$ and $\mu_k \lambda_k \to 0$. Choose an index sequence $\{k_i\}$ such that

$$\sum_{i=0}^{\infty} \mu_{k_i} \lambda_{k_i} < \infty.$$

Now define $\{s_\nu\}$ by setting

$$s_\nu = \begin{cases} \lambda_{k_i} & \text{if } \nu = \nu_{k_i},\ i = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\limsup s_\nu = L$ and, for any positive integer m,

$$t_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu \leqq \sum_{i=0}^{m-1} p_{n,\nu_{k_i}} \lambda_{k_i} + \sum_{i=m}^{\infty} \mu_{k_i} \lambda_{k_i}.$$

Hence t_n is finite for $n = 0, 1, 2, \ldots$ and, by (3),

$$0 \leqq \limsup t_n \leqq \sum_{i=m}^{\infty} \mu_{k_i} \lambda_{k_i} = \gamma_m \text{ say.}$$

Since $\gamma_m \to 0$, it follows that $t_n \to 0$, and hence that $\{s_\nu\}$ is P-convergent to zero.

(ii) Suppose $\{s_\nu\}$ is a sequence of non-negative numbers with $\limsup s_\nu \geqq 2\gamma > 0$ and such that

$$t_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu < \infty \quad \text{for } n = 0, 1, 2, \ldots.$$

There is an index sequence $\{\nu_k\}$ such that

$$s_{\nu_k} > \gamma \quad \text{for } k = 0, 1, 2, \ldots.$$

Let $\{n_k\}$ be a sequence of integers such that $p_{n_k,\nu_k} = \max_{n\geqq 0} p_{n,\nu_k}$. Suppose now that (9) does not hold, so that there is a positive number δ such that $p_{n_k,\nu_k} > \delta$ for k sufficiently large. Then

$$\infty > t_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu \geqq \gamma \sum_{i=0}^{\infty} p_{n,\nu_i} \geqq \gamma p_{n,\nu_k}$$

and so for sufficiently large k,

$$t_{n_k} \geqq \gamma p_{n_k,\nu_k} > \gamma \delta.$$

Since $\sum_{i=0}^{\infty} p_{n,\nu_i}$ converges, we have $p_{n,\nu_i} \to 0$ as $i \to \infty$. Thus the sequence $\{n_k\}$ cannot be bounded, for if it were, we would have $p_{n_k,\nu_k} \to 0$ as $k \to \infty$. Consequently t_n does not tend to zero, i. e. $\{s_\nu\}$ is not P-convergent to zero.

Corollary 1. *Let P satisfy (3). Then (9) is necessary and sufficient for there to be a non-convergent sequence which is $[P]_\lambda$-convergent to zero.*

Corollary 2. *Let P satisfy (3), and suppose that Q is a matrix such that for every sequence $\{\sigma_\nu\}$ there is a sequence $\{s_\nu\}$ for which (1) holds. Then (9) is a necessary and sufficient condition for there to be a sequence which is not Q-convergent, but which is $[P, Q]_\lambda$-convergent to zero.*

The next theorem follows easily from Theorem 5.

Theorem 6. *Let P satisfy* (3). *Then* (9) *is necessary and sufficient for there to be a divergent sequence of zeros and ones which is P-convergent to zero.*

It is interesting to compare Theorem 6 with the following theorem established by Agnew [1].

Theorem (Agnew). *If P satisfies* (2) *and*

$$\lim_{n\to\infty}\max_{\nu\geq 0} p_{n,\nu} = 0, \tag{10}$$

then there is at least one divergent sequence of zeros and ones, which is P-convergent.

The relation between our conditions (3) and (9) and Agnew's conditions (2) and (10) is clarified by the following scholium, the proof of which is elementary.

Scholium. *The following two sets of conditions on the matrix P are equivalent:*

(i) $\lim\limits_{\nu\to\infty} p_{n,\nu} = 0,\ n = 0, 1, 2, \ldots$ *and* $\lim\limits_{n\to\infty}\max\limits_{\nu\geq 0} p_{n,\nu} = 0.$

(ii) $\lim\limits_{n\to\infty} p_{n,\nu} = 0,\ \nu = 0, 1, 2, \ldots$ *and* $\lim\limits_{\nu\to\infty}\max\limits_{n\geq 0} p_{n,\nu} = 0.$

Clearly condition (2) implies that $\lim\limits_{\nu\to\infty} p_{n,\nu} = 0$ for $n = 0, 1, 2, \ldots$. Thus by combining Theorem 6 with the scholium we can strengthen Agnew's theorem by replacing condition (2) in the hypothesis by the weaker condition:

$$\lim_{\nu\to\infty} p_{n,\nu} = 0 \text{ for } n = 0, 1, 2, \ldots.$$

Another consequence of Theorem 5 is the following 'Mazur-Orlicz type theorem' for strong summability.

Theorem 7. *Let P satisfy* (3). *If only bounded sequences are* $[P]_\lambda$*-convergent, then only convergent sequences are* $[P]_\lambda$*-convergent.*

Proof. By Theorem 5 (i) with $L = \infty$ we find that $\liminf\limits_{\nu\to\infty}\max\limits_{n\geq 0} p_{n,\nu} > 0$. Thus if $\sum\limits_{\nu=0}^{\infty} p_{n,\nu} \mid s_\nu - s\mid^\lambda$ is finite for $n = 0, 1, 2, \ldots$ and tends to zero as n tends to infinity, it follows from Theorem 5 (ii) that $\limsup \mid s_\nu - s\mid^\lambda = 0$, so that $s_\nu \to s$ as required.

The purpose of Theorem 8 is to obtain conditions under which the inclusion relations in Theorem 1 (ii) and (iii) are strict. To facilitate discussion we introduce the following definitions.

The matrix P is called an *S-matrix* if it satisfies (3), and if there is an index sequence $\{\nu_k\}$ such that

$$\mu_k = \max_{n\geq 0} p_{n,\nu_k} > 0 \quad \text{for } k = 0, 1, 2, \ldots \tag{11}$$

and

$$\lim \mu_k = 0. \tag{12}$$

The matrix P is called an *S*-matrix* if it satisfies (3), (11) and (12), and if there is an unbounded sequence of non-negative integers $\{n_k\}$ such that $\mu_k = p_{n_k,\nu_k}$ for $k = 0, 1, 2, \ldots$.

Let P be given. For each non-negative integer k, let $P^{(k)} = \{p_{n,\nu}^{(k)}\}$ be the matrix whose elements satisfy

$$p_{n,\nu}^{(k)} = p_{n+k,\nu} \quad \text{for } n, \nu = 0, 1, 2, \ldots.$$

Thus $P^{(k)}$ is the matrix obtained from P by deleting the first k rows.

It is elementary to show that P is an S^*-matrix if and only if $P^{(k)}$ is an S-matrix for every non-negative integer k. Clearly an S^*-matrix is an S-matrix; and an S-matrix P satisfies (9), a condition appearing in Theorem 5 and Corollary 1. A matrix P, having no column consisting entirely of zeros, and satisfying (3) is an S-matrix; if it is also triangular, then it is an S^*-matrix. The set of S^*-matrices is thus reasonably large.

Theorem 8. *Let P be an S-matrix. Then, for any* $\alpha > 1$, *there is a sequence* $\{s_\nu\}$ *of non-negative numbers which is P-convergent to zero, whereas*

$$\sup_{n\geq 0} \tau_n = \infty \tag{13}$$

where

$$\tau_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu^\alpha.$$

If P is an S-matrix, then the conclusion can be strengthened by the replacement of* (13) *by*

$$\limsup \tau_n = \infty. \tag{14}$$

Proof. Suppose that P is an S-matrix, and let μ_k and $\{\nu_k\}$ be as set out in the definition of S-matrix. Define $\lambda_k = \mu_k^{-\beta}$ where $1/\alpha < \beta < 1$. Then $\lambda_k \to \infty$, $\lambda_k\mu_k = \mu_k^{1-\beta} \to 0$ and $\lambda_k^\alpha \mu_k = \mu_k^{1-\alpha\beta} \to \infty$. Defining an index sequence $\{k_i\}$ and a sequence $\{s_\nu\}$ as in the proof of Theorem 5 (i), we see that

$$t_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu$$

is finite for $n = 0, 1, 2, \ldots$ and tends to zero as n tends to infinity. On the other hand,

$$\tau_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu^\alpha \geq p_{n,\nu_{k_i}} \lambda_{k_i}^\alpha \quad \text{for } i = 0, 1, 2, \ldots.$$

In particular, if n_k is an integer such that $p_{n_k,\nu_k} = \mu_k$, then

$$\tau_{n_{k_i}} \geq \mu_{k_i} \lambda_{k_i}^\alpha \to \infty \quad \text{as } i \to \infty.$$

Hence (13) holds. If $n_k \to \infty$, then (14) holds.

Suppose finally that $\{n_k\}$ is unbounded but not properly divergent. Then there is a subsequence $\{n_k'\}$ of $\{n_k\}$ and a corresponding subsequence $\{\nu_k'\}$ of $\{\nu_k\}$ such that $n_k' \to \infty$ and

$$0 < p_{n_k',\nu_k'} = \max_{n\geq 0} p_{n,\nu_k'} \to 0 \quad \text{as } k \to \infty.$$

Hence, by what has been proved above, there is a sequence $\{s_\nu\}$ of non-negative numbers, which is P-convergent to zero, and for which (14) holds.

Corollary 3. *If P is an S-matrix, and* $\lambda > \mu > 0$, *then there is a sequence* $\{s_\nu\}$ *which is* $[P]_\mu$*-convergent to zero, but which is not* $[P]_\lambda$*-bounded.*

Corollary 4. *Let P be an S-matrix, and suppose that Q is a matrix such that for every sequence* $\{\sigma_\nu\}$ *there is a sequence* $\{s_\nu\}$ *for which* (1) *holds. If* $\lambda > \mu > 0$, *then there is a sequence* $\{s_\nu\}$ *which is* $[P, Q]_\mu$*-convergent but which is not* $[P, Q]_\lambda$*-bounded.*

Corollary 5. *If P is an S*-matrix and* $\lambda > \mu > 0$, *then there is a sequence* $\{s_\nu\}$ *which, for every non-negative integer k, is* $[P^{(k)}]_\mu$*-convergent to zero, but is not* $[P^{(k)}]_\lambda$*-bounded.*

5. Properties of $[c_P]_\lambda$.

Theorem 9. *Let P satisfy* (3) *and* (6). *If* $[c_P]_\lambda$ *contains a divergent sequence, then it contains both bounded and unbounded divergent sequences.*

Proof. Using Theorem 7 we find that $[c_P]_\lambda$ contains an unbounded sequence $\{s_\nu\}$ of non-negative numbers which is $[P]_\lambda$-convergent to zero. Define a sequence $\{s'_\nu\}$ by setting $s'_\nu = \min\{1, s_\nu\}$. Then $\{s'_\nu\}$ is bounded, is $[P]_\lambda$-convergent to zero and, by Theorem 3 (i), is divergent.

Our next theorem is concerned with the inclusions

$$c < [c_P]_\lambda < c_P \quad \text{for all } \lambda \geqq 1,$$

which Theorem 2 shows to hold when P satisfies (2) and (3), and in particular when P is regular.

Theorem 10. *There are regular matrices P for which any one of the following statements holds for every* $\lambda \geqq 1$;

(i) $$c \subsetneqq [c_P]_\lambda \subsetneqq c_P;$$

(ii) $$c = [c_P]_\lambda \subsetneqq c_P;$$

(iii) $$c \subsetneqq [c_P]_\lambda = c_P.$$

Proof. (i) Let P be the matrix associated with Cesàro summability of order 1. Then P satisfies (3) and (9), and so Corollary 1 shows that $c \neq [c_P]_\lambda$. Also P satisfies (6), and the sequence $\{s_\nu\}$ given by $s_{2n} = 1$ and $s_{2n+1} = 0$ $(n = 0, 1, 2, \ldots)$ is P-convergent to $\frac{1}{2}$. Thus by Theorem 3 (i), we see that $[c_P]_\lambda \neq c_P$.

(ii) Let P be a regular matrix for which c_P contains an unbounded sequence, but no bounded divergent sequence (for example take $p_{0,0} = 1$, $p_{n,n} = \frac{1}{3}$, $p_{n,n-1} = \frac{2}{3}$ for $n = 1, 2, 3, \ldots$ and $p_{n,\nu} = 0$ otherwise). Thus $c \neq c_P$; and Theorem 9 shows that $[c_P]_\lambda$ contains no divergent sequences, so that $c = [c_P]_\lambda$.

(iii) Let P be the matrix such that for every sequence $\{s_\nu\}$

$$t_n = \sum_{\nu=0}^{\infty} p_{n,\nu} s_\nu = s_{2n} \quad \text{for } n = 0, 1, 2, \ldots.$$

Then P is regular, $c \neq c_P$ and $[c_P]_\lambda = c_P$.

For the purpose of the next theorem we recall the definition of section-boundedness.

A matrix P is said to be *section-bounded* if for every sequence $\{s_\nu\} \in c_P$, we have

$$\sup_{n,m} \left| \sum_{\nu=0}^{m} p_{n,\nu} s_\nu \right| < \infty.$$

Theorem 11. *If P satisfies* (2) *but is not section-bounded, then for every* $\lambda \geqq 1$, *the set* $c_P - [m_P]_\lambda$ *(and a fortiori the set* $c_P - [c_P]_\lambda$*) is not empty.*

Proof. Since P is not section-bounded, there is a sequence $\{s_\nu\} \in c_P$ for which $\sup_{n,m} \left| \sum_{\nu=0}^{m} p_{n,\nu} s_\nu \right| = \infty$. This sequence cannot belong to $[m_P]_\lambda$, for if it did, by Theorem 1 (iii), we would have

$$\sup_{n \geqq 0} \sum_{\nu=0}^{\infty} p_{n,\nu} \, | s_\nu | = M < \infty$$

and hence

$$\sup_{n,m} \left| \sum_{\nu=0}^{m} p_{n,\nu} s_\nu \right| \leqq M.$$

Theorem 12. *Let the matrix P be regular and triangular. If*

(15) $$p_{n,\nu} \geqq p_{n+1,\nu} \quad \text{for } n \geqq \nu,\ \nu = 0, 1, 2, \ldots,$$

(16) $$p_{n,n} \to 0,$$

(17) $$\sum_{\nu=0}^{n} p_{n,\nu} \leqq \sum_{\nu=0}^{n+1} p_{n+1,\nu} \quad \text{for } n = 0, 1, 2, \ldots;$$

then there is a divergent sequence of zeros and ones which is P-convergent to $\frac{1}{2}$, *but is not* $[P]_\lambda$*-convergent for any* $\lambda \geqq 1$.

Proof. We use the notation

$$u_n = \sum_{\nu=0}^{n} p_{n,\nu} s_\nu \quad \text{for } n = 0, 1, 2, \ldots.$$

If $0 \leqq s_\nu \leqq 1$ for $\nu = 0, 1, 2, \ldots$ then, by (15) and (17) we have

(18) $$u_{n-1} - p_{n,n}(1 - s_n) \leqq u_n \leqq u_{n-1} + p_{n,n} s_n \quad \text{for } n = 1, 2, \ldots.$$

We now define the sequence $\{s_\nu\}$ inductively by setting $s_0 = 1$, and for $\nu > 0$ setting $s_\nu = 1$ if $u_{\nu-1} < \frac{1}{2}$ and $s_\nu = 0$ if $u_{\nu-1} \geqq \frac{1}{2}$. As a consequence of the regularity of P, we find that $\{s_\nu\}$ is divergent.

Let $\varepsilon > 0$. Choose an integer N such that $p_{n,n} < \varepsilon$ for all $n \geqq N$. Suppose

(19) $$\frac{1}{2} - \varepsilon < u_{n-1} < \frac{1}{2} + \varepsilon$$

holds for some integer $n > N$. Then either $\frac{1}{2} \leqq u_{n-1} < \frac{1}{2} + \varepsilon$, in which case $s_n = 0$, and so by (18)

$$\frac{1}{2} - \varepsilon < u_{n-1} - p_{n,n} \leqq u_n \leqq u_{n-1} < \frac{1}{2} + \varepsilon;$$

or $\frac{1}{2} - \varepsilon < u_{n-1} < \frac{1}{2}$, in which case $s_n = 1$, and so again by (18)

$$\frac{1}{2} - \varepsilon < u_{n-1} \leqq u_n < u_{n-1} + p_{n,n} < \frac{1}{2} + \varepsilon.$$

Thus (19) holds with $n - 1$ replaced by n. Since $u_n - u_{n-1} \to 0$ by (18) and $u_n \geqq \frac{1}{2} \geqq u_{n-1}$ for infinitely many n, it follows that there is an integer $n_0 > N$ for which (19) holds with $n = n_0$. Thus by induction (19) holds for all $n \geqq n_0$ and so $u_n \to \frac{1}{2}$, i. e. $\{s_n\}$ is P-convergent to $\frac{1}{2}$. But by Theorem 2 (ii) and Theorem 3, $\{s_n\}$ cannot be $[P]_\lambda$-convergent for any $\lambda \geqq 1$.

We now give some examples of matrices which satisfy the hypotheses of Theorem 12.

Let $\{p_n\}$ be a sequence of positive numbers, and let $P_n = \sum_{\nu=0}^{n} p_\nu$.

(i) Let P be given by

$$p_{n,\nu} = \begin{cases} \dfrac{p_\nu}{P_n} & \text{for } \nu \leqq n \\ 0 & \text{for } n > \nu. \end{cases}$$

Then P is the matrix associated with the 'weighted mean' method $(\overline{N}, p_n)$ (see [3]). If $P_n \to \infty$ and $\frac{p_n}{P_n} \to 0$, then P satisfies the hypotheses of Theorem 12, and P is also section-bounded.

(ii) Let P be given by

$$p_{n,\nu} = \begin{cases} \dfrac{p_{n-\nu}}{P_n} & \text{for } \nu \leqq n \\ 0 & \text{for } \nu > n. \end{cases}$$

Then P is the matrix associated with the Nörlund summability method (N, p_n) (see [3]). If $\dfrac{p_n}{P_n} \to 0$, and if $\{p_n\}$ is a monotonic non-increasing sequence, then P satisfies the hypotheses of Theorem 12. Some of these matrices are section-bounded.

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