DIVERGENCE CRITERIA FOR POSITIVE SERIES

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Suppose throughout that f is a mapping of the set of positive integers into itself, and that $\{\lambda_n\}$ is a sequence of real non-negative numbers.

Using a combinatorial argument, K. A. Post [1] recently established the following result:

Let

(1) $$f(n+1)-f(n) \geqq n+1, \quad n=1,2,\cdots,$$

and suppose that the sequence $\{a_n\}$ *satisfies*

(2) $$0 < a_n \leqq a_{n+1} + a_{f(n)}, \quad n = 1,2,\cdots.$$

Then $\Sigma\, a_n = \infty$.

Post notes Erdös's observation that if $f(n) \leqq cn^2$, $0 < c < \frac{1}{2}$, then (2) does not, in general, imply the divergence of $\Sigma\, a_n$.

Our first theorem extends the scope of the above divergence criterion by showing that (1) and (2) can be replaced by more general inequalities.

Theorem 1. *Let*

(3) $$\lambda_n \leqq 1, \qquad n = 1,2,\cdots,$$

(4) $$f(n+1)-f(n) \geqq n\lambda_n + 1, \quad n=1,2,\cdots,$$

and suppose that the sequence $\{a_n\}$ *satisfies*

(5) $$0 < a_n \leqq a_{n+1} + \lambda_n a_{f(n)}, \quad n = 1,2,\cdots.$$

Then $\Sigma\, a_n = \infty$.

Our second theorem shows that for a decreasing sequence $\{a_n\}$, condition (3) of Theorem 1 is redundant when a slightly modified version of condition (4) holds.

Theorem 2. *Let*

(6) $$f(n+1)-f(n) \geqq n\lambda_{n+1} + 1, \quad n = 1,2,\cdots,$$

and suppose that $\{a_n\}$ *is a decreasing sequence satisfying* (5). *Then* $\Sigma\, a_n = \infty$.

Remarks. (i) Post's combinatorial argument cannot be used in the proof of Theorem 1 because, in general, we shall have $\lambda_n < 1$ for some values of n.

(ii) Condition (6) cannot be replaced by (4) in Theorem 2. Indeed, if $\{a_n\}$ is *any* given sequence of positive numbers, we can define f and $\{\lambda_n\}$ by induction so that (4) and (5) hold: Let $f(1) = 1$ and suppose $\lambda_1, \lambda_2, \cdots, \lambda_{m-1}$ and $f(1), f(2), \cdots, f(m)$ are known. First define λ_m so that (5) holds for $n = m$, and then define $f(m+1)$ so that (4) is satisfied for $n = m$.

(iii) Conditions (4) and (5) alone do not, in general, imply the divergence of $\Sigma\, a_n$, even when λ_n is constant. This we can demonstrate by means of the following example: Let $\lambda_n \equiv \lambda > 1$ and let f satisfy (4). Let $f^0(m) = m$ and $f^\nu(m) = f(f^{\nu-1}(m))$. For $n = 1,2,\cdots$, let $k = k_n$ be the largest non-negative integer for which $f^k(r) = n$. Having thus determined k, a simple argument shows that $r = r_n$ is also uniquely determined. We define $\{a_n\}$ by $a_n = \lambda^{-k}2^{-r}$. It is easily seen that (5) holds. But $\Sigma\, a_n \leqq \Sigma_{k,r}\lambda^{-k}2^{-r} < \infty$.

(iv) In neither Theorem 1 nor Theorem 2 can the coefficient λ_n in (5) be replaced, in general, by any larger number even for a decreasing sequence $\{a_n\}$. For, let $\lambda_n \equiv \lambda > 0$, let $a_n = 1/n\log n(\log\log n)^2$ for $n \geqq 2$, and let $f(n)$ be defined by $f(1) = 1$ and $f(n+1) - f(n) = 2 + [\lambda n]$ for $n \geqq 1$. Then, for arbitrary $\varepsilon > 0$, we have $a_n \leqq a_{n+1} + (\lambda + \varepsilon)a_{f(n)}$ if n is large enough. But $\Sigma\, a_n < \infty$.

Proof of Theorem 1. If $\lambda_n \equiv 0$ then $a_n \geqq a_1 > 0$ for all n, and the required conclusion is trivial. Assume therefore that $\lambda_{N-1} > 0$. Then by (4), $f(N) > N$.

Now, from (5) we have by iteration that, for $n \geqq 1$, $f(n) < m < f(n+1)$,

$$a_m \geqq \max\{a_{f(n)} - \sum_{r=f(n)}^{f(n+1)-1} \lambda_r a_{f(r)},\ 0\} = b_n$$

say, and hence, by (4), we have that

(7) $$\sum_{m=f(n)}^{f(n+1)-1} a_m \geqq a_{f(n)} + \{f(n+1)-f(n)-1\}b_n \geqq a_{f(n)} + n\lambda_n b_n$$
$$\geqq a_{f(n)} + \sum_{k=N}^{n} \lambda_n b_n.$$

Suppose that $\Sigma\, a_m < \infty$. Summing on both sides of (7) for $N \leqq n < \infty$, we have, after interchanging the order of summations on the right, that

(8) $$\sum_{m=f(N)}^{\infty} a_m \geqq \sum_{n=N}^{\infty} a_{f(n)} + \sum_{k=N}^{\infty}\sum_{n=k}^{\infty} \lambda_n b_n.$$

From the definition of b_n and since $\lambda_n \leqq 1$, it follows that

(9) $$\sum_{n=k}^{\infty} \lambda_n b_n \geqq \sum_{n=k}^{\infty} \lambda_n a_{f(n)} - \sum_{r=f(k)}^{\infty} \lambda_r a_{f(r)} = \sum_{n=k}^{f(k)-1} \lambda_n a_{f(n)}.$$

Further, by (5),

$$\sum_{n=k}^{f(k)-1} \lambda_n a_{f(n)} \geqq a_k - a_{f(k)},$$

whence, from (8) and (9), $\Sigma_{m=f(N)}^{\infty}\, a_m \geqq \Sigma_{k=N}^{\infty}\, a_k$. The last inequality is impossible, since $f(N) > N$ and $a_N > 0$. Therefore $\Sigma\, a_n = \infty$.

Proof of Theorem 2. As in the proof of Theorem 1, we may assume that $\lambda_{N-1} > 0$, so that $f(N) > N$. From (5) we get for $N \leqq n \leqq M$,

$$a_{f(n)} \leqq a_{f(M)+1} + \sum_{k=f(n)}^{f(M)} \lambda_k a_{f(k)}. \tag{10}$$

Multiplying both sides of (10) by $f(n)-f(n-1)$ and summing for $N \leqq n \leqq M$ we obtain

$$\sum_{n=N}^{M} \{f(n)-f(n-1)\}a_{f(n)} \leqq f(M)\ a_{f(M)+1} + \sum_{n=N}^{M} \{f(n)-f(n-1)\} \sum_{k=f(n)}^{f(M)} \lambda_k a_{f(k)}$$

$$\leqq f(M)a_{f(M)+1} + \sum_{k=f(N)}^{f(M)} \lambda_k a_{f(k)} \sum_{\substack{n \geqq N \\ f(n) \leqq k}} \{f(n)-f(n-1)\}$$

$$\leqq f(M)a_{f(M)+1} + \sum_{k=f(N)}^{f(M)} \lambda_k a_{f(k)}\{k-f(N-1)\}.$$

Now by (6), $\lambda_k\{k-f(N-1)\} \leqq \lambda_k(k-1) < f(k)-f(k-1)$, whence

$$\sum_{n=N}^{M} \{f(n)-f(n-1)\}a_{f(n)} < f(M)a_{f(M)+1} + \sum_{k=f(N)}^{f(M)} \{f(k)-f(k-1)\}a_{f(k)}. \tag{11}$$

Suppose now that $\sum a_n < \infty$. Then, since $\{a_n\}$ is a decreasing sequence,

$$na_n \to 0 \quad \text{as} \quad n \to \infty, \tag{12}$$

$$\sum_{n=N}^{\infty} \{f(n)-f(n-1)\}a_{f(n)} \leqq \sum_{n=N}^{\infty} \sum_{\nu=f(n-1)}^{f(n)-1} a_\nu \leqq \sum_{n=1}^{\infty} a_n < \infty. \tag{13}$$

Letting $M \to \infty$ in (11), we get, on account of (12) and (13), that

$$\sum_{n=N}^{\infty} \{f(n)-f(n-1)\}a_{f(n)} \leqq \sum_{k=f(N)}^{\infty} \{f(k)-f(k-1)\}a_{f(k)}.$$

But this is impossible, since $f(N) > N$. Therefore $\sum a_n = \infty$.

The following questions may be of interest:

Given an increasing integer-valued function g, what properties must f have in order that $0 < a_n \leqq a_{g(n)} + a_{f(n)}$ be a divergence criterion?

For what pairs of mappings g, f of the set of integers into itself is it true that, for some integer x, all values $f(x), g(x), f(f(x)), f(g(x)), g(f(x)), g(g(x)), f(f(f(x)))$, $\cdots$ are different?

The second question arises naturally in connection with Post's combinatorial lemma.

Reference

1. K. A. Post, A combinatorial lemma involving a divergence criterion for series of positive terms, this MONTHLY, 77(1970) 1085–1087.