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CONVERGENCE CRITERIA FOR BOUNDED SEQUENCES

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1. Introduction

Let $\{K_n\}$ be a sequence of complex numbers, let

$$K(z) = \sum_{n=0}^{\infty} K_n z^n$$

and let

$$k_0 = K_0,\ k_n = K_n - K_{n-1} \quad (n = 1, 2, \ldots).$$

Let D be the open unit disc $\{z: |z| < 1\}$, let $\bar{D}$ be its closure and let $\partial D = \bar{D} - D$.

The primary object of this paper is to prove the two theorems stated below, the first of which generalises a result of Copson (1).

Theorem 1. *If*

$$\sum_{n=0}^{\infty} |K_n| < \infty, \tag{1}$$

$$K(z) \neq 0 \text{ on } \partial D, \tag{2}$$

and if

$$\{a_n\} \text{ is a bounded sequence} \tag{3}$$

such that, for some positive integer N,

$$\sum_{r=0}^{n} k_r a_{n-r} \geqq 0 \quad (n = N, N+1, \ldots), \tag{4}$$

then $\{a_n\}$ *is convergent.*

In essence, Copson's theorem is the above result with conditions (1) and (2) replaced by the single condition

$$-1 = K_0 < K_1 < \ldots < K_{N-1} < K_N = K_{N+r} = 0 \quad (r = 1, 2, \ldots). \tag{C}$$

If (C) holds, then (1) is trivially satisfied, and $K(z)$ is a polynomial satisfying (2), since $K(1) < 0$ and, for $z = e^{i\theta}$, $0 < \theta < 2\pi$,

$$\text{Re}\,(1-z)K(z) = -\sum_{r=1}^{N} k_r(1 - \cos r\theta) < 0.$$

The next theorem shows that condition (2) is necessary for the validity of Theorem 1 when $K(z)$ is subject to certain additional conditions: in particular, it shows that (2) is necessary when $K(z)$ is analytic on $\bar{D}$ and $K(1) \neq 0$.

Theorem 2. *If $K(z) = p(z)q(z)$ where $p(z)$ is a polynomial and*

$$q(z) = \sum_{n=0}^{\infty} q_n z^n,$$

and if

$$\sum_{n=0}^{\infty} | q_n | < \infty, \tag{5}$$

$$q(z) \neq 0 \text{ on } \bar{D}, \tag{6}$$

$$K(\zeta) = 0,\ \zeta \neq 1,\ |\zeta| = 1, \tag{7}$$

then there is a bounded divergent sequence $\{a_n\}$ and a positive integer N such that

$$\sum_{r=0}^{n} k_r a_{n-r} = 0 \quad (n = N, N+1, \ldots). \tag{8}$$

2. Proof of Theorem 1

By (1), $K(z)$ is analytic on D and continuous on $\bar{D}$. Hence, by (2), $K(z)$ can have at most a finite number of zeros in D; and consequently

$$K(z) = p(z)q(z) \tag{9}$$

where $p(z)$ is a polynomial with no zeros in the complement of D, and $q(z)$ is analytic on D and continuous and non-zero on $\bar{D}$.

Let

$$a(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and let

$$u(z) = q(z)a(z), \tag{10}$$

$$v(z) = p(z)u(z). \tag{11}$$

Since, by (3), $a(z)$ is analytic on D, so also are $u(z)$ and $v(z)$.

Let $\{q_n\}$, $\{u_n\}$, $\{v_n\}$ be the sequences such that

$$q(z) = \sum_{n=0}^{\infty} q_n z^n, \quad u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=0}^{\infty} v_n z^n$$

for all z in D.

Since $v(z) = K(z)a(z)$, we have that

$$v_n = \sum_{r=0}^{n} K_r a_{n-r}$$

and hence, by (1) and (3), that $\{v_n\}$ is bounded. Further, by (4), we have that

$$v_n - v_{n-1} = \sum_{r=0}^{n} k_r a_{n-r} \geqq 0 \quad (n = N, N+1, \ldots). \tag{12}$$

It follows that

$$v_n \to v \tag{13}$$

where v is finite.

We prove next that $\{q_n\}$ satisfies (5), and that

$$u_n \to u \tag{14}$$

where u is finite.

Case (i). $p(z) = cz^m \quad (m = 0, 1, \ldots)$.

It is evident that (5) and (14) hold in this case.

Case (ii). $p(z) = \alpha - z,\ 0 < |\alpha| < 1$.

By (9), $K(\alpha) = 0$ and $q(z) = (\alpha - z)^{-1}K(z)$. Hence

$$\alpha q_n = \sum_{r=0}^{n} \alpha^{r-n} K_r = -\sum_{r=n+1}^{\infty} \alpha^{r-n} K_r,$$

and so, by (1), we have that

$$\sum_{n=0}^{\infty} | q_n | \leqq \sum_{r=1}^{\infty} | K_r | \sum_{n=0}^{r-1} | \alpha |^{r-1-n} \leqq \frac{1}{1 - |\alpha|} \sum_{r=1}^{\infty} | K_r | < \infty.$$

Also, by (11), $v(\alpha) = 0$ and $u(z) = (\alpha - z)^{-1} v(z)$. Hence, by (13), we have that

$$u_n = -\sum_{r=n+1}^{\infty} \alpha^{r-n-1} v_r = -\sum_{r=0}^{\infty} \alpha^r v_{n+1+r} \to -\frac{v}{1-\alpha} \text{ as } n \to \infty.$$

Thus, (5) and (14) hold in Case (ii).

Application of Case (i) followed by repeated applications of Case (ii) establishes (5) and (14) in the remaining case:

$$p(z) = cz^m(\alpha_1 - z)(\alpha_2 - z)\ldots(\alpha_j - z), \quad 0 < |\alpha_1| < 1, \quad 0 < |\alpha_2| < 1, \ldots, 0 < |\alpha_j| < 1.$$

Finally, since $q(z)$ has no zeros on $\bar{D}$ and (5) holds, we have, by the Wiener-Lévy Theorem ((2), p. 246), that there is a sequence $\{c_n\}$ such that

$$\frac{1}{q(z)} = \sum_{n=0}^{\infty} c_n z^n \quad (z \in \bar{D}) \tag{15}$$

and

$$\sum_{n=0}^{\infty} | c_n | < \infty. \tag{16}$$

By (10), $a(z) = u(z)/q(z)$, and hence, by (14) and (15), we have that

$$a_n = \sum_{r=0}^{n} c_r u_{n-r} \to u \sum_{r=0}^{\infty} c_r \quad \text{as } n \to \infty.$$

3. Proof of Theorem 2

Define a sequence $\{a_n\}$ and a function $a(z)$ by

$$a(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{q(z)(\zeta - z)} \quad (z \in D); \tag{17}$$

and let

$$w_n = \sum_{r=0}^{n} k_r a_{n-r},$$

$$w(z) = \sum_{n=0}^{\infty} w_n z^n.$$

Then

$$w(z) = (1-z)K(z)a(z) = \frac{(1-z)p(z)}{\zeta - z}$$

and, by (6) and (7), $\zeta - z$ is a factor of the polynomial $p(z)$. Consequently $w(z)$ is a polynomial, of degree $N-1$ say, and (8) follows.

Further, by the Wiener-Lévy Theorem, hypotheses (5) and (6) imply conditions (15) and (16). Hence, by (17), we have that

$$\zeta^{n+1} a_n = \zeta^n \sum_{r=0}^{n} c_r \zeta^{r-n} \to \frac{1}{q(\zeta)} \quad \text{as } n \to \infty.$$

Since $q(\zeta) \neq 0$, it follows that $\{a_n\}$ is bounded but not convergent.

4. Remarks

1. The proof of Theorem 1 shows that conditions (1) and (2) imply that $K(z)$ must satisfy all the hypotheses of Theorem 2 preceding hypothesis (7).

2. The following theorem is a corollary of Theorems 1 and 2.

Theorem 3. *If $K(z)$ is analytic on $\bar{D}$ and $K(1) \neq 0$, then condition* (2) *is necessary and sufficient for every bounded sequence $\{a_n\}$ satisfying* (4), *for some positive integer N, to be convergent.*

A direct proof of Theorem 3 that avoids the Wiener-Lévy theorem and other complications can readily be constructed from parts of the proofs of Theorems 1 and 2.

3. *Theorem* 1 *remains valid when condition* (4) *is replaced by*

$$\sum_{r=0}^{n} k_r a_{n-r} \in Q \quad (n = N, N+1, \ldots) \tag{18}$$

where Q is any closed quadrant of the plane.

To establish this we need only modify the proof of Theorem 1 to the extent of changing " $\geqq 0$ " in (12) to " $\in Q$ ". Condition (18) is slightly more general than (4) and somewhat more appropriate in the context of complex sequences.

REFERENCES

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