ON RIESZ AND GENERALISED CESÀRO SUMMABILITY

D. BORWEIN

1. *Introduction*

Let $\{\lambda_n\}$ be a strictly increasing unbounded sequence with $\lambda_0 \geqslant 0$, let

$$\mu = p+\delta \quad (p = 0, 1, \ldots;\ 0 \leqslant \delta < 1),$$

and let $\sum_{n=0}^{\infty} a_n$ be an arbitrary series. Write

$$A^{\mu}(w) = \sum_{\lambda_\nu < w} (w-\lambda_\nu)^{\mu} a_\nu,$$

$$\pi_n{}^{\mu}(t) = \begin{cases} (\lambda_{n+1}-t)^{\delta} & (p = 0), \\ (\lambda_{n+1+p}-t)^{\delta} \prod_{i=1}^{p} (\lambda_{n+i}-t) & (p \geqslant 1), \end{cases}$$

$$C_n{}^{\mu} = \sum_{\nu=0}^{n} \pi_n{}^{\mu}(\lambda_\nu)\, a_\nu.$$

The series $\sum a_n$ is said to be summable to s by

(i) the Riesz method (R, λ, μ), if $w^{-\mu} A^{\mu}(w) \to s$ as $w \to \infty$,

(ii) the generalised Cesàro method (C, λ, μ), if $C_n{}^{\mu}/\pi_n{}^{\mu}(0) \to s$.

It is known that the inclusion $(R, \lambda, \mu) \subseteq (C, \lambda, \mu)$ holds for $\mu \geqslant 0$ **[3, 6]**, i.e. every series summable (R, λ, μ) is summable (C, λ, μ) to the same sum; and that the reverse inclusion

$$(C, \lambda, \mu) \subseteq (R, \lambda, \mu) \tag{1}$$

is valid in the cases (a)**[5]** $0 \leqslant \mu \leqslant 1$, (b)**[7]** $\mu = 2, 3, \ldots$, (c)**[1]** $\mu \geqslant 0$, $\lambda_n = n$. Apart from the special case (c), the only known result **[2**; Theorem 4] concerning inclusion (1) for non-integral $\mu > 1$ is that it holds when $1 < \mu < 2$ provided the sequence $\{\lambda_n\}$ satisfies the conditions

$$\frac{\lambda_{n+1}}{\lambda_n} \downarrow$$

and

$$\frac{\lambda_{n+2}-\lambda_{n+1}}{\lambda_{n+1}-\lambda_n} \downarrow. \tag{2}$$

In this paper it is proved, *inter alia*, that inclusion (1) holds in the range $1 < \mu < 2$ if $\lambda_n = \lambda(n)$ $(n \geqslant n_0)$, where λ is a logarithmico-exponential function (see **[4]**) such that $\lambda(x) \to \infty$ as $x \to \infty$ and $\lambda(x-1)/\lambda(x)$ is ultimately increasing. Examples of sequences $\{\lambda_n\}$, satisfying these conditions but not condition (2), are given by $\lambda_n = \log(1+n)$ and $\lambda_n = n^{\alpha}$ $(0 < \alpha < 1)$.

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2. *The main results*

Suppose throughout this section that

$$0 < \delta < 1.$$

In addition to the notations already introduced, we shall also use the following:

$$s_n = \sum_{\nu=0}^{n} a_\nu,$$

$$k_n = \lambda_{n+1} - \lambda_n,$$

$$c_n(t) = -\frac{d}{dt}\{(\lambda_{n+1}-t)(\lambda_{n+2}-t)^\delta - (\lambda_n - t)(\lambda_{n+1}-t)^\delta\} \quad (0 \leqslant t < \lambda_{n+1}), \tag{3}$$

$$c_{n,\nu} = \int_{\lambda_\nu}^{\lambda_{\nu+1}} c_n(t)\,dt \quad (0 \leqslant \nu \leqslant n). \tag{4}$$

Then, as in [2], we have

$$C_n^{1+\delta} - C_{n-1}^{1+\delta} = \sum_{\nu=0}^{n} c_{n,\nu} s_\nu \quad (n \geqslant 1).$$

Note that

$$-\frac{d}{dt}(\lambda_n - t)(\lambda_{n+1}-t)^\delta = (1+\delta)(\lambda_{n+1}-t)^\delta - \delta k_n (\lambda_{n+1}-t)^{\delta-1} \quad (0 \leqslant t < \lambda_{n+1}). \tag{5}$$

For $j = 0, 1, 2, \ldots$, define Ω_j to be the set of continuous, non-negative unboundedly increasing functions λ on $[j, \infty)$ such that $0 < \lambda'(x) < \infty$ on the open set $U_j = \bigcup_{i=j}^{\infty} (i, i+1)$, and

$$\lambda(x-1)/\lambda(x) \text{ is increasing in } [j+1, \infty), \tag{6}$$

$$\lambda(x)\,\lambda'(x-1)/\lambda'(x) \text{ is increasing in } U_{j+1}, \tag{7}$$

$$\lambda'(x-1) = O(\lambda'(x)) \text{ in } U_{j+1}. \tag{8}$$

The first four of the following five theorems should be compared with like-numbered theorems in [2].

THEOREM 1. *If* $\lambda \in \Omega_0$, $\lambda_n = \lambda(n)$ $(n = 0, 1, \ldots)$ *and if*

$$\xi_n > 0, \quad \frac{\lambda_{n+1}^\delta}{\xi_n} \downarrow 0, \tag{9}$$

then

$$c_{n,\nu} > 0 \quad (0 \leqslant \nu \leqslant n), \tag{10}$$

$$\frac{c_{n,\nu}}{c_{n-1,\nu}} \leqslant \frac{c_{n,\nu-1}}{c_{n-1,\nu-1}} \quad (1 \leqslant \nu \leqslant n-1), \tag{11}$$

$$c_{n,0} = o(\xi_n), \tag{12}$$

$$\frac{c_{n,0}}{\xi_n} \leqslant M \frac{c_{r,0}}{\xi_r} \quad (0 \leqslant r \leqslant n, \ M \text{ a positive constant}). \tag{13}$$

Proof. Define $k(u)$ to be the function on $[\lambda_1, \infty)$ such that

$$k(\lambda(x)) = \lambda(x) - \lambda(x-1) \quad (x \geqslant 1); \tag{14}$$

and let

$$\phi = \phi(u, t) = (1+\delta)(u-t)^{\delta} - \delta k(u)(u-t)^{\delta-1},$$

$$\psi = \psi(u, t) = \frac{\partial \phi}{\partial t} \quad (0 \leqslant t < u,\ u \geqslant \lambda_1).$$

Then $k(\lambda_{n+1}) = k_n$ $(n = 0, 1, \ldots)$, and hence, in view of (3) and (5), we have

$$c_n(t) = \phi(\lambda_{n+2}, t) - \phi(\lambda_{n+1}, t) \quad (0 \leqslant t < \lambda_{n+1},\ n \geqslant 0). \tag{15}$$

Differentiating (14) we find that

$$1 - k'(\lambda(x)) = \frac{\lambda'(x-1)}{\lambda'(x)} > 0 \quad (n+1 < x < n+2,\ n = 0, 1, \ldots), \tag{16}$$

and hence that

$$\frac{\partial \phi}{\partial u} = \delta(1 - k'(u) + \delta)(u-t)^{\delta-1} + \delta(1-\delta)\, k(u)(u-t)^{\delta-2} > 0$$

$$(0 \leqslant t < u,\ \lambda_{n+1} < u < \lambda_{n+2}). \tag{17}$$

It follows from (15) and (17) that $c_n(t) > 0$ $(0 \leqslant t < \lambda_{n+1})$, and consequently, by (4), that $c_{n,\nu} > 0$ $(0 \leqslant \nu \leqslant n)$, i.e. conclusion (10) holds.

Differentiating (17) with respect to t, we get

$$\frac{\partial \psi}{\partial u} = \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial u}\right) = \frac{1-\delta}{u-t}\left\{\frac{\partial \phi}{\partial u} + \delta k(u)(u-t)^{\delta-2}\right\} \quad (0 \leqslant t < u,\ \lambda_{n+1} < u < \lambda_{n+2}). \tag{18}$$

Since

$$\frac{c_{n,\nu}}{c_{n-1,\nu}} = \frac{c_n(t_\nu)}{c_{n-1}(t_\nu)} \quad (\lambda_\nu < t_\nu < \lambda_{\nu+1},\ 0 \leqslant \nu \leqslant n-1),$$

inequality (11) will be established if we can shcw that $c_n(t)/c_{n-1}(t)$ decreases as t increases in $(0, \lambda_n)$, and to do this it suffices to prove

$$\frac{c_n'(t)}{c_n(t)} \leqslant \frac{c_{n-1}'(t)}{c_{n-1}(t)} \quad (0 \leqslant t < \lambda_n,\ n \geqslant 1). \tag{19}$$

In view of (15), (17) and (18), we have

$$\frac{c_n'(t)}{c_n(t)} = \frac{\psi(\lambda_{n+2}, t) - \psi(\lambda_{n+1}, t)}{\phi(\lambda_{n+2}, t) - \phi(\lambda_{n+1}, t)} = \left[\frac{\partial \psi}{\partial u}\Big/\frac{\partial \phi}{\partial u}\right]_{u=u_n}$$

$$= \frac{1-\delta}{u_n - t}\left\{1 + \frac{1}{1-\delta+(1-k'(u_n)+\delta)(u_n-t)/k(u_n)}\right\}$$

$$(0 \leqslant t < \lambda_{n+1} < u_n < \lambda_{n+2}). \tag{20}$$

Further, for $x \in U_1$, $\lambda(x) > t \geqslant 0$, we have, by (16), that

$$\left(1-k'(\lambda(x))+\delta\right)\frac{(\lambda(x)-t)^2}{k(\lambda(x))} = \left(\frac{\lambda'(x-1)}{\lambda'(x)}+\delta\right)\frac{(\lambda(x)-t)^2}{\lambda(x)-\lambda(x-1)}$$

$$= \left(\frac{\lambda(x)\,\lambda'(x-1)}{\lambda'(x)}+\delta\lambda(x)\right)\frac{1}{1-\lambda(x-1)/\lambda(x)}\left(1-\frac{t}{\lambda(x)}\right)^2$$

which increases as x increases in U_1, since $\lambda(x)$ increases and satisfies conditions (6) and (7) with $j = 0$. It follows that

$$\left(1-k'(u)+\delta\right)\frac{(u-t)^2}{k(u)}$$

increases as u increases in $(t, \infty) \cap \bigcup_{i=1}^{\infty} (\lambda_i, \lambda_{i+1})$, and (19) is a consequence of this and (20).

It remains to establish conclusions (12) and (13).

In virtue of (4), (15), (16) and (17), we have

$$c_{n,0} = k_0\, c_n(v_n) = k_0\, k_{n+1} \frac{\partial \phi}{\partial u}\bigg]_{u=w_n,\, t=v_n}$$

$$= \delta k_0\, k_{n+1}(w_n-v_n)^{\delta-1}\left\{1-k'(w_n)+\delta+(1-\delta)\frac{k(w_n)}{w_n-v_n}\right\}$$

$$= \delta k_0\, k_{n+1}(\lambda(x_n)-v_n)^{\delta-1}\left\{\frac{\lambda'(x_n-1)}{\lambda'(x_n)}+\delta+(1-\delta)\frac{\lambda(x_n)-\lambda(x_n-1)}{\lambda(x_n)-v_n}\right\}$$

$$(\lambda_0 < v_n < \lambda_1 < \lambda_{n+1} < w_n = \lambda(x_n) < \lambda_{n+2}).$$

Hence, by (6) and (8), there are positive constants B, B_1, B_2, b such that

$$c_{n,0} \leqslant B_1\, k_{n+1}(\lambda_{n+1}-\lambda_1)^{\delta-1}$$

$$\leqslant B_2\, k_{n+1}\, \lambda_{n+1}^{\delta-1}$$

$$\leqslant B\frac{k_{n+1}}{\lambda_{n+2}}\lambda_{n+1}^{\delta} \quad (n \geqslant 1), \tag{21}$$

and

$$c_{n,0} \geqslant b k_{n+1}\, \lambda_{n+2}^{\delta-1}$$

$$\geqslant b\frac{k_{n+1}}{\lambda_{n+2}}\lambda_{n+1}^{\delta} \quad (n \geqslant 0). \tag{22}$$

It follows from (21) and (9) that

$$c_{n,0} = O\left(\xi_n \frac{k_{n+1}}{\lambda_{n+2}}\frac{\lambda_{n+1}^{\delta}}{\xi_n}\right) = o(\xi_n),$$

which is conclusion (12).

Finally, by (6), (9), (21) and (22), we have, for $0 \leqslant r \leqslant n-1$,

$$\frac{c_{n,0}}{\xi_n} \leqslant B \frac{k_{n+1}}{\lambda_{n+2}} \frac{\lambda_{n+1}^{\delta}}{\xi_n} \leqslant B \frac{k_{r+1}}{\lambda_{r+2}} \frac{\lambda_{r+1}^{\delta}}{\xi_r} \leqslant \frac{B}{b} \frac{c_{r,0}}{\xi_r},$$

i.e. conclusion (13) holds. This completes the proof of Theorem 1.

THEOREM 2. *If* $\lambda \in \Omega_0$, $\lambda_n = \lambda(n)$ $(n = 0, 1, \ldots)$,

$$\xi_n > 0, \quad \frac{\lambda_{n+1}^{\delta}}{\xi_n} \downarrow 0, \quad \frac{k_n}{\xi_n} \downarrow,$$

$$C_n^{1+\delta} = o(\xi_n),$$

then

$$s_n = o\left(\frac{\xi_n}{k_n^{1+\delta}}\right),$$

$$A^{\delta}(w) = o\left(\frac{\xi_\chi}{k_\chi}\right) \quad (w \to \infty),$$

$$A^{1+\delta}(w) = o(\xi_\chi) \quad (w \to \infty),$$

where $\chi = \chi(w)$ *is the integer such that* $\lambda_\chi < w \leqslant \lambda_{\chi+1}$.

Proof. Essentially the same as the proof of Theorem 2 in [**2**], with the present Theorem 1 and identity (20) respectively replacing Theorem 1 and identity (21) of [**2**].

The next theorem is a simple consequence of Theorem 2 (see the proof of Theorem 3 in [**2**]).

THEOREM 3. *If* $\lambda \in \Omega_0$, $\lambda_n = \lambda(n)$ $(n = 0, 1, \ldots)$ *and* $C_n^{1+\delta} = o(\lambda_{n+1} \lambda_{n+2}^{\delta})$, *then* $A^{1+\delta}(w) = o(w^{1+\delta})$ $(w \to \infty)$.

THEOREM 4. *If* $\lambda \in \Omega_j$, $\lambda_n = \lambda(n)$ $(n = j, j+1, \ldots)$, *and* $1 < \mu < 2$, *then*

$$(C, \lambda, \mu) \subseteq (R, \lambda, \mu).$$

Proof. Case (i). $j = 0$. Since both (C, λ, μ) and (R, λ, μ) are regular methods of summability (see [**2**; Lemma 4]), this case follows immediately from Theorem 3.

Case (ii). $j = 1, 2, \ldots$. Let

$$\lambda^*(x) = \lambda(x+j) \quad (x \geqslant 0), \quad \lambda_n^* = \lambda^*(n) = \lambda_{n+j} \quad (n = 0, 1, \ldots).$$

Then $\lambda^* \in \Omega_0$. Further, if $\sum_{n=0}^{\infty} a_n$ is summable (C, λ, μ) to s, then $\sum_{n=0}^{\infty} a_{n+j}$ is summable (C, λ^*, μ) to $s - s_{j-1}$ and hence, by case (i), is summable (R, λ^*, μ) to $s - s_{j-1}$, so that $\sum_{n=0}^{\infty} a_n$ is summable (R, λ, μ) to s. Thus case (ii) is established.

THEOREM 5. *If* $\lambda_n = \lambda(n)$ $(n = n_0, n_0+1, \ldots)$, *where* λ *is a logarithmico-exponential function such that* $\lambda(x) \to \infty$ *as* $x \to \infty$ *and* $\lambda(x-1)/\lambda(x)$ *is ultimately increasing, and* $1 < \mu < 2$, *then*

$$(C, \lambda, \mu) \subseteq (R, \lambda, \mu).$$

Proof. By known properties of logarithmico-exponential functions (see [**4**]) $\lambda(x)$ is ultimately increasing. Also, since $\lambda(x-1)/\lambda(x)$ is ultimately increasing, we have that, for $x \geqslant x_0$ say,

$$\lambda(x-1) > 0,\ \lambda'(x-1) > 0,\ \frac{\lambda'(x-1)}{\lambda(x-1)} - \frac{\lambda'(x)}{\lambda(x)} \geqslant 0,$$

so that

$$\lambda(x)\frac{\lambda'(x-1)}{\lambda'(x)} \geqslant \lambda(x-1) \to \infty \text{ as } x \to \infty.$$

The logarithmico-exponential function $\lambda(x)\,\lambda'(x-1)/\lambda'(x)$ is thus ultimately increasing. Further, since

$$0 < \lim_{x\to\infty} \frac{\lambda(x-1)}{\lambda(x)} \leqslant 1,$$

we have that [**4**; p. 34]

$$\frac{\lambda'(x-1)}{\lambda'(x)} = O(1) \quad (x \geqslant x_0).$$

It follows that $\lambda \in \Omega_j$ for j sufficiently large, and Theorem 5 is thus a consequence of Theorem 4.

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Department of Mathematics,
University of Western Ontario,
London, Canada.