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1. Introduction. Suppose throughout that $\alpha > 0$, β is real, and N is a non-negative integer such that $\alpha N + \beta > 0$. A series $\sum_0^\infty a_n$ of complex terms is said to be summable (B, α, β) to l if, as $x \to \infty$,

$$\alpha e^{-x} \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)} \to l,$$

where $s_n = a_0 + a_1 + \ldots + a_n$. The Borel-type summability method (B, α, β) is regular, i.e., all convergent series are summable (B, α, β) to their natural sums; and $(B, 1, 1)$ is the standard Borel exponential method B.

Our aim in this paper is to prove the following Tauberian theorem.

THEOREM. *If*

(i) $\rho \geqq -\frac{1}{2}$, $a_n = o(n^\rho)$, *and*

(ii) $\sum_0^\infty a_n$ *is summable* (B, α, β) *to* l,

then the series is summable by the Cesàro method $(C, 2\rho + 1)$ *to* l.

The case $\alpha = \beta = 1$ of the theorem is known (**3**, Theorem 147), and the case $\alpha > 1$ is a consequence of this case and the following established result (**1**, result (I); **2**, Lemma 4).

(I) *If* $\alpha > \gamma > 0$ *and, for any non-negative integer* $M > -\delta/\gamma$,

$$\sum_{n=M}^{\infty} \frac{a_n x^n}{\Gamma(\gamma n + \delta)}$$

is convergent for all x, *then hypothesis* (ii) *implies that* $\sum_0^\infty a_n$ *is summable* (B, γ, δ) *to* l.

The proof in this paper of the theorem, however, makes no appeal to result (I) and is valid for all $\alpha > 0$.

The theorem remains true if hypothesis (ii) is replaced by

(ii)′ $\sum_0^\infty a_n$ *is summable* (B', α, β) *to* l,

by which it is meant that, as $y \to \infty$,

$$\int_0^y e^{-x}\, dx \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)} \to l - s_{N-1} \qquad (s_{-1} = 0).$$

This is a consequence of the following known result (**2**, Theorem 2).

(II) *A series is summable* $(B, \alpha, \beta + 1)$ *to* l *if and only if it is summable* (B', α, β) *to* l.

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2. Preliminary results.

LEMMA 1. (i) $x^v \Gamma(y - v) \geqq \Gamma(y)$ *if* $x \geqq y > v \geqq 0$,

(ii) $x^v \Gamma(y - v) \leqq \Gamma(y)$ *if* $v \geqq 0$, $0 < x \leqq y - v - 1$.

Proof. Let $\psi(v) = x^v \Gamma(y - v)$. In case (i), we have, by standard results (**4**, §§ 12.3, 12.31):

$$\begin{aligned}
\frac{\psi'(v)}{\psi(v)} &= \log x - \frac{\Gamma'(y-v)}{\Gamma(y-v)} \\
&= \log x - \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-(y-v)t}}{1 - e^{-t}}\right] dt \\
&\geqq \log x - \int_0^\infty \frac{e^{-t} - e^{-(y-v)t}}{t}\, dt \\
&= \log x - \log(y - v) \\
&\geqq 0,
\end{aligned}$$

so that $\psi(v) \geqq \psi(0)$, as required.

Similarly, in case (ii) we have:

$$\begin{aligned}
\frac{\psi'(v)}{\psi(v)} &= \log x - \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-(y-v-1)t}}{e^t - 1}\right] dt \\
&\leqq \log x - \int_0^\infty \frac{e^{-t} - e^{-(y-v-1)t}}{t}\, dt \\
&= \log x - \log(y - v - 1) \\
&\leqq 0,
\end{aligned}$$

from which the required inequality follows.

LEMMA 2 (cf. **3**, Theorem 137). *Let* $x > 0$, *let*

$$u_n = u_n(x) = \alpha e^{-x} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta)} \qquad (n = N, N + 1, \ldots),$$

and let

$$0 < \delta < 1/\alpha, \quad \gamma = \tfrac{1}{3}(\alpha\delta)^2, \quad \tfrac{1}{2} < \zeta < \tfrac{2}{3}, \quad 0 < \eta < 2\zeta - 1.$$

Then

(a) $\displaystyle\sum_{n=N}^{\infty} u_n \to 1$ *as* $x \to \infty$;

(b) $u_n \leqq u_{n+1}$ *when* $n \leqq \dfrac{x}{\alpha} - \dfrac{\beta}{\alpha} - 1$, *and*

$u_{n+1} \leqq u_n$ *when* $n \geqq \dfrac{x}{\alpha} + \dfrac{1 - \beta}{\alpha}$;

(c) $\displaystyle\sum_{|n - x/\alpha| > \delta x} u_n = O(e^{-\gamma x})$;

(d) $\displaystyle\sum_{|n - x/\alpha| > x^\zeta} u_n = O(e^{-x^\eta})$;

(e) $u_n = \dfrac{\alpha}{\sqrt{(2\pi x)}} e^{-\alpha^2 (n - x/\alpha)^2/2x} \{1 + O(x^{3\zeta-2})\}$ *when* $\left|n - \dfrac{x}{\alpha}\right| \leqq x^\zeta$.

Proof. Part (a). This result is well known (see **1**, p. 130).

Part (b). Since

$$\frac{u_{n+1}}{u_n} = \frac{x^{\alpha}\Gamma(\alpha n + \beta)}{\Gamma(\alpha n + \beta + \alpha)},$$

the required results follow from Lemma 1 with $v = \alpha$, $y = \alpha n + \beta + \alpha$.

Part (c). Let n_1 and n_2 be the integers such that

$$n_1 > \frac{x}{\alpha} + \delta x \geqq n_1 - 1 \quad \text{and} \quad n_2 < \frac{x}{\alpha} - \delta x \leqq n_2 + 1.$$

By Stirling's theorem, we have:

$$\Gamma(\alpha n + \beta) = (2\pi)^{1/2} e^{-\alpha n} (\alpha n)^{\alpha n + \beta - 1/2} \left\{1 + O\left(\frac{1}{n}\right)\right\},$$

and hence

$$u_{n_1} = O\left[\frac{e^{-x} x^{\alpha n_1 + \beta - 1}}{e^{-\alpha n_1} (\alpha n_1)^{\alpha n_1 + \beta - 1/2}}\right] = O\left(e^{\alpha n_1 - x} x^{-1/2} \left(\frac{x}{\alpha n_1}\right)^{\alpha n_1 + \beta - 1/2}\right)$$

$$= O\left(e^{\alpha\delta x} \left(\frac{x}{\alpha n_1}\right)^{\alpha n_1}\right) = O(e^{\alpha\delta x - \alpha n_1 \log(\alpha n_1/x)})$$

$$= O(e^{\alpha\delta x - (x + \alpha\delta x)\log(1+\alpha\delta)}) = O(e^{-\Delta_1 x}),$$

where

$$\Delta_1 = -\alpha\delta + (1 + \alpha\delta)\log(1 + \alpha\delta) = \frac{(\alpha\delta)^2}{1\cdot 2} - \frac{(\alpha\delta)^3}{2\cdot 3} + \frac{(\alpha\delta)^4}{3\cdot 4} - \ldots > \frac{1}{3}(\alpha\delta)^2.$$

Similarly,

$$u_{n_2} = O(e^{-\Delta_2 x}),$$

where

$$\Delta_2 = \alpha\delta + (1 - \alpha\delta)\log(1 - \alpha\delta) = \frac{(\alpha\delta)^2}{1\cdot 2} + \frac{(\alpha\delta)^3}{2\cdot 3} + \ldots > \frac{1}{2}(\alpha\delta)^2.$$

Next, for $r \geqq 0$, $x \geqq 2(1 - \beta)/\alpha\delta$, we have, by Lemma 1 (ii) with $v = \alpha r$, $y = \alpha n_1 + \beta + \alpha r$:

$$\frac{u_{n_1+r}}{u_{n_1}} = \frac{x^{\alpha r}\Gamma(\alpha n_1 + \beta)}{\Gamma(\alpha n_1 + \beta + \alpha r)} \leqq (1 + \tfrac{1}{2}\alpha\delta)^{-\alpha r},$$

since $0 < x(1 + \frac{1}{2}\alpha\delta) \leqq \alpha n_1 + \beta - 1$. It follows that

$$\sum_{n - x/\alpha > \delta x} u_n = \sum_{r=0}^{\infty} u_{n_1+r} \leqq u_{n_1} \sum_{r=0}^{\infty} (1 + \tfrac{1}{2}\alpha\delta)^{-r} = O(e^{-\Delta_1 x}) = O(e^{-\gamma x}).$$

Finally, by part (b), we have:

$$\sum_{n - x/\alpha < -\delta x} u_n = \sum_{n < x/\alpha - \delta x} u_n \leqq x u_{n_2} = O(x e^{-\Delta_2 x}) = O(e^{-\gamma x}).$$

This completes the proof of part (c). We shall prove part (e) before part (d).

Part (e). Let $h = n - x/\alpha$, so that $|h| \leqq x^{\zeta}$.

By Stirling's theorem, we have:

$$\log\Gamma(\alpha n + \beta) = \tfrac{1}{2}\log 2\pi - \alpha n + (\alpha n + \beta - \tfrac{1}{2})\log \alpha n + O\left(\frac{1}{n}\right)$$

$$= \tfrac{1}{2}\log 2\pi - x - \alpha h$$

$$+ (\alpha h + x + \beta - \tfrac{1}{2})\log(\alpha h + x) + O\left(\frac{1}{x}\right)$$

$$= \tfrac{1}{2}\log 2\pi - x - \alpha h + (\alpha n + \beta - \tfrac{1}{2})\log x$$

$$+ (\alpha h + x + \beta - \tfrac{1}{2})\left\{\frac{\alpha h}{x} - \frac{\alpha^2 h^2}{2x^2} + O\left(\frac{|h|^3}{x^3}\right)\right\} + O\left(\frac{1}{x}\right)$$

$$= \tfrac{1}{2}\log 2\pi - x - \alpha h + (\alpha n + \beta - \tfrac{1}{2})\log x + \alpha h + \frac{\alpha^2 h^2}{2x}$$

$$+ O\left(\frac{1}{x}\right) + O\left(\frac{|h|}{x}\right) + O\left(\frac{|h|^3}{x^2}\right)$$

$$= \tfrac{1}{2}\log 2\pi - x + (\alpha h + x + \beta - \tfrac{1}{2})\log x + \frac{\alpha^2 h^2}{2x} + O(x^{3\zeta - 2})$$

since $\frac{1}{2} < \zeta < \frac{2}{3}$ and $|h| \leqq x^{\zeta}$.

Consequently,

$$\log u_n = \log \alpha - x + (\alpha n + \beta - 1)\log x - \log\Gamma(\alpha n + \beta)$$

$$= \tfrac{1}{2}\log\frac{\alpha^2}{2\pi x} - \frac{\alpha^2 h^2}{2x} + O(x^{3\zeta - 2}),$$

and therefore

$$u_n = \frac{\alpha}{\sqrt{(2\pi x)}} e^{-\alpha^2 h^2/2x}\{1 + O(x^{3\zeta - 2})\},$$

as required.

Part (d). Since $e^{-\gamma x} = O(e^{-x^{\eta}})$, it suffices, in view of Part (c), to prove that

$$\sum_{\delta x \geqq |n - x/\alpha| > x^{\zeta}} u_n = O(e^{-x^{\eta}}).$$

By Parts (b) and (e), the largest term in this sum is $O(e^{-\alpha^2 x^{2\zeta - 1}/2})$, and the required estimate is an immediate consequence.

3. Cesàro sums. In this section we prove some lemmas about the Cesàro sums s_n^{λ} of a given series $\sum_0^{\infty} a_n$. These are defined by the formula:

$$s_n^{\lambda} = \sum_{\nu=0}^{n} \binom{\nu + \lambda}{\nu} a_{n-\nu},$$

so that $s_n^{-1} = a_n$, $s_n^0 = s_n = a_0 + a_1 + \ldots + a_n$, and generally,

$$s_n^{\lambda+\delta} = \sum_{\nu=0}^{n} \binom{\nu + \delta - 1}{\nu} s_{n-\nu}^{\lambda}.$$

LEMMA 3 (cf. **3**, Theorem 146). *If* $k > 0$,

(1) $$\phi_k(x) = \alpha^k \sum_{n=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(\alpha n + k)} \frac{x^{\alpha n}}{n!},$$

$\sum_{n=N}^{\infty} (a_n t^{\alpha n}/\Gamma(\alpha n + \beta))$ *is convergent for all positive* t, *and* $a_n = 0$ *for* $n < N$, *then, for* $x > 0$,

(2) $$\alpha^k \sum_{n=N}^{\infty} s_n^k \frac{x^{\alpha n+\beta+k-1}}{\Gamma(\alpha n+\beta+k)} = \frac{1}{\Gamma(k)} \int_0^x (x-t)^{k-1} \phi_k(x-t)\, dt \sum_{n=N}^{\infty} s_n \frac{t^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}.$$

Proof. The convergence of $\sum_{n=N}^{\infty} (a_n t^{\alpha n}/\Gamma(\alpha n+\beta))$ for all positive t is equivalent to the convergence of $\sum_{n=N}^{\infty} (s_n t^{\alpha n}/\Gamma(\alpha n+\beta))$ for all positive t (**2**, Lemma 4). The right-hand side of (2) is thus equal to

$$\frac{\alpha^k}{\Gamma(k)} \int_0^x (x-t)^{k-1}\, dt \sum_{n=N}^{\infty} \frac{s_n t^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m+k)}{\Gamma(\alpha m+k)} \frac{(x-t)^{\alpha m}}{m!}$$

$$= \frac{\alpha^k}{\Gamma(k)} \sum_{n=N}^{\infty} \frac{s_n}{\Gamma(\alpha n+\beta)} \sum_{m=0}^{\infty} \frac{\Gamma(m+k)}{\Gamma(\alpha m+k) m!} \int_0^x t^{\alpha n+\beta-1} (x-t)^{\alpha m+k-1}\, dt$$

$$= \frac{\alpha^k}{\Gamma(k)} \sum_{n=N}^{\infty} s_n \sum_{m=0}^{\infty} \frac{\Gamma(m+k) x^{\alpha n+\alpha m+\beta+k-1}}{m!\Gamma(\alpha n+\alpha m+\beta+k)}$$

$$= \alpha^k \sum_{n=N}^{\infty} s_n \sum_{m=N}^{\infty} \binom{m-n+k-1}{m-n} \frac{x^{\alpha m+\beta+k-1}}{\Gamma(\alpha m+\beta+k)}$$

$$= \alpha^k \sum_{m=N}^{\infty} \frac{x^{\alpha m+\beta+k-1}}{\Gamma(\alpha m+\beta+k)} \sum_{n=N}^{m} \binom{m-n+k-1}{m-n} s_n$$

$$= \alpha^k \sum_{m=N}^{\infty} \frac{x^{\alpha m+\beta+k-1}}{\Gamma(\alpha m+\beta+k)} s_m^k,$$

as required.

LEMMA 4. *If* $k \geq 0$ *and* $\sum_0^{\infty} a_n$ *is summable* (B, α, β) *to* l, *then*

(3) $$\Gamma(k+1)\alpha^{k+1} e^{-x} \sum_{n=N}^{\infty} s_n^k \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta+k)} \to l \quad \text{as } x \to \infty.$$

Proof. The case $k = 0$ is immediate. Suppose that $k > 0$. If $\sum_0^{\infty} a_n$ is summable (C, k) to l, i.e. if

$$s_n^k \sim \frac{n^k l}{\Gamma(k+1)} \quad \text{as } n \to \infty,$$

then

$$\frac{\alpha^k s_n^k \Gamma(k+1)}{\Gamma(\alpha n+\beta+k)} \sim \frac{l}{\Gamma(\alpha n+\beta)} \quad \text{as } n \to \infty,$$

and (3) follows by the regularity of the (B, α, β) method.

There is, therefore, no loss in generality in assuming that

$$a_n = 0 \quad \text{for} \quad n < N.$$

Then, by Lemma 3, it suffices to prove that

(4) $$kx^{-k} \int_0^x (x-t)^{k-1} \phi_k(x-t) e^{-(x-t)} \sigma(t)\, dt \to l \quad \text{as } x \to \infty,$$

where

$$\sigma(t) = \alpha e^{-t} \sum_{n=N}^{\infty} s_n \frac{t^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}.$$

By hypothesis, we have:

(5) $$\sigma(t) \to l \quad \text{as} \quad t \to \infty.$$

Further, since

$$\frac{\alpha^k \Gamma(n+k)}{\Gamma(\alpha n+k) n!} \sim \frac{\alpha}{\Gamma(\alpha n+1)} \quad \text{as } n \to \infty,$$

we have by (1) and the regularity of the $(B, \alpha, 1)$ method, that

(6) $$e^{-x}\phi_k(x) \to 1 \quad \text{as} \quad x \to \infty.$$

A straightforward application of a standard result (**3**, Theorem 6) yields (4) as a consequence of (5) and (6).

LEMMA 5. *If* $\sum_0^{\infty} a_n$ *is summable* (B, α, β) *to* 0 *and*

(7) $$s_n^{k-\mu} = o(n^{\lambda}) \qquad (k \geqq 0,\ 0 < \mu \leqq 1,\ \lambda > -1,\ \lambda + \mu > 0),$$

then

(8) $$s_n^k = o(n^k) + o(n^{\lambda+\mu/2}).$$

Proof. It follows from (7), by a known result (**3**, Theorem 144), that

(9) $$s_n^k = o(n^{\lambda+\mu}),$$

and that, if $0 < H < 1/\alpha$ and $|n - x/\alpha| < Hx$, then

(10) $$s_n^k - s_{[x/\alpha]}^k = o\{(|n - x/\alpha|^{\mu} + 1)x^{\lambda}\}$$

uniformly as $x \to \infty$. Let $\frac{1}{2} < \zeta < \frac{2}{3}$, and write

$$\alpha e^{-x} \sum_{n=N}^{\infty} (s_n^k - s_{[x/\alpha]}^k) \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta+k)} = \alpha e^{-x}\left[\sum_{N \leqq n < x/\alpha - x^{\zeta}} + \sum_{x/\alpha - x^{\zeta} \leqq n \leqq x/\alpha + x^{\zeta}} + \sum_{n > x/\alpha + x^{\zeta}}\right] = S_1 + S_2 + S_3.$$

Then

$$S_1 + S_2 + S_3 + \alpha e^{-x} s_{[x/\alpha]}^k \sum_{n=N}^{\infty} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta+k)} = \alpha e^{-x} \sum_{n=N}^{\infty} s_n^k \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta+k)} = o(1) \quad \text{as } x \to \infty$$

by Lemma 4, and hence we have:

(11) $$S_1 + S_2 + S_3 + x^{-k}s^k_{[x/\alpha]}(1 + o(1)) = o(1) \quad \text{as} \quad x \to \infty.$$

Next, by (9) and Lemma 2(d),

(12) $$S_1 + S_3 = O\left[e^{-x} \sum_{N \leq n < x/\alpha - x^\zeta} x^{\lambda+\mu} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta + k)}\right] + O\left[e^{-x} \sum_{n > x/\alpha + x^\zeta} n^{\lambda+\mu} \frac{x^{\alpha n+\beta-1}}{\Gamma(\alpha n + \beta + k)}\right]$$
$$= O\left[x^{\lambda+\mu-k}e^{-x} \sum_{N \leq n < x/\alpha - x^\zeta} \frac{x^{\alpha n+\beta+k-1}}{\Gamma(\alpha n + \beta + k)}\right] + O\left[x^{\lambda+\mu-k}e^{-x} \sum_{n > x/\alpha + x^\zeta} \frac{x^{\alpha n+\beta+k-\lambda-\mu-1}}{\Gamma(\alpha n + \beta + k - \lambda - \mu)}\right]$$
$$= O(x^{\lambda+\mu-k}e^{-x^\eta}) \qquad (0 < \eta < 2\zeta - 1)$$
$$= o(1) \quad \text{as } x \to \infty.$$

Further, by (10) and Lemma 2(e),

(13) $$S_2 = o\left[x^{\lambda-k}e^{-x} \sum_{|n - x/\alpha| \leq x^\zeta} \left(\left|n - \frac{x}{\alpha}\right|^\mu + 1\right) \frac{x^{\alpha n+\beta+k-1}}{\Gamma(\alpha n + \beta + k)}\right]$$
$$= o\left[x^{\lambda-k} \sum_{|h_n| \leq x^\zeta} (|h_n|^\mu + 1) \frac{\alpha}{\sqrt{(2\pi x)}} e^{-\alpha^2 h_n{}^2/2x}\right] \qquad (h_n = n - x/\alpha)$$
$$= o\left(x^{\lambda-k-1/2} \int_{-\infty}^{\infty} (|t|^\mu + 1)e^{-\alpha^2 t^2/2x}\, dt\right)$$
$$= o(x^{\lambda-k+\mu/2}) + o(x^{\lambda-k})$$
$$= o(x^{\lambda-k+\mu/2}) \quad \text{as } x \to \infty.$$

It follows from (11), (12), and (13) that

$$s^k_{[x/\alpha]}(1 + o(1)) = o(x^k) + o(x^{\lambda+\mu/2}) \quad \text{as } x \to \infty,$$

and the required conclusion (8) is an immediate consequence.

4. Proof of the theorem. Suppose, without loss of generality, that $l = 0$. By hypothesis (i), we have that (7) holds with $k = 0$, $\mu = 1$, and $\lambda = \rho$. Hence, by Lemma 5, we have:

(14) $$s_n = s_n^0 = o(n^{\rho+1/2}),$$

since $\rho + \frac{1}{2} \geqq 0$.

Suppose that $m\mu = 2\rho + 1$, where m is an integer and $0 < \mu \leqq 1$. We shall prove that

(15) $$s_n^{r\mu} = o(n^{\rho+1/2+r\mu/2})$$

for $r = 0, 1, \ldots, m$. By (14), we see that (15) holds for $r = 0$. Assume that it holds for a given $r < m$, so that (7) holds with

$$k = (r + 1)\mu, \qquad \lambda = \rho + \tfrac{1}{2} + \tfrac{1}{2}r\mu.$$

Since

$$\tfrac{1}{2}(r + 1)\mu \leqq \tfrac{1}{2}m\mu = \rho + \tfrac{1}{2},$$

it follows, by Lemma 5, that

$$s_n^{(r+1)\mu} = o(n^{(r+1)\mu}) + o(n^{\rho+1/2+(r+1)\mu/2}) = o(n^{\rho+1/2+(r+1)\mu/2}),$$

which is (15) with $r + 1$ replacing r.

Hence, (15) holds for $r = 0, 1, \ldots, m$; in particular, the case $r = m$ yields:

$$s_n^{2\rho+1} = o(n^{2\rho+1}),$$

i.e. $\sum_0^\infty a_n$ is summable $(C, 2\rho + 1)$ to 0.

This completes the proof of the theorem.

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