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On Riesz and Generalised Cesàro Summability of Arbitrary Positive Order

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It is the purpose of this paper to give a concise account[1] of inclusion relations involving Riesz and generalised Cesàro summability, and also to finalise one aspect of this problem by proving a best-possible inclusion theorem.

For κ a non-negative real number, $\{\lambda_n\}$ a strictly increasing unbounded real sequence with $\lambda_0 \geqq 0$, the Riesz sum of an arbitrary real or complex series $\sum a_\nu$ is

$$A^\kappa(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^\kappa a_\nu \qquad (\omega \geqq 0);$$

the series $\sum a_\nu$ is summable by the Riesz method (R, λ, κ) to s if $\omega^{-\kappa} A^\kappa(\omega) \to s$ as $\omega \to \infty$. When $\omega \to \infty$ through the sequence $\{\lambda_n\}$, we obtain the definition of "discrete" Riesz summability (R^*, λ, κ), and we may then relax the restriction on κ to $\kappa > -1$; thus $\sum a_\nu$ is summable (R^*, λ, κ) to s if $\lambda_n^{-\kappa} A^\kappa(\lambda_n) \to s$. When RIESZ [16] introduced his "typical means", he found that the method with the discrete matrix, denoted here by (R^*, λ, κ), had, for $\lambda_n = n$ and for higher values of κ, properties totally unlike those of the Cesàro method (C, κ), but that the close relationship with (C, κ) could be restored by introducing the continuous parameter ω. However, this causes difficulties in many instances (particularly in inclusion theorems and summability factors) because the "continuous" (R, λ, κ) method has no inverse; this has led to the recent introduction of related generalised Cesàro methods.

Such a method $(\bar{C}, \lambda, \kappa)$ was first defined by JURKAT [8], who proved a number of properties of this method, subject to various restrictions on the sequence $\{\lambda_n\}$: let

$$\kappa = p + \delta \qquad (p = 0, 1, 2, \ldots;\ 0 < \delta \leqq 1),$$

$$\bar{C}_n^0 = s_n = \sum_{\nu=0}^{n} a_\nu, \qquad \bar{C}^\kappa[s_n] = \bar{C}_n^\kappa,$$

and $\bar{C}^\kappa[1]$ be obtained from $\bar{C}_n^\kappa$ by putting $s_n = 1$ for every n, or (equivalently) $a_0 = 1$, $a_\nu = 0$ $(\nu \neq 0)$;

$$\bar{C}_n^\kappa = \frac{1}{(p+1)\,\Gamma(\delta+1)} \sum_{\nu=0}^{n} \Delta_\nu (\lambda_{n+1} - \lambda_\nu)^\delta \cdot \frac{\lambda_{\nu+p+1} - \lambda_\nu}{\lambda_{\nu+1} - \lambda_\nu}\, \bar{C}_\nu^p.$$

[1] This account has been given by D.C. RUSSELL in a talk at the International Congress of Mathematicians, Moscow, August 1966.

The generalised Cesàro mean is then defined by

$$\bar{t}_n^0 = s_n, \qquad \bar{t}_n^\kappa = \bar{C}^\kappa[s_n]/\bar{C}^\kappa[1],$$

and $\sum a_\nu$ is summable $(\bar{C}, \lambda, \kappa)$ to s if $\bar{t}_n^\kappa \to s$.

However, we have found that a different definition of the summability method for non-integral κ yields results with fewer restrictions on $\{\lambda_n\}$ (in some cases with no restrictions). Accordingly we define the generalised Cesàro mean

$$\dot{t}_n^0 = s_n, \qquad \dot{t}_n^\kappa = \dot{t}_n^{p+\delta} = \sum_{\nu=0}^{n} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_\nu}{\lambda_{n+p}}\right) \left(1 - \frac{\lambda_\nu}{\lambda_{n+p+1}}\right)^\delta a_\nu;$$

the series $\sum a_\nu$ is summable $(\dot{C}, \lambda, \kappa)$ to s if $\dot{t}_n^\kappa \to s$. A definition of $(\dot{C}, \lambda, -1)$ has been given by MADDOX [12].

Several authors have investigated inclusion relations between the summability methods defined above. While there are also a number of Tauberian results, we consider here only full inclusion, in the sense that no restriction on the series $\sum a_\nu$ (other than its summability) is postulated. Results on absolute summability are also omitted. We write $A \subseteq B$ to mean that each series summable $-A$ is also summable $-B$ to the same value; if at the same time we also have $B \subseteq A$ then we write $A \sim B$. We use $A \subset B$ to mean that B is strictly stronger than A, and denote convergence (the identity transformation) by I; we denote throughout

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}, \qquad \Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n).$$

(a) *Relations between* (C, λ, κ) *and* $(\bar{C}, \lambda, \kappa)$.

The two definitions coincide when $\lambda_0 = 0$ and either $0 \leqq \kappa \leqq 1$ or $\kappa = p$ (a non-negative integer); consequently, for any $\{\lambda_n\}$,

$$(C, \lambda, \kappa) \sim (\bar{C}, \lambda, \kappa) \qquad (0 \leqq \kappa \leqq 1),$$

$$(C, \lambda, p) \sim (\bar{C}, \lambda, p) \qquad (p = 0, 1, 2, \ldots).$$

(b) *Relations involving* (R^*, λ, κ).

$(R, \lambda, \kappa) \subseteq (R^*, \lambda, \kappa)$	$(\kappa \geqq 0)$	(trivial),
$(R^*, \lambda, \kappa) \subseteq (R, \lambda, \kappa)$	$(0 \leqq \kappa \leqq 1)$	JURKAT [7],
$(R^*, \lambda, \kappa) \equiv (C, \lambda, \kappa)$	$(0 \leqq \kappa \leqq 1)$	(definition),
$(R^*, \lambda, \kappa) \subseteq (R, \lambda, \kappa)$	if $1 < \kappa < \log 3/\log 2 = 1 \cdot 58 \ldots$ and if $\lambda_{n+1}/\lambda_n \searrow$, $\Delta\lambda_{n+1}/\Delta\lambda_n \to 1$	PEYERIMHOFF [15],
$(R^*, \lambda, 2) \subseteq (R, \lambda, 2) \Rightarrow (R, \lambda, 2) \sim I$		KUTTNER [11],
$\Lambda_n = O(1) \Rightarrow (R^*, \lambda, 2) \sim I$		KUTTNER [10],
$\kappa > 2 \Rightarrow I \subset (R^*, c^n, \kappa)$	for some $c = c(\kappa) > 1$	KUTTNER [10].

(c) *The case* $\lambda_n = n$.

$(R, n, \kappa) \sim (C, \kappa)$	$(\kappa \geqq 0)$	RIESZ [17],
$(R^*, n, \kappa) \sim (C, \kappa)$	$-1 < \kappa \leqq 1$	RIESZ [18]
	$1 < \kappa < 2$	KUTTNER [9],
$(R^*, n, \kappa) \nsubseteq (C, \kappa)$	$\kappa = 2, 3$	RIESZ [18]
	$\kappa = 2, 3, 4, \ldots$	PEYERIMHOFF [14]
	$\kappa \geqq 2$	KUTTNER [9],
$(\bar{C}, n, p) \equiv (C, n, p) \equiv (C, p)$	$(p = 0, 1, 2, \ldots)$	(definition),
$(\bar{C}, n, \kappa) \sim (C, \kappa)$	$(\kappa \geqq 0)$	JURKAT [8],
$(\dot{C}, n, \kappa) \sim (C, \kappa)$	$(\kappa \geqq 0)$	BORWEIN [3].

(d) *Relations between* (R, λ, κ) *and* $(\bar{C}, \lambda, \kappa)$.

The cases $0 \leqq \kappa \leqq 1$ and κ an integer are (by (a) above) contained in the results for (C, λ, κ). Otherwise we have:

$(R, \lambda, \kappa) \sim (\bar{C}, \lambda, \kappa)$ if $\kappa > 1$, κ non-integral, and if $\Lambda_n \nearrow$, $|\Delta\lambda_n|$ monotonic, $n\,|\Delta\lambda_n|^{\kappa - [\kappa]} \nearrow$ JURKAT [8].

(e) *Relations between* (R, λ, κ) *and* (C, λ, κ).

The case $0 \leqq \kappa \leqq 1$ is already covered in (b) above. In addition, we have:

$$(C, \lambda, p) \subseteq (R, \lambda, p) \qquad (p = 2, 3, 4, \ldots)$$

without restriction on $\{\lambda_n\}$ RUSSELL [19];

$$(C, \lambda, \kappa) \subseteq (R, \lambda, \kappa) \qquad (1 < \kappa < 2)$$

when

$$\lambda_{n+1}/\lambda_n \searrow, \qquad \Delta\lambda_{n+1}/\Delta\lambda_n \searrow$$

BORWEIN [4].

In the opposite direction:

$$(R, \lambda, p) \subseteq (C, \lambda, p) \qquad (p = 2, 3, 4, \ldots)$$

when

$$\Lambda_n \nearrow$$

JURKAT [8],

or when

$$0 < a \leqq \lambda_{n+1} - \lambda_n \leqq b < \infty$$

BURKILL [5],

or (which includes both these results) when

$\Lambda_{n-1} = O(\Lambda_n)$ (and without restriction on $\{\lambda_n\}$ when $p = 2$) RUSSELL [19],

or (which is independent of the previous case) when

$$\lambda_{n+1} = O(\lambda_n)$$

BORWEIN [2],

and, finally, without restriction on $\{\lambda_n\}$. MEIR [13].

The non-integral case corresponding to this last result has not hitherto been considered (except by JURKAT for $(\bar{C}, \lambda, \kappa)$), and it is the object of this paper to deal decisively with this by proving the following theorem, in which there is no restriction on $\{\lambda_n\}$ other than the basic assumption of unbounded monotonicity.

Theorem. $(R, \lambda, \kappa) \subseteq (C, \lambda, \kappa) \qquad (\kappa \geqq 0)$.

This will incidentally show that (C, λ, κ) is regular for every $\kappa \geqq 0$ (see also BORWEIN [4, Lemma 4]). Since the theorem is trivially true for $0 \leqq \kappa \leqq 1$, we may suppose that $\kappa = p + \delta$, where p is a positive integer and $0 < \delta \leqq 1$. We require the following lemma.

Lemma. *Let p be a fixed positive integer. For $n = 1, 2, 3, \ldots$, there is an integer $m = m(n)$, $n \leqq m \leqq n+p$, and numbers c_{nj}, ω_{nj} $(j = 0, 1, \ldots, p)$ such that*

$$(1) \qquad |c_{nj}| \leqq K_p, \quad \lambda_m \leqq \omega_{nj} \leqq \lambda_{m+1} \qquad (j = 0, 1, \ldots, p),$$

and

$$(2) \qquad \left(1 - \frac{x}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{x}{\lambda_{n+p}}\right) = \sum_{j=0}^{p} c_{nj} \left(1 - \frac{x}{\omega_{nj}}\right)^p.$$

Proof of the Lemma. This is due to BORWEIN [2] and MEIR [13][2]; since the details are of importance in the sequel, we sketch them here.

Case (i). If $\lambda_{n+p+1}/\lambda_n \leqq (p+1)^{p+1}$ take m to be an integer, $n \leqq m \leqq n+p$, such that

$$(3) \qquad \lambda_{m+1} - \lambda_m = \max_{n \leqq i \leqq n+p} (\lambda_{i+1} - \lambda_i)$$

and define

$$\omega_{nj} = \lambda_m + \frac{j+1}{p+1}(\lambda_{m+1} - \lambda_m) \qquad (j = 0, 1, \ldots, p).$$

BORWEIN [2] shows that

$$(\lambda_{n+1} - x) \ldots (\lambda_{n+p} - x) = \sum_{j=0}^{p} y_{nj} (\omega_{nj} - x)^p$$

where

$$|y_{nj}| \leqq (p+1)!\,(p+1)^{2p} \qquad (j = 0, 1, \ldots, p);$$

thus

$$|c_{nj}| \equiv \left| \frac{y_{nj}\, \omega_{nj}^p}{\lambda_{n+1} \cdots \lambda_{n+p}} \right| \leqq |y_{nj}| \left(\frac{\lambda_{n+p+1}}{\lambda_n}\right)^p \leqq (p+1)!\,(p+1)^{3p+p^2},$$

and (1) and (2) follow.

Case (ii). If $\lambda_{n+p+1}/\lambda_n > (p+1)^{p+1}$ take m to be the integer, $n \leqq m \leqq n+p$, such that

$$(4) \qquad \frac{\lambda_{m+1}}{\lambda_m} > p+1, \quad \text{and} \quad \frac{\lambda_{i+1}}{\lambda_i} \leqq p+1 \qquad (n \leqq i \leqq m-1),$$

[2] We are indebted to Dr. MEIR for showing us the manuscript of this paper prior to publication, and for allowing us to quote his result.

and define

$$\omega_{nj} = (j+1)\,\lambda_m \qquad (j = 0, 1, \ldots, p).$$

This definition is given by MEIR [13], who shows that if the c_{nj} are then defined by (2), it can be deduced that (1) holds.

Proof of the Theorem. On putting $x = \lambda_\nu$ in (2), multiplying through by $(1 - \lambda_\nu/\lambda_{n+p+1})^\delta a_\nu$ and summing from $\nu = 0$ to m, we obtain

$$(5) \qquad t_n^\kappa \equiv t_n^{p+\delta} = \sum_{j=0}^{p} c_{nj}\, \omega_{nj}^{-p} \lambda_{n+p+1}^{-\delta} \sum_{\nu=0}^{m} (\omega_{nj} - \lambda_\nu)^p (\lambda_{n+p+1} - \lambda_\nu)^\delta a_\nu.$$

Denote, for clarity, $\omega = \omega_{nj}$, $t = \lambda_{n+p+1}$; then the inner sum on the right of (5) is

$$S_{nj} \equiv \sum_{\nu=0}^{m} (\omega - \lambda_\nu)^p (t - \lambda_\nu)^\delta a_\nu$$

$$= \int_0^\omega (\omega - u)^p (t-u)^\delta \, dA(u)$$

$$= \frac{(-1)^p}{(p-1)!} \int_0^\omega A^{p-1}(u) \left(\frac{d}{du}\right)^p [(\omega - u)^p (t-u)^\delta]\, du$$

on integrating by parts p times. A further integration by parts now gives, on using Leibniz' formula for differentiating a product,

$$(6) \qquad S_{nj} = A^p(\omega)(t - \omega)^\delta + \sum_{r=0}^{p} k_r \int_0^\omega A^p(u)(\omega - u)^r (t-u)^{\delta - r - 1}\, du$$

$$\equiv S'_{nj} + S''_{nj}, \qquad \text{say},$$

where the coefficients k_r depend only on r and κ.

We suppose, as we may without loss of generality, that $\sum a_\nu$ is summable (R, λ, κ) to zero, namely that $A^{p+\delta}(u) = o(u^{p+\delta})$. Then, by the limitation theorem for Riesz means, in a form given by BORWEIN [1, Lemma 2; in o-form] we have

$$A^p(\omega) = o(\omega^p A_m^\delta) \qquad (\text{since } \lambda_m \leqq \omega \leqq \lambda_{m+1}).$$

Using this estimate in the definition (6) of S'_{nj}, we obtain

$$T'_n \equiv \sum_{j=0}^{p} c_{nj}\, \omega^{-p} t^{-\delta} S'_{nj} = o(1) \sum_{j=0}^{p} c_{nj} A_m^\delta (1 - \omega/t)^\delta.$$

We now consider the choices of m detailed in the proof of the lemma, and note that $\lambda_n \leqq \lambda_m \leqq \omega \leqq \lambda_{m+1} \leqq \lambda_{n+p+1} = t$. Then in case (i), from (3),

$$A_m \left(1 - \frac{\omega}{t}\right) \leqq \frac{\lambda_{m+1}(\lambda_{n+p+1} - \lambda_m)}{\lambda_{n+p+1}(\lambda_{m+1} - \lambda_m)} \leqq p+1;$$

while in case (ii) we use (4) and note that $\lambda_{m+1}/\lambda_m > p+1$ implies $\Lambda_m < (p+1)/p \leqq p+1$, so that

$$\Lambda_m \left(1-\frac{\omega}{t}\right) < p+1.$$

Hence, by (1),

$$T_n' = o(1) \sum_{j=0}^{p} c_{nj}(p+1)^\delta = o(1). \tag{7}$$

Turning to S_{nj}'' we see that, for each r in $0 \leqq r \leqq p$, $[(\omega-u)/(t-u)]^r$ decreases as u increases in $(0, \omega)$; consequently, by the Second Mean Value Theorem for integrals, there is a $\xi = \xi(\omega, t)$ in $0 \leqq \xi \leqq \omega$ such that

$$\begin{aligned}
&\left| \int_0^\omega A^p(u)(\omega-u)^r (t-u)^{\delta-r-1}\, du \right| \\
&\quad = \left(\frac{\omega}{t}\right)^r \left| \int_0^\xi A^p(u)(t-u)^{\delta-1}\, du \right| \\
&\quad = o(\omega^{p+\delta}).
\end{aligned}$$

In the last step we have used $\omega/t \leqq 1$, the hypothesis $A^{p+\delta}(u) = o(u^{p+\delta})$, and the o-form of the Riesz Mean-Value Theorem (see, for example, CHANDRESEKHARAN and MINAKSHISUNDARAM [6, Lemma 1.42]). Employing this result in the definition (6) of S_{nj}'' and making use of (1), we obtain

$$T_n'' \equiv \sum_{j=0}^{p} c_{nj}\, \omega^{-p} t^{-\delta} S_{nj}'' = \sum_{j=0}^{p} c_{nj}\, \omega^{-p} t^{-\delta} \cdot o(\omega^{p+\delta}) = o(1). \tag{8}$$

Finally, substitution of the estimates for T_n' and T_n'' given by (7) and (8) into (5) gives

$$t_n^\kappa = T_n' + T_n'' = o(1),$$

so that $\sum a_\nu$ is summable (C, λ, κ) to zero, and the proof is complete.

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