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ON A METHOD OF SUMMABILITY EQUIVALENT TO THE CESÀRO METHOD

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1. *Introduction*

Suppose throughout that $\mu = m + \delta$ where m is a non-negative integer and $0 \leqslant \delta < 1$, and let

$$\Pi_\mu(x) = m!\binom{x+m}{m}(x+m+1)^\delta = (x+1)\ldots(x+m)(x+m+1)^\delta.$$

We define a new method of summability (C^*, μ) as follows:

A series $\sum\limits_0^\infty a_n$ is said to be summable (C^*, μ) to s if

$$\sigma_n = \frac{1}{\Pi_\mu(n)} \sum_{r=0}^{n} \Pi_\mu(n-r)a_r \to s \text{ as } n \to \infty;$$

if, in addition,

$$\sum_{n=0}^{\infty} |\sigma_n - \sigma_{n+1}| < \infty,$$

the series is said to be absolutely summable (C^*, μ) to s. When μ is an integer the method (C^*, μ) reduces to the standard Cesàro method (C, μ). The "discontinuous" Riesz method (R^*, α), defined as above with $(x+1)^\alpha$ $(\alpha > -1)$ in place of $\Pi_\mu(x)$, is identical with (C^*, α) in the range $0 \leqslant \alpha \leqslant 1$. It is known that in the range $-1 < \alpha < 2$ the methods (R^*, α) and (C, α) are both equivalent and absolutely equivalent (see §2 for an explanation of the terminology), but that these equivalences break down whenever $\alpha \geqslant 2$ (Riesz [9], Peyerimhoff [8], Kuttner [5]).

The object of this note is to prove the following theorem, the known cases of which are $\mu = 0, 1, \ldots$ and $0 < \mu < 1$.

THEOREM. *The methods (C, μ) and (C^*, μ) are both equivalent and absolutely equivalent for all $\mu \geqslant 0$.*

2. *Nörlund methods*

Let $\{p_n\}$, $\{q_n\}$ be sequences of non-negative real numbers with $p_0 > 0$, $q_0 > 0$, let

$$P_n = \sum_{r=0}^{n} p_r, \quad Q_n = \sum_{r=0}^{n} q_r$$

and let

$$P(z) = \sum_{r=0}^{\infty} P_n z^n, \quad Q(z) = \sum_{r=0}^{\infty} Q_n z^n.$$

When both power series have non-zero radii of convergence, we define

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associated sequences $\{k_n\}$, $\{l_n\}$ and functions $k(z)$, $l(z)$ by the identities

$$\frac{Q(z)}{P(z)} = \sum_{n=0}^{\infty} k_n z^n = k(z), \quad \frac{P(z)}{Q(z)} = \sum_{n=0}^{\infty} l_n z^n = l(z).$$

The Nörlund method (N, p) is defined in the same way as the method (C^*, μ) but with

$$\sigma_n = \frac{1}{P_n} \sum_{r=0}^{n} P_{n-r} a_r;$$

and (N, q) is similarly defined.

The method (N, p) is said to be regular (absolutely regular) if every convergent (absolutely convergent) series with sum s is summable (absolutely summable) by (N, p) to s. A necessary and sufficient condition for (N, p) to be regular is (see [**2**; p. 64])

$$\frac{p_n}{P_n} \to 0, \tag{1}$$

in which case the series defining $P(z)$ is convergent for $|z| < 1$. Necessary and sufficient conditions for (N, p) to be absolutely regular (Mears [**6**], Knopp and Lorentz [**4**]) are (1) and

$$\sup_{r \geqslant 1} \sum_{n=1}^{\infty} \left| \frac{P_n}{P_{n+r}} - \frac{P_{n-1}}{P_{n+r-1}} \right| < \infty. \tag{2}$$

The methods (N, p) and (N, q) are said to be equivalent (absolutely equivalent) if every series summable (absolutely summable) by either of them to s is summable (absolutely summable) by the other to s.

It is known (Riesz [**9**], Miesner [**7**; Cor. 1]; see also [**2**; p. 67]) that if the methods (N, p) and (N, q) are regular (absolutely regular) and if

$$\sum_{0}^{\infty} |k_n| < \infty \text{ and } \sum_{0}^{\infty} |l_n| < \infty,$$

then the two methods are equivalent (absolutely equivalent). It is also known (see [**10**; p. 246]) that if $\sum_{n=0}^{\infty} |r_n| < \infty$ and $r(z) = \sum_{n=0}^{\infty} r_n z^n$ is not zero for $|z| \leqslant 1$, then the Taylor expansion of $\dfrac{1}{r(z)}$ is absolutely convergent for $|z| = 1$.

We combine the above two results in the following lemma.

Lemma 1. *If the methods (N, p) and (N, q) are regular (absolutely regular) and if*

$$\sum_{n=0}^{\infty} |l_n| < \infty$$

and $l(z)$ is not zero for $|z| \leqslant 1$, then the methods are equivalent (absolutely equivalent).

3. *The methods (C, μ) and $(C^* \mu)$*

From now on we shall suppose that

$$0 < \delta < 1$$

and that

$$P_n = \Pi_\mu(n) = m! \binom{n+m}{m} (n+m+1)^\delta, \quad Q_n = \binom{n+\mu}{n},$$

so that

$$P(z) = \sum_{n=0}^{\infty} \Pi_\mu(n) z^n, \quad Q(z) = (1-z)^{-\mu-1}$$

and

$$l(z) = \sum_{n=0}^{\infty} l_n z^n = (1-z)^{\mu+1} P(z).$$

The methods (C^*, μ) and (C, μ) are thus the Nörlund methods (N, p) and (N, q) respectively.

It is familiar that (C, μ) is both regular and absolutely regular; and (C^*, μ) is regular since

$$\frac{\Pi_\mu(n-1)}{\Pi_\mu(n)} \to 1 \text{ as } n \to \infty.$$

Further, setting

$$\psi(x) = \frac{\Pi_\mu(x)}{\Pi_\mu(x+r)},$$

we see that, for $x \geqslant 0$, $r = 1, 2, \ldots$,

$$\frac{\psi'(x)}{\psi(x)} = \frac{1}{x+1} + \ldots + \frac{\delta}{x+m+1} - \left(\frac{1}{x+1+r} + \ldots + \frac{\delta}{x+m+1+r} \right) > 0,$$

so that, for $n = 1, 2, \ldots$,

$$\frac{\Pi_\mu(n)}{\Pi_\mu(n+r)} > \frac{\Pi_\mu(n-1)}{\Pi_\mu(n+r-1)}.$$

It follows that (2) is satisfied with $P_n = \Pi_\mu(n)$ and hence that (C^*, μ) is absolutely regular.

Lemma 2. *For $x > -m-1$,*

(i) $\Pi_\mu(x) = \sum_{r=0}^{m} (-1)^r j_r (x+m+1)^{\mu-r}$ *where each $j_r > 0$,*

(ii) $(-1)^k \left(\dfrac{d}{dx}\right)^{m+1+k} \Pi_\mu(x) > 0, \quad (k = 0, 1, \ldots).$

Proof. Conclusion (i) is immediate. By (i),

$$\left(\frac{d}{dx}\right)^{m+1} \Pi_\mu(x) = \sum_{r=0}^{m} (-1)^r j_r (\delta - r) \ldots (\delta + m - r)(x+m+1)^{\delta-1-r}$$

$$= \sum_{r=0}^{m} b_r (x+m+1)^{\delta-1-r},$$

where each $b_r > 0$ since $0 < \delta < 1$; and conclusion (ii) follows.

LEMMA 3.
$$\sum_{n=0}^{\infty} |l_n| < \infty.$$

Proof. For $|z|<1$, by Lemma 2 (i),
$$l(z)=\sum_{n=0}^{\infty} l_n z^n = \sum_{r=0}^{m} (-1)^r j_r (1-z)^r l_r(z)$$
where
$$l_r(z)=\sum_{n=0}^{\infty} l_{n,r} z^n = (1-z)^{\mu-r+1} \sum_{n=0}^{\infty} (n+m+1)^{\mu-r} z^n.$$
By a trivial modification of a known result (Miesner [7; Lemma 5])
$$\sum_{n=0}^{\infty} |l_{n,r}| < \infty \quad (r=0,1,\ldots,m),$$
and the required conclusion follows.

4. *Proof of the theorem*

In view of Lemmas 1 and 3 it suffices to show that $l(z)$ is not zero for $|z| \leqslant 1$.

Let
$$F(z)=(1-z)^{-\delta} l(z) = (1-z)^{m+1} \sum_{n=0}^{\infty} \Pi_\mu(n) z^n.$$
Then, for $|z|<1$,
$$F(z)=\sum_{n=0}^{\infty} c_n z^n,$$
where
$$c_n = (-1)^{m+1} \sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} \Pi_\mu(r+n-m-1) \quad (n=0,1,\ldots),$$
since $\Pi_\mu(-k)=0$ for $k=1,2,\ldots,m+1$.

By a standard result, we have for $n=0,1,\ldots,$
$$c_n = \left[\left(\frac{d}{dx}\right)^{m+1} \Pi_\mu(x)\right]_{x=x_1} \text{ and } c_n - c_{n+1} = -\left[\left(\frac{d}{dx}\right)^{m+2} \Pi_\mu(x)\right]_{x=x_2},$$
where $n-m-1<x_1<n$ and $n-m-1<x_2<n+1$.

Hence, by Lemma 2, $c_n > c_{n+1} > 0$ for $n=0,1,\ldots,$ and consequently, by a theorem of Kakeya [3] (see also [5]),
$$(1-z)^{-\delta} l(z) \text{ is not zero when } |z| \leqslant 1, \quad z \neq 1.$$
It remains only to show that $l(1) \neq 0$ and this can be done as follows. By a known result (see [1])
$$l(1) = \lim_{x\to 1-} l(x) = \lim_{x \to 1-} (1-x)^{\mu+1} \sum_{n=0}^{\infty} \frac{\Pi_\mu(n)}{\binom{n+\mu}{n}} \binom{n+\mu}{n} x^n$$
$$= \lim_{n\to\infty} \frac{\Pi_\mu(n)}{\binom{n+\mu}{n}} = \Gamma(\mu+1) > 0.$$

References

1. D. Borwein, " On a scale of Abel-type summability methods ", *Proc. Cambridge Phil. Soc.*, 53 (1957), 318–322.
2. G. H. Hardy, *Divergent series* (Oxford, 1949).
3. S. Kakeya, " On the limits of the roots of an algebraic equation with positive coefficients ", *Tôhoku Math. J.*, 2 (1912), 140–142.
4. K. Knopp and G. G. Lorentz, "Beiträge zur absoluten Limitierung", *Archiv der Math.*, 2 (1949), 10–16.
5. B. Kuttner, " On discontinuous Riesz means of type *n* ", *Journal London Math. Soc.*, 37 (1962), 354–364.
6. F. M. Mears, "Absolute regularity and the Nörlund mean ", *Annals of Math.*, 38 (1937), 594–601.
7. W. Miesner, "The convergence fields of Nörlund means", *Proc. London Math. Soc.* (3), 15 (1965), 495–507.
8. A. Peyerimhoff, " On convergence fields of Nörlund means ", *Proc. American Math. Soc.*, 7 (1956), 335–347.
9. M. Riesz, " Sur l'équivalence de certaines méthodes de sommation ", *Proc. London Math. Soc.* (2), 22 (1924), 412–419.
10. A. Zygmund, *Trigonometric series, Vol.* 1. (Cambridge, 1959).

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