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ON BOREL-TYPE METHODS

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The object of this paper is to examine Borel-type methods of summability, to define strong Borel-type methods and investigate their relations with ordinary and absolute Borel-type methods.

1. Introduction. Suppose throughout that σ, $a_n (n = 0, 1, \cdots)$ are arbitrary complex numbers, that $\alpha > 0$, that β is real and that N is a positive integer greater than $-\beta/\alpha$. Let x be a real variable in the range $[0, \infty)$: in all limits and order relationship involving x, it is to be understood that $x \to \infty$.

Define $s_n = \sum_{\nu=0}^{n} a_\nu$, $s_{-1}=0$, $\sigma_N=\sigma-s_{N-1}$.

Borel-type sums are defined as follows:

$$a_{\alpha,\beta}(x) = \sum_{n=N}^{\infty} \frac{a_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}; \quad s_{\alpha,\beta}(x) = \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}.$$

It is known that the convergence of either series for all $x \geqq 0$ implies the convergence, for all $x \geqq 0$ of the other (See [1]).

Borel-type means are defined as follows:

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t) dt ,$$

$$S_{\alpha,\beta}(x) = \alpha e^{-x} s_{\alpha,\beta}(x), \quad T_{\alpha,\beta}(x) = \alpha e^{-x} a_{\alpha,\beta}(x).$$

Borel-type methods are defined as follows:

1. Summability:

(i) If $A_{\alpha,\beta}(x) \to \sigma_N$, we say that $s_n \to \sigma(B', \alpha, \beta)$.

(ii) If $S_{\alpha,\beta}(x) \to \sigma$, we say that $s_n \to \sigma(B, \alpha, \beta)$.

2. Absolute Summability:

(i) If $A_{\alpha,\beta}(x)\to\sigma_N$ and $A_{\alpha,\beta}(x)$ is of bounded variation with respect to x in the range $[0,\infty)$ we say that $s_n\to\sigma|B',\alpha,\beta|$.

(ii) If $S_{\alpha,\beta}(x)\to\sigma$ and $S_{\alpha,\beta}(x)$ is of bounded variation with respect to x in the range $[0,\infty)$, we say that $s_n\to\sigma|B,\alpha,\beta|$.

3. Strong Summability:

(i) If

$$\int_0^x e^t|A_{\alpha,\beta-1}(t)-\sigma_N|\,dt=o(e^x),$$

we say that $s_n\to\sigma[B',\alpha,\beta]$.

(ii) If

$$\int_0^x e^t|S_{\alpha,\beta-1}(t)-\sigma|\,dt=o(e^x),$$

we say that $s_n\to\sigma[B,\alpha,\beta]$.

4. Boundedness:

(i) If $A_{\alpha,\beta}(x)=O(1)$, we say that $s_n=O(1)$ (B',α,β).

(ii) If $S_{\alpha,\beta}(x)=O(1)$, we say that $s_n=O(1)$ (B,α,β).

5. Strong Boundedness:

(i) If

$$\int_0^x e^t|A_{\alpha,\beta-1}(t)-\sigma_N|\,dt=O(e^x),$$

we say that $s_n=O(1)$ $[B',\alpha,\beta]$.

(ii) If

$$\int_0^x e^t|S_{\alpha,\beta-1}(t)-\sigma|\,dt=O(e^x),$$

we say that $s_n=O(1)$ $[B,\alpha,\beta]$.

The summability methods $(B,1,1)$ and $(B',1,1)$ are the Borel exponential and the Borel integral methods respectively as given in 'Divergent Series'. (See [3] p.182). The Borel-type summability methods (B,α,β) and (B',α,β) are due to Borwein (See [2]). The ideas of absolute summability are also due to Borel himself. (See [3] p.184).

We assume henceforth that the series defining $a_{\alpha,\beta}(x)$, $s_{\alpha,\beta}(x)$ are convergent for all $x\geqq 0$ and, since the choice of N is clearly immaterial, that $\alpha N+\beta\geqq 2$; so that the functions

$$a_{\alpha,\beta-1}(x),\ a_{\alpha,\beta}(x),\ s_{\alpha,\beta-1}(x),\ s_{\alpha,\beta}(x)$$

are all continuous for $x\geqq 0$. Also, we assume, without loss of generality, that $a_0=a_1=\cdots=a_{N-1}=0$, so that $\sigma_N=\sigma$.

Given a function, $f(x)$, continuous for $x\geqq 0$ we write for $x\geqq 0$,

$$f_0(x)=f(x);\quad f_\delta(x)=\{\Gamma(\delta)\}^{-1}\int_0^x (x-t)^{\delta-1}f(t)\,dt\quad(\delta>0).$$

This section ends with the statement of several known results which will be used throughout the paper:

I *If* $\gamma>0$ *and* $f(x)=s_{\alpha,\beta}(x)$, *then, for* $x>0$

$$f_\gamma(x)=s_{\alpha,\beta+\gamma}(x)=\sum_{n=N}^{\infty}\frac{s_n x^{\alpha n+\beta+\gamma-1}}{\Gamma(\alpha n+\beta+\gamma)}.$$

II $a_{\alpha,\beta}(x)=s_{\alpha,\beta}(x)-s_{\alpha,\beta+\alpha}(x)$.

III $$\int_0^x e^{-t}a_{\alpha,\beta}(t)\,dt$$

$$=\{\Gamma(\alpha)\}^{-1}\int_0^x e^{-u}s_{\alpha,\beta}(u)\,du\int_{x-u}^{\infty}t^{\alpha-1}e^{-t}\,dt.\qquad\text{(See [1])}$$

2. Preliminary Results.

The following lemmas are required:

LEMMA 1.

(i) $$\frac{d}{dx}A_{\alpha,\beta}(x)=e^{-x}a_{\alpha,\beta}(x),$$

(ii) $$\frac{d}{dx}S_{\alpha,\beta}(x)=\alpha e^{-x}\{s_{\alpha,\beta-1}(x)-s_{\alpha,\beta}(x)\},$$

(iii) $$\frac{d}{dx}T_{\alpha,\beta}(x)=\alpha e^{-x}\{a_{\alpha,\beta-1}(x)-a_{\alpha,\beta}(x)\},$$

(iv) $A_{\alpha,\beta-1}(x)=e^{-x}a_{\alpha,\beta}(x)+\int_0^x e^{-t}a_{\alpha,\beta}(t)dt=A_{\alpha,\beta}(x)+A'_{\alpha,\beta}(x)$,

(v) $S_{\alpha,\beta-1}(x)=\alpha e^{-x}s_{\alpha,\beta-1}(x)=S_{\alpha,\beta}(x)+S'_{\alpha,\beta}(x)$,

(iv) $T_{\alpha,\beta-1}(x)=\alpha e^{-x}a_{\alpha,\beta-1}(x)=T_{\alpha,\beta}(x)+T'_{\alpha\beta}(x)$.

The proofs are immediate.

LEMMA 2. *If*

$$\int_0^\infty |f(t)|\,dt<\infty$$

then

$$\int_0^x e^t|f(t)|\,dt=o(e^x).$$

PROOF. If

$$F(x)=\int_x^\infty |f(t)|\,dt$$

then

$$\int_0^x e^t|f(t)|\,dt=-F(0)+e^xF(x)-\int_0^x e^tF(t)dt=o(e^x).$$

LEMMA 3. *If* $f(x)\in BV_x[0,\infty)$* *and* $\int_0^\infty |\varphi(t)|\,dt<\infty$, *then*

$$w(x)=\int_0^x f(x-t)\varphi(t)dt\in BV_x[0,\infty).$$

PROOF. $w(x)=\int_{-\infty}^\infty f^*(x-t)\varphi^*(t)dt$

where

$$f^*(u)=\begin{cases}f(u) & (u\geqq 0)\\ 0 & (u<0)\end{cases}$$

$$\varphi^*(u)=\begin{cases}\varphi(u) & (u\geqq 0)\\ 0 & (u<0)\end{cases}$$

*From now on, by "$BV_x[0,\infty)$" we mean "is of bounded variation with respect to x in the range $[0,\infty)$".

Then, for $0<x_0<x_1<\cdots<x_n$,

$$\sum_{r=1}^n |w(x_r)-w(x_{r-1})|$$

$$\leq\int_{-\infty}^\infty\left\{\sum_{r=1}^n |f^*(x_r-t)-f^*(x_{r-1}-t)|\right\}|\varphi^*(t)|\,dt$$

$$\leq V_0^\infty f\int_0^\infty |\varphi(t)|\,dt$$

where $V_0^\infty f$ is the total variation of f in the range $[0,\infty)$. Hence, the result follows. This is a special case of the lemma in Tatchell [4].

LEMMA 4. *Suppose that* $f(x)$ *is continuous for* $x\geq 0$.

(i) *Suppose that* $\delta>0$, *then*
$e^{-x}f_\delta(x)\in BV_x[0,\infty)$ *whenever* $e^{-x}f(x)\in BV_x[0,\infty)$.

(ii) *Suppose that* $\delta>0$, *then*

$$\int_0^x e^{-t}f_\delta(t)dt\in BV_x[0,\infty) \quad \textit{whenever}$$

$$\int_0^x e^{-t}f(t)dt\in BV_x[0,\infty).$$

PROOF. Let $F(x)=\int_0^x e^{-t}f(t)dt$.

Then

$$\Gamma(\delta)e^{-x}f_\delta(x)=\int_0^x e^{-t}f(t)\varphi(x-t)dt$$

and

$$\Gamma(\delta)\int_0^x e^{-t}f_\delta(t)dt=\int_0^x F(t)\varphi(x-t)\,dt$$

where

$$\varphi(u)=u^{\delta-1}e^{-u}.$$

The result follows from lemma 3. Compare lemma 2 in Borwein [1].

LEMMA 5. *If* $\int_0^x |f(t)|\,dt = o(e^x)$,

then

$$\int_0^x |f_\delta(t)|\,dt = o(e^x) \text{ whenever } \delta > 0.$$

PROOF. Let $g(t) = |f(t)|$. Hence $g_\delta(t) \geq |f_\delta(t)|$. By lemma 2(a) in Borwein [1], it follows that if $g_1(x) = o(e^x)$, then $g_{1+\delta}(x) = o(e^x)$ and hence the result follows.

3. Theorems. This section is divided into two parts. The first contains theorems which give relations between methods of the same type: that is "B" methods or "B'" methods. The second contains theorems giving relations between the "B" and "B'" methods.

3.1 THEOREM 1.

(i) *If* $s_n \to \sigma|B, \alpha, \beta|$ *then* $s_n \to \sigma(B, \alpha, \beta)$.
(ii) *If* $s_n \to \sigma|B', \alpha, \beta|$ *then* $s_n \to \sigma(B', \alpha, \beta)$.

The proof is immediate from the definitions.

Each of the other theorems in this section can be stated in parts corresponding to the parts of theorem 1. We shall only state the part corresponding to (i) in each case; the proofs of the other parts are obtained by replacing "$S_{\alpha,\beta}(x)$" by "$A_{\alpha,\beta}(x)$" and "σ" by "σ_N" respectively.

THEOREM 2. *If* $s_n \to \sigma|B, \alpha, \beta|$ *then* $s_n \to \sigma[B, \alpha, \beta]$.

PROOF. We have $S_{\alpha,\beta}(x) - \sigma = o(1)$ and $\int_0^\infty |S'_{\alpha,\beta}(t)|\,dt < \infty$.

Hence, in view of lemma 2, it follows that,

$$\begin{aligned}
&\int_0^x e^t |S_{\alpha,\beta-1}(t) - \sigma|\,dt \\
&= \int_0^x e^t |S'_{\alpha,\beta}(t) + S_{\alpha,\beta}(t) - \sigma|\,dt \\
&= \int_0^x e^t |S'_{\alpha,\beta}(t) + o(1)|\,dt \\
&= o(e^x).
\end{aligned}$$

This completes the proof of theorem 2.

THEOREM 3. *If* $s_n \to \sigma[B, \alpha, \beta]$ *then* $s_n \to \sigma(B, \alpha, \beta)$.

PROOF. We have $\int_0^x e^t |S_{\alpha,\beta-1}(t) - \sigma|\,dt = o(e^x)$.

Hence, it follows that,

$$\begin{aligned}
|S_{\alpha,\beta}(x) - \sigma| &= e^{-x}|e^x S_{\alpha,\beta}(x) - e^x\sigma| \\
&= e^{-x}\left|\int_0^x e^t\{S_{\alpha,\beta}(t) + S'_{\alpha,\beta}(t)\}\,dt - \sigma e^x\right| \\
&= e^{-x}\left|\int_0^x e^t\{S_{\alpha,\beta-1}(t) - \sigma\}\,dt - \sigma\right| \\
&\leq e^{-x}\int_0^x e^t|S_{\alpha,\beta-1}(t) - \sigma|\,dt + |\sigma|e^{-x} \\
&= o(1).
\end{aligned}$$

This completes the proof of theorem 3.

THEOREM 4. *If* $s_n = O(1)\ [B, \alpha, \beta]$ *then* $s_n = O(1)\ (B, \alpha, \beta)$.

The proof is similar to that of theorem 3.

The following two results are immediate from the definitions:

THEOREM 5. *If* $s_n \to \sigma(B, \alpha, \beta)$ *then* $s_n \to \sigma[B, \alpha, \beta+1]$.

THEOREM 6. *If* $s_n = O(1)\ (B, \alpha, \beta)$ *then* $s_n = O(1)\ [B, \alpha, \beta+1]$.

Results giving relations between methods of the same type are:

THEOREM 7. *If* $s_n \to \sigma(B, \alpha, \beta)$ *then* $s_n \to \sigma(B, \alpha, \beta+\delta)$ $(\delta > 0)$.

This is due to Borwein [2], result (II).

THEOREM 8. *If* $s_n = O(1)\ (B, \alpha, \beta)$ *then* $s_n = O(1)\ (B, \alpha, \beta+\delta)$ $(\delta > 0)$.

This follows from analogues of lemmas 1 and 2 in [1] with $O(\cdot)$ instead of $o(\cdot)$.

THEOREM 9. *If* $s_n \to \sigma[B, \alpha, \beta]$ *then* $s_n \to \sigma[B, \alpha, \beta+\delta]$ $(\delta > 0)$.

This follows from lemma 5.

THEOREM 10. *If* $s_n = O(1)$ $[B, \alpha, \beta]$ *then* $s_n = O(1)$ $[B, \alpha, \beta+\delta]$ $(\delta > 0)$.

This follows from the analogue of lemma 5 with $O(\cdot)$ instead of $o(\cdot)$.

Finally, in this section, there are two theorems giving the exact relation between the strong and ordinary cases:

THEOREM 11. $s_n \to \sigma[B, \alpha, \beta]$ *if and only if* $s_n \to \sigma(B, \alpha, \beta)$ *and*

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = o(e^x).$$

PROOF. We have that $S_{\alpha,\beta-1}(t) - \sigma = S_{\alpha,\beta}(t) - \sigma + S'_{\alpha,\beta}(t)$.
Whence

(a) $|S_{\alpha,\beta-1}(t) - \sigma| \leqq |S_{\alpha,\beta}(t) - \sigma| + |S'_{\alpha,\beta}(t)|$,

(b) $|S'_{\alpha,\beta}(t)| \leqq |S_{\alpha,\beta}(t) - \sigma| + |S_{\alpha,\beta-1}(t) - \sigma|$.

NECESSITY. Suppose that $s_n \to \sigma[B, \alpha, \beta]$. Then, it follows from theorem 3 that $s_n \to \sigma(B, \alpha, \beta)$. That is $S_{\alpha,\beta}(t) - \sigma = o(1)$ and, further

$$\int_0^x e^t |S_{\alpha,\beta}(t) - \sigma|\, dt = o(e^x),$$

Thus, using (b), it follows that

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = o(e^x).$$

SUFFICIENCY. Suppose on the contrary that $S_{\alpha,\beta}(t) - \sigma = o(1)$ and that

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = o(e^x).$$

Using (a), it follows at once that

$$\int_0^x e^t |S_{\alpha,\beta-1}(t) - \sigma|\, dt = o(e^x).$$

THEOREM 12. $s_n = O(1)\,[B, \alpha, \beta]$ *if and only if* $s_n = O(1)$ (B, α, β) *and*

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = O(e^x).$$

The proof is similar to that of theorem 11.

3.2. Theorems in this section are stated in full.

THEOREM 13. $s_n \to \sigma(B, \alpha, \beta)$ *if and only if* $a_n \to 0$ (B, α, β) *and* $s_n \to \sigma(B', \alpha, \beta)$.

This theorem is due to Borwein [2].

THEOREM 14. $s_n \to \sigma\ |B, \alpha, \beta|$ *if and only if* $a_n \to 0$ $|B, \alpha, \beta|$ *and* $s_n \to \sigma |B', \alpha, \beta|$.

PROOF. (i) NECESSITY. Suppose that $s_n \to \sigma |B, \alpha, \beta|$. Then we have that

$$S_{\alpha,\beta}(x) = \alpha e^{-x} s_{\alpha,\beta}(x) = \sigma + o(1) \tag{14.1}$$

$$S_{\alpha,\beta}(x) = \alpha e^{-x} s_{\alpha,\beta}(x) \in BV_x[0, \infty). \tag{14.2}$$

First, in view of theorem 13, and (14.1), we obtain that

$$s_n \to \sigma(B', \alpha, \beta) \text{ and } a_n \to 0(B, \alpha, \beta).$$

Further, in view of (14.2) and lemma 4(i), we have that, since $\alpha > 0$,

$$e^{-x} s_{\alpha,\beta+\alpha}(x) \in BV_x[0, \infty).$$

Hence

$$e^{-x} a_{\alpha,\beta}(x) = e^{-x}\{s_{\alpha,\beta}(x) - s_{\alpha,\beta+\alpha}(x)\} \in BV_x[0, \infty),$$

and so it follows that $a_n \to 0 |B, \alpha, \beta|$.

Also

$$A_{\alpha,\beta}(x) = \int_0^x e^t a_{\alpha,\beta}(t)\,dt = \int_0^x e^{-u} s_{\alpha,\beta}(u)\varphi(x-u)\,du,$$

where $\varphi(v) = \int_v^\infty t^{\alpha-1} e^{-t}\, dt$.

Since $\int_0^\infty \varphi(v)\,dv = \Gamma(\alpha+1) < \infty$, it follows from (14.2) and lemma 4(a) that $A_{\alpha,\beta}(x) \in BV_x[0, \infty)$, and further that $s_n \to \sigma |B', \alpha, \beta|$.

(ii) SUFFICIENCY. Suppose that $s_n \to \sigma |B', \alpha, \beta|$ and $a_n \to 0 |B, \alpha, \beta|$. Then we have that

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t)\,dt = \sigma + o(1) \tag{14.4}$$

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t)\,dt \in BV_x[0, \infty) \tag{14.5}$$

$$T_{\alpha,\beta}(x) = \alpha e^{-x} a_{\alpha,\beta}(x) = o(1) \tag{14.6}$$

$$T_{\alpha,\beta}(x) = \alpha e^{-x} a_{\alpha,\beta}(x) \in BV_x[0, \infty). \tag{14.7}$$

First, in view of theorem 13, it follows from (14.4) and (14.6) that $s_n \to \sigma$ (B, α, β). Further, using the notation in the second part of the proof of the theorem in [1], we obtain that

$$e^{-x}s_{\alpha,\beta+\delta}(x) = \int_0^x B(x-t)\varphi(t)dt$$

where (A) $\delta = k\alpha > 4$ where k is a positive integer,

(B) $\varphi(x) = \dfrac{d}{dx}\{e^{-x}f_{\delta-1}(x)\}$, where $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{\delta n}}{\Gamma(\delta n+1)}$, and satisfies

$$\int_0^\infty |\varphi(x)|\,dx < \infty,$$

(C) $B(x) = \int_0^x e^{-t}\{s_{\alpha,\beta}(t) - s_{\alpha,\beta+\delta}(t)\}dt.$

Now, from (14.5), we obtain that

$$\int_0^x e^{-t}\{s_{\alpha,\beta}(t) - s_{\alpha,\beta+\alpha}(t)\}dt \in BV_x[0,\infty),$$

and further, in view of lemma 4 (ii), that

$$\int_0^x e^{-t}\{s_{\alpha,\beta}(t) - s_{\alpha,\beta+\delta}(t)\}dt \in BV_x[0,\infty),$$

and hence it follows that $B(x-t)$ is of bounded variation with respect to x in the range $[t,\infty)$ uniformly for $t \geq 0$.

Hence, in view of lemma 3, it follows that

$$e^{-x}s_{\alpha,\beta+\delta}(x) \in BV_x[0,\infty).$$

Also, in view of (14.7) and lemma 4 (i), we obtain that

$$e^{-x}\{s_{\alpha,\beta}(x) - s_{\alpha,\beta+\delta}(x)\} \in BV_x[0,\infty)$$

since $\delta = k\alpha$, and so,

$$e^{-x}s_{\alpha,\beta}(x) \in BV_x[0,\infty).$$

Hence $s_n \to \sigma|B,\alpha,\beta|$.
This completes the proof of theorem 14.

THEOREM 15. *$s_n \to \sigma\,[B,\alpha,\beta]$ if and only if $a_n \to 0[B,\alpha,\beta]$ and $s_n \to \sigma[B',\alpha,\beta]$.*

PROOF. (i) NECESSITY, Suppose that $s_n \to \sigma[B,\alpha,\beta]$. By theorem 3 it follows that $s_n \to \sigma(B,\alpha,\beta)$, and so, by theorem 13, that $s_n \to \sigma(B',\alpha,\beta)$ and $a_n \to 0(B,\alpha,\beta)$. Further, from theorem 11, we have that

$$\int_0^x e^t|S'_{\alpha,\beta}(t)|\,dt = o(e^x);$$

that is

$$\int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta-1}(t)|\,dt = o(e^x),$$

and further, in view of lemma 5, that

$$\int_0^x |s_{\alpha,\beta+\alpha}(t) - s_{\alpha,\beta+\alpha-1}(t)|\,dt = o(e^x).$$

Thus,

$$\int_0^x |a_{\alpha,\beta}(t) - a_{\alpha,\beta-1}(t)|\,dt$$

$$\leq \int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta-1}(t)|\,dt + \int_0^x |s_{\alpha,\beta+\alpha}(t) - s_{\alpha,\beta+\alpha+1}(t)|\,dt$$

$$= o(e^x);$$

that is

$$\int_0^x e^t|T'_{\alpha,\beta}(t)|\,dt = o(e^x),$$

and so it follows that $a_n \to 0\,[B,\alpha,\beta]$.
This means that

$$\int_0^x e^t|T_{\alpha,\beta-1}(t)|\,dt = o(e^x),$$

and so, since

$$A_{\alpha,\beta}(t) = T_{\alpha,\beta}(t) = T_{\alpha,\beta-1}(t) - T'_{\alpha,\beta}(t)$$

it follows that

$$\int_0^x e^t|A'_{\alpha,\beta}(t)|\,dt = o(e^x).$$

Thus, $s_n \to \sigma[B',\alpha,\beta]$.

(ii) SUFFICIENCY. Suppose that $s_n \to \sigma[B', \alpha, \beta]$ and $a_n \to 0\ [B, \alpha, \beta]$. By theorem 3 it follows that $s_n \to \sigma(B, \alpha, \beta)$ and $a_n \to 0\ (B, \alpha, \beta)$, and so, by theorem 13, that $s_n \to \sigma(B, \alpha, \beta)$ and further, from theorem 7, that $s_n \to \sigma(B, \alpha, \beta+\nu)$ whenever $\nu \geqq 0$. Thus

$$s_{\alpha,\beta+\delta}(x) - s_{\alpha,\beta+\delta-1}(x) = o(e^x) \tag{15.1}$$

where $\delta = k\alpha \geqq 1$, k being a positive integer.

Further, from theorem 11, we obtain that

$$\int_0^x e^t |A'_{\alpha,\beta}(t)|\, dt = \int_0^x |a_{\alpha,\beta}(t)|\, dt = o(e^x)$$

and

$$\int_0^x e^t |T'_{\alpha,\beta}(t)|\, dt = \int_0^x \alpha |a_{\alpha,\beta}(t) - a_{\alpha,\beta-1}(t)|\, dt = o(e^x)$$

whence

$$\int_0^x |a_{\alpha,\beta-1}(t)|\, dt = o(e^x).$$

It follows from lemma 5, that

$$\int_0^x |s_{\alpha,\beta-1}(t) - s_{\alpha,\beta+\delta-1}(t)|\, dt = o(e^x)$$

and

$$\int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta+\delta}(t)|\, dt = o(e^x).$$

Thus, from these two results and (15.1) it follows that

$$\alpha \int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta-1}(t)|\, dt = \int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = o(e^x)$$

and so, by theorem 11, we further have that $s_n \to \sigma[B, \alpha, \beta]$. This completes the proof of theorem 15.

THEOREM 16. *$s_n \to \sigma(B', \alpha, \beta)$ if and only if $s_n \to \sigma(B, \alpha, \beta+1)$.*

This theorem is due to Borwein [2].

THEOREM 17. *$s_n \to \sigma|B', \alpha, \beta|$ if and only if $s_n \to \sigma|B, \alpha, \beta+1|$.*

PROOF. (i) NECESSITY. Suppose that $s_n \to \sigma|B', \alpha, \beta|$. Then we have that

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t)\, dt = \sigma + o(1) \tag{17.1}$$

$$A_{\alpha,\beta}(x) = \int_0^x e^{-t} a_{\alpha,\beta}(t)\, dt \in BV_x[0, \infty) \tag{17.2}$$

First, from (17.1) and in view of theorem 16, it follows that

$$S_{\alpha,\beta+1}(x) = \alpha e^{-x} s_{\alpha,\beta+1}(x) = \sigma + o(1),$$

that is $s_n \to \sigma(B, \alpha, \beta+1)$.

Further, using the argument in theorem 14, we obtain that

$$e^{-x} s_{\alpha,\beta+\delta}(x) \in BV_x[0, \infty)$$

where $\delta = k\alpha$ and k is a positive integer. So, in view of lemma 3(i), it follows that

$$e^{-x} s_{\alpha,\beta+\delta+1}(x) \in BV_x[0, \infty).$$

Now, for $x > 0$,

$$\int_0^x e^{-t} a_{\alpha,\beta}(t)\, dt$$

$$= -e^{-x} a_{\alpha,\beta+1}(x) + \int_0^x e^{-t} a_{\alpha,\beta+1}(t)\, dt \tag{17.3}$$

Also, from (17.2) and in view of lemma 4(ii), it follows that

$$\int_0^x e^{-t} a_{\alpha,\beta+1}(t)\, dt \in BV_x[0, \infty),$$

and hence, from (17.2) and (17.3) that

$$e^{-x} a_{\alpha,\beta+1}(x) \in BV_x[0, \infty).$$

Thus

$$e^{-x}\{s_{\alpha,\beta+1}(x)-s_{\alpha,\beta+\alpha+1}(x)\}\in BV_x[0,\infty),$$

and further, since $\alpha>0$, in view of lemma 4(i),

$$e^{-x}\{s_{\alpha,\beta+1}(x)-s_{\alpha,\beta+\delta+1}(x)\}\in BV_x[0,\infty)$$

where, as above, $\delta=k\alpha$, k being a positive integer. Hence

$$e^{-x}s_{\alpha,\beta+1}(x)\in BV_x[0,\infty)$$

and so, we obtain that $s_n\to\sigma|B,\alpha,\beta+1|$.

(ii) SUFFICIENCY. Suppose that $s_n\to\sigma|B,\alpha,\beta+1|$. Then, by theorem 14, $s_n\to\sigma|B',\alpha,\beta+1|$ and $a_n\to 0|B,\alpha,\beta+1|$, and so, from theorem 1, $s_n\to\sigma(B',\alpha,\beta+1)$ and $a_n\to 0(B,\alpha,\beta+1)$. Thus, by theorem 16, $s_n\to\sigma(B,\alpha,\beta)$. Further, we have

$$T_{\alpha,\beta+1}(x)=\alpha e^{-x}a_{\alpha,\beta+1}(x)\in BV_x[0,\infty)$$

and

$$A_{\alpha,\beta+1}(x)=\int_0^x e^{-t}a_{\alpha,\beta+1}(t)dt\in BV_x[0,\infty).$$

From (17.3) and these results, it immediately follows that

$$A_{\alpha,\beta}(x)=\int_0^x e^{-t}a_{\alpha,\beta}(t)dt\in BV_x[0,\infty),$$

and so, we have that $s_n\to\sigma|B',\alpha,\beta|$.

This completes the proof of theorem 17.

THEOREM 18. $s_n\to\sigma[B',\alpha,\beta]$ *if and only if* $s_n\to\sigma[B,\alpha,\beta+1]$.

PROOF. (i) NECESSITY. Suppose that $s_n\to\sigma[B',\alpha,\beta]$. From theorem 3, it follows that $s_n\to\sigma(B',\alpha,\beta)$ and so, from theorem 16, that $s_n\to\sigma(B,\alpha,\beta+1)$. Further, from theorem 11, we have

$$\int_0^x e^t|A'_{\alpha,\beta}(t)|\,dt=\alpha\int_0^x|a_{\alpha,\beta}(t)|\,dt$$
$$=o(e^x).$$

Thus, in view of lemma 5, we have

$$\int_0^x|a_{\alpha,\beta+1}(t)|\,dt=o(e^x),$$

and hence

$$\int_0^x|a_{\alpha,\beta+1}(t)-a_{\alpha,\beta}(t)|\,dt=o(e^x). \tag{18.1}$$

Thus

$$\int_0^x|\{s_{\alpha,\beta+1}(t)-s_{\alpha,\beta}(t)\}-\{s_{\alpha,\beta+\alpha+1}(t)-s_{\alpha,\beta+\alpha}(t)\}|\,dt=o(e^x),$$

and further, in view of lemma 5, we obtain that

$$\int_0^x|\{s_{\alpha,\beta+1}(t)-s_{\alpha,\beta}(t)\}-\{s_{\alpha,\beta+\delta+1}(t)-s_{\alpha,\beta+\delta}(t)\}|\,dt=o(e^x)$$

where $\delta=k\alpha$, k being a positive integer.

Also, starting from (18.1) and arguing as in the second part of the proof of theorem 15, we obtain that

$$\int_0^x|s_{\alpha,\beta+\delta+1}(t)-s_{\alpha,\beta+\delta}(t)|\,dt=o(e^x).$$

Hence

$$\alpha\int_0^x|s_{\alpha,\beta+1}(t)-s_{\alpha,\beta}(t)|\,dt=\int_0^x e^t|S'_{\alpha,\beta}(t)|\,dt$$
$$=o(e^x),$$

and so, in view of theorem 11, we obtain that $s_n\to\sigma[B,\alpha,\beta+1]$.

(ii) SUFFICIENCY. Suppose that $s_n\to\sigma[B,\alpha,\beta+1]$. From theorem 3, it follows that $s_n\to\sigma(B,\alpha,\beta+1)$ and so, from theorem 16, that $s_n\to\sigma(B',\alpha,\beta)$. Further, from theorem 15, we obtain that $s_n\to\sigma[B',\alpha,\beta+1]$ and $a_n\to 0\,[B,\alpha,\beta+1]$, and so we have, in view of theorem 11

$$\int_0^x e^t |T'_{\alpha,\beta+1}(t)|\, dt = \alpha \int_0^x |a_{\alpha,\beta+1}(t) - a_{\alpha,\beta}(t)|\, dt = o(e^x)$$

and

$$\int_0^x e^t |A_{\alpha,\beta+1}(t)|\, dt = \alpha \int_0^x |a_{\alpha,\beta+1}(t)|\, dt = o(e^x).$$

It follows immediately that

$$\int_0^x e^t |S'_{\alpha,\beta}(t)|\, dt = \alpha \int_0^x |a_{\alpha,\beta}(t)|\, dt = o(e^x),$$

and so, from theorem 11, that $s_n \to \sigma[B', \alpha, \beta]$.
This completes the proof of theorem 18.

It is interesting to note that the proof of theorem 18 nowhere uses the full strength of the hypothesis $s_n \to \sigma[B', \alpha, \beta]$. In fact, the weaker hypothesis $s_n \to \sigma(B', \alpha, \beta)$ will do, for by theorem 14 and lemma 5(a), we can obtain that

$$\int_0^x |s_{\alpha,\beta}(t) - s_{\alpha,\beta-1}(t) - s_{\alpha,\beta+\delta}(t) + s_{\alpha,\beta+\delta-1}(t)|\, dt = o(e^x)$$

This gives rise to the interesting result:

THEOREM 19. *$s_n \to \sigma[B, \alpha, \beta]$ if and only if $s_n \to \sigma(B', \alpha, \beta)$ and $a_n \to 0$ $[B, \alpha, \beta]$.*

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