# A NON-ENUMERABLE EXCEPTIONAL SET

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Let $\{x_n\}$ be a sequence of real positive numbers not converging to 0 and let $A$ be the set of all real numbers $a$ for which $\{x_n\}$ converges to 0 (mod $a$). The problem posed in this Journal of showing that the exceptional set $A$ must be of measure 0 was solved by I. J. Schoenberg [1964, 332] who asked whether or not $A$ is necessarily enumerable. In this note I show that $A$ need not be enumerable.

Let $x_n = 2^{n-1}(n-1)!$ and, for $n = 1, 2, \cdots; k = 0, \pm 1, \cdots$, let $I(k, n)$ be the closed interval

$$\left[\frac{k - \frac{1}{4n}}{x_n}, \frac{k + \frac{1}{4n}}{x_n}\right].$$

Then $I(k, n)$ contains the three intervals $I(j, n+1)$, $j = 2nk-1, 2nk, 2nk+1$, all other intervals $I(j, n+1)$ being disjoint from $I(k, n)$. Also $I(k, n)$ and $I(2nk, n+1)$ have a common centre $k/x_n = 2nk/x_{n+1}$.

Let

$$E_n = \bigcup_{-x_n < k < x_n} I(k, n); \qquad Y_n = E_1 \cap E_2 \cap \cdots \cap E_n.$$

There is clearly a set $N_n$ of $3^{n-1}$ integers such that

$$Y_n = \bigcup_{k \in N_n} I(k, n).$$

The set $Y = \bigcap_{n=1}^{\infty} Y_n$ is closed and contains the set $H$ of all centres of $I(k, n)$ with $k \in N_n$. Further, each point $y \in Y$ lies in infinitely many of the intervals $I(k, n)$ $(k \in N_n)$ and consequently every neighborhood of $y$ contains points of $H$. Hence $Y$ is nonempty and perfect and so cannot be enumerable.

If $y \in Y$, $y \neq 0$, then for every integer $n \geqq 1$ there is an integer $k_n \in N_n$ such that $y \in I(k_n, n)$; i.e. $|yx_n - k_n| \leqq 1/4n$ and so

$$x_n \to 0 \pmod{1/y}.$$

It follows that in this case the exceptional set $A$ is not enumerable.

In a note which appeared in this MONTHLY, (1964, p. 804), after the present article was accepted for publication, Dr. Paul Erdös demonstrated *inter alia* the non-enumerability of the above set $A$. The proof here given is somewhat simpler than his.

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